NO SET CARRIES EXACTLY THREE DENSE LINEAR ORDERS WITHOUT ENDPOINTS

A PROOF IN A WEAK ZERMELO THEORY WITHOUT CHOICE OR REPLACEMENT

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Abstract

Let Z_{sep} be the theory consisting of Extensionality, Pairing, Infinity, Union, Power Set, and the full Separation schema. Neither Choice, Replacement, Foundation, nor any form of Countable Choice is assumed. For a standard finite $n \geq 1$, the notation $s(X) = n$ abbreviates a first-order formula saying that X carries n, but not $n+1$, pairwise nonisomorphic dense linear orders without endpoints.

We prove

$$Z_{\text{sep}} \vdash \neg \exists X\, (s(X) = 3).$$

The first part is a countable-benchmark argument. If $\omega \hookrightarrow X$ for a DLO carrier X, a choice-free monotone-subsequence construction produces a countable set $B \subseteq X$ whose complement is again a DLO. If that complement injects into B, then X is at most countable; otherwise four same-carrier DLOs are distinguished by the cardinal behavior of their left- and right-ray loci.

For the second part, exact three supplies a self-dual DLO type. A nontrivial increasing automorphism immediately gives an injection $\omega \hookrightarrow X$. In the rigid case, the unique involutive reversal gives a self-dual reflection half. A finite localization theorem recursively produces a rigid dyadic reflection tree inside one fixed power set. Its center set is at most countable. If the half is not at most countable, one point outside all centers determines a branch, and successive local reflections form an injective ω-sequence. Both alternatives contradict exact three.

The result applies to every carrier and therefore gives a negative answer to Shelah's exact-three question for models of $\mathrm{Th}(\mathbb{Q}, <)$ on one underlying set. The general finite-spectrum problem is not resolved here.

Keywords. dense linear orders; axiom of choice; Dedekind-finite sets; categoricity; weak set theory.

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Contents

1. Introduction

A **dense linear order without endpoints**, abbreviated **DLO**, is a nonempty strict linear order $(X, <)$ such that:

(1) whenever $x < y$, there is z with $x < z < y$;
(2) no point is least;
(3) no point is greatest.

A DLO carrier is infinite: every nonempty finite linear order has a least and a greatest point by Lemma 2.9 below.

Throughout this paper,

$$X \text{ is at most countable} \iff X \hookrightarrow \omega,$$

and

$$X \text{ is Dedekind-infinite} \iff \omega \hookrightarrow X.$$

These two conditions need not be complements without Choice. We use the word "uncountable" only when quoting the terminology of prior work.

In the language $\{<\}$, the DLO axioms form the complete theory $\mathrm{Th}(\mathbb{Q},<)$ and admit quantifier elimination; see Marker 2002. Thus a structure with universe X is a model of $\mathrm{Th}(\mathbb{Q},<)$ exactly when its order relation is a DLO on X, and structure isomorphisms are precisely order isomorphisms.

In Definition 7.2(2), Shelah glosses categoricity in the power $|X|$ as uniqueness up to isomorphism among models with universe X. Claim 7.10(2)–(3) makes this same-universe convention explicit when counting several pairwise nonisomorphic models, and Question 7.11 asks whether it is consistent with ZF that $\mathrm{Th}(\mathbb{Q},<)$ have exactly three models in some uncountable power Shelah 2009. Transporting structures along bijections of their carriers shows that, for a fixed finite spectrum, this is equivalent to the same-carrier formulation used here. No hypothesis such as $X \cong X \times X$ is imposed in Question 7.11, so the theorem applies in particular to the narrower reasonable powers of Definition 7.5.

The immediate catalyst for this work was Elliot Glazer's public report of a 24-hour Sol experiment on the broader finite-spectrum problem and the accompanying conversation transcript Glazer 2026a, Glazer 2026b. The transcript explicitly proposed that an exact-three spectrum is impossible and included an earlier public candidate proof outline based on circular cuts, the loci of points with countable left or right rays, an asserted two-sided core, and a 3×3 family of left and right endpoint markers. The proof given here is self-contained. It does not use the transcript's universal core-extraction or circular-cut steps. It retains the ray-locus idea in the fully case-split countable-benchmark four-type theorem, and handles the remaining exact-three problem by a different chain: the symmetric-flank theorem, labeled finite localization, and a rigid dyadic reflection tree whose off-center reflections produce an injective ω-sequence. The transcript is therefore cited for provenance and motivation, not as a premise whose correctness is assumed below.

Immediately before Question 7.11, Shelah studies an Ehrenfeucht expansion of DLO by an increasing sequence of constants. Those constants already provide an injection of ω into the carrier. For the pure DLO language no such sequence is named, and the main role of the rigid reflection tree below is to produce an internal ω-sequence without making countably many arbitrary choices.

For background on choiceless cardinal comparison and Dedekind-finite sets, see Jech 1973 and Howard–Rubin 1998. Walczak–Typke studies first-order structures on weakly Dedekind-finite sets Walczak–Typke 2005, and Truss surveys related interactions between Choice and model-theoretic structures

Truss 2019. These results provide context but are not used as premises of the proof.

The proof is organized around two statements.

Theorem 1.1 (Dedekind-infinite spectrum alternative). If a DLO carrier X admits an injection $\omega \hookrightarrow X$, then

$$X \hookrightarrow \omega \quad \text{or} \quad s(X) \geq 4. \tag{1}$$

In the first alternative, countable back-and-forth gives $s(X) = 1$.

Theorem 1.2 (Exact-three carrier dichotomy). If $s(X) = 3$, then

$$X \hookrightarrow \omega \quad \text{or} \quad \omega \hookrightarrow X. \tag{2}$$

If the first alternative holds, countable back-and-forth gives $s(X) = 1$. If the second holds, Theorem 1.1 gives either $X \hookrightarrow \omega$ or $s(X) \geq 4$; the first of these again gives $s(X) = 1$. Either conclusion contradicts $s(X) = 3$.

The load-bearing ingredients are the countable-benchmark four-type theorem, the common-host self-duality criterion, and the sharp finite inequality

$$t \leq q = 2^{n+1} - 1,$$

where t is the number of center points and q is the number of remaining DLO cells in the dyadic localization. The foundational coding used in the weak theory is developed in Appendices A and B.

A companion Lean 4 development formalizes all 57 numbered items in this paper and the claimed proof-theoretic endpoint. It includes an object-language encoding of Z_{sep}, a natural-deduction calculus, and Gödel completeness by a Henkin construction. Its final declaration is `ExactThreeDLO.FO.zsep_proves_not_spec3Sentence`; its type asserts `Provable Zsep` of the negation of `spec3Sentence`. The source-to-Lean correspondence and archived software release are available in Isthmus 2026.

2. Formal setting and notation

Definition 2.1 (The base theory). Z_{sep} consists of:

- Extensionality;
- Pairing;
- Infinity;
- Union;
- Power Set;
- the full Separation schema.

No use is made of Choice, Countable Choice, Dependent Choice, Replacement, or Foundation.

The construction of ω, the induction scheme on ω, ordered pairs, Cartesian products, and the arithmetic used below are recorded in Appendix A. Finite functions and finite sequences are coded inside fixed power sets. In particular,

$$2^{<\omega} := \{s \in \mathcal{P}(\omega \times 2) : s \text{ is a function with domain some } n \in \omega\} \tag{3}$$

is a set.

Definition 2.2 (Order and cardinal-comparison notation). For a linear order $L = (X, <)$:

- L^* denotes the reverse order;
- $(-\infty, x)_L$ and $(x, \infty)_L$ denote the strict left and right rays;
- $A \hookrightarrow B$ means that an injection from A to B exists;
- $A \not\hookrightarrow B$ means that no such injection exists;
- a set is **finite** if it is bijective with some finite ordinal, and **infinite** otherwise;
- a set is **at most countable** if it injects into ω;
- a set is **Dedekind-infinite** if ω injects into it;
- for every $m \in \omega$, the notation $|A| \le m$ abbreviates the existence of an injection $A \to m$, and $|A| = m$ abbreviates the existence of a bijection $A \to m$.

Finite order sums are taken over disjoint underlying blocks. Empty blocks are omitted. Whenever empty blocks are omitted, every newly created adjacent boundary is checked against the criterion of Lemma 2.7. A singleton block is denoted by $\mathbf{1}$ or by its unique point.

Definition 2.3 (Internal isomorphism). For orders (A, R) and (B, S), an isomorphism is a set $f \subseteq A \times B$ which is a bijection and satisfies

$$x \, R \, y \iff f(x) \, S \, f(y).$$

An anti-isomorphism is an isomorphism to the reverse order. All orders and all witnessing functions in this paper are internal sets of the ambient model.

Definition 2.4 (Finite spectrum formulas). For each standard finite $n \ge 1$, let $\mathrm{Spec}_{\ge n}(X)$ be the first-order assertion that there are DLO relations

$$R_0, \dots, R_{n-1} \subseteq X \times X$$

which are pairwise nonisomorphic. Put

$$\mathrm{Spec}_{=n}(X) := \mathrm{Spec}_{\geq n}(X) \wedge \neg\, \mathrm{Spec}_{\geq n+1}(X). \tag{4}$$

We write

$$s(X) \geq n$$

and

$$s(X) = n$$

only as abbreviations for these formulas. No quotient set of isomorphism classes and no cardinal-valued object $s(X)$ is formed.

Let $\Phi_n(X)$ be the finite-representative assertion that there are pairwise nonisomorphic DLOs $R_0, \dots, R_{n-1}$ on X and every DLO on X is isomorphic to exactly one of them.

Lemma 2.5 (Exact finite spectra and representative families). For every standard finite $n \geq 1$,

$$\Phi_n(X) \quad \Longleftrightarrow \quad s(X) = n. \tag{5}$$

Proof. Assume $\Phi_n(X)$. If $S_0, \dots, S_n$ were pairwise nonisomorphic DLOs on X, exactness and pairwise nonisomorphism of the representatives would assign to each S_j a unique index $i_j < n$ with $S_j \cong R_{i_j}$. The graph $j \mapsto i_j$ is obtained by Separation from $(n+1) \times n$. Corollary 2.10 gives two distinct $j, j' \leq n$ with $i_j = i_{j'}$. Composition of the internal isomorphisms would make S_j and $S_{j'}$ isomorphic, a contradiction. Hence $s(X) = n$.

Conversely, assume $s(X) = n$ and choose n pairwise nonisomorphic DLOs $R_0, \dots, R_{n-1}$. If another DLO R were nonisomorphic to all of them, then

$$R_0, \dots, R_{n-1}, R$$

would witness $s(X) \geq n + 1$. Hence every DLO is isomorphic to one of the chosen representatives, and uniqueness follows from their pairwise nonisomorphism. □

Lemma 2.6 (Bounded unique recursion). Let A be a set, let $a_0 \in A$, and let $\Psi(n, a, b, p)$ be a formula. Assume that for every $n < \omega$ there is a unique finite function

$$\sigma_n : n + 1 \longrightarrow A$$

such that

$$\sigma_n(0) = a_0$$

and

$$\Psi(i, \sigma_n(i), \sigma_n(i+1), p) \qquad (i < n). \tag{6}$$

Then there is a unique function $f : \omega \to A$ satisfying

$$f(0) = a_0$$

and

$$\Psi(n, f(n), f(n+1), p) \qquad (n < \omega).$$

Its graph is a set in Z_{sep}.

Proof. Finite histories are coherent. The restriction

$$\sigma_{n+1} \restriction (n+1)$$

satisfies the length-$(n+1)$ history condition, and therefore uniqueness gives

$$\sigma_{n+1} \restriction (n+1) = \sigma_n. \tag{7}$$

Define

$$G := \{(n, a) \in \omega \times A : \exists \sigma \, (\sigma : n+1 \to A \wedge \sigma \text{ satisfies (6)} \wedge \sigma(n) = a)\}.$$

Finite functions from an initial segment of ω into A are coded inside a fixed power set built from $\omega \times A$; see Lemma B.1, which uses only the product construction of Proposition A.4. Thus G exists by Separation on $\omega \times A$. Uniqueness makes G a function, and (7) gives the required step relation.

If f' is any other solution, then $f' \restriction (n+1)$ is the unique length-$(n+1)$ history for every n. Hence $f' = G$, proving uniqueness. □

Lemma 2.7 (Finite block-sum DLO criterion). Let

$$X = B_0 \sqcup \cdots \sqcup B_{m-1}$$

be a finite partition into nonempty linearly ordered blocks, and give X the order sum

$$B_0 + \cdots + B_{m-1}.$$

Assume:

(1) every nonsingleton block is dense internally;
(2) at every adjacent boundary $B_i \mid B_{i+1}$, the left block has no greatest point or the right block has no least point;
(3) the first block has no least point;
(4) the last block has no greatest point.

Then the order sum is a DLO.

Proof. If two points lie in one nonsingleton block, use internal density. If they lie in adjacent blocks, use condition 2. If their blocks are separated by at least one intermediate block, any point in an intermediate nonempty block lies between them. Conditions 3 and 4 give the absence of endpoints. □

Empty-block convention. If a displayed order sum contains empty blocks, first delete them and then verify condition 2 for every boundary in the reduced block list; deleting an empty block may create a new adjacent boundary.

Lemma 2.8 (Finite ranking). Let $m \in \omega$ and let $S \subseteq m$. There are unique $t \leq m$ and a unique strictly increasing bijection

$$e : t \longrightarrow S. \tag{8}$$

Consequently:

(1) every subset of a finite ordinal is finite;
(2) every nonempty subset of a finite ordinal has a greatest element;
(3) if $u : Y \to m$ is injective, then Y is bijective with some finite ordinal $t \leq m$.

Proof. We use induction on m.

For $m = 0$, one has $S = \emptyset$, $t = 0$, and the empty function is the unique increasing bijection.

Assume the statement for m, and let $S \subseteq m + 1$.

- If $m \notin S$, apply the induction hypothesis to $S \subseteq m$. For uniqueness, let $d : u \to S$ be any strictly increasing bijection. Finite induction gives $d(i) \geq i$ for every $i < u$. Hence $u \leq m$, so the uniqueness clause of the induction hypothesis gives $u = t$ and $d = e$.
- If $m \in S$, apply the induction hypothesis to $S \setminus \{m\}$. If

$$e : t \longrightarrow S \setminus \{m\}$$

is the unique increasing bijection, extend it by $e^+(t) = m$. Then $e^+ : t+1 \to S$ is increasing and onto. Conversely, if $d : u \to S$ is an increasing bijection, then $u > 0$ and, because m is the greatest member of S,

$$d(u-1) = m.$$

The restriction $d \restriction (u-1)$ is an increasing bijection onto $S \setminus \{m\}$. Since its values lie in m, the same inequality $d(i) \geq i$ gives $u-1 \leq m$. The induction hypothesis therefore gives $u-1 = t$ and $d \restriction t = e$, so $d = e^+$.

All functions involved are subsets of the fixed finite product $(m+1) \times (m+1)$.

Item 1 is the existence part. For item 2, if $S \neq \emptyset$, then its increasing enumeration has nonzero domain t, and $e(t-1)$ is the greatest element. For item 3, put $S = u[Y] \subseteq m$. The injection $u : Y \to S$ is a bijection; compose it with the inverse of the increasing bijection just constructed. $\square$

Lemma 2.9 (Finite linear-order enumeration). Let $m \in \omega$, let A be a set, let $u : m \to A$ be a bijection, and let $\prec$ be a strict linear order on A. Then there is a unique order isomorphism

$$e : (m, \in) \longrightarrow (A, \prec). \tag{9}$$

If $m > 0$, the points $e(0)$ and $e(m-1)$ are respectively the unique least and greatest elements of $(A, \prec)$.

Proof. We use induction on m. The case $m = 0$ is immediate.

Assume the assertion for m, let $u : m+1 \to A$ be a bijection, and put

$$a := u(m), \qquad A_0 := A \setminus \{a\}.$$

The restriction $u \restriction m : m \to A_0$ is a bijection. By the induction hypothesis there is a unique increasing bijection

$$e_0 : (m, \in) \longrightarrow (A_0, \prec).$$

Let

$$J := \{i \in m : a \prec e_0(i)\}.$$

If $J \neq \emptyset$, let j be its least element, supplied by Lemma 2.8; if $J = \emptyset$, put $j = m$. Insert a at position j:

$$e(i) = \begin{cases} e_0(i), & i < j, \\ a, & i = j, \\ e_0(i-1), & j < i < m+1. \end{cases}$$

For $i < j$, minimality of j and trichotomy give $e_0(i) \prec a$; for $j < m$, one has $a \prec e_0(j)$, and monotonicity of e_0 handles all later values. Thus e is an increasing bijection.

In any increasing bijection $d : m+1 \to A$, the position of a is forced to be j. Deleting that coordinate yields an increasing bijection from m onto A_0, which equals e_0 by the induction hypothesis. Hence $d = e$. The endpoint assertion follows from the increasing enumeration. □

Corollary 2.10 (Finite pigeonhole principle). For every $m \in \omega$:

(1) every injection $f : m \to m$ is surjective;
(2) there is no injection $m+1 \to m$.

Moreover, if $m, n \in \omega$ and $m \cong n$, then $m = n$.

Proof. We prove item 1 by induction on m. The case $m = 0$ is immediate. Assume every injection $m \to m$ is surjective, and let $f : m+1 \to m+1$ be injective. Put $a = f(m)$, and let $\tau : m+1 \to m+1$ be the transposition of a and m (the identity if $a = m$); its graph is obtained by Separation on $(m+1) \times (m+1)$. Then $g := \tau \circ f$ is injective and satisfies $g(m) = m$. Injectivity gives $g(x) \neq m$ for $x < m$, so $g \restriction m : m \to m$ is injective and therefore surjective by the induction hypothesis. Thus g, and hence f, is surjective.

For item 2, the case $m = 0$ is immediate because there is no function $1 \to 0$. Suppose $m > 0$ and $h : m+1 \to m$ were injective. Then $h \restriction m : m \to m$ would be surjective by item 1. Hence $h(m) = h(k)$ for some $k < m$, contradicting injectivity.

For the final assertion, finite ordinals are comparable by Proposition A.2. If $m < n$, then $m+1 \subseteq n$; composing this inclusion with a bijection $n \to m$ would give an injection $m+1 \to m$, contrary to item 2. The case $n < m$ is symmetric. □

Lemma 2.11 (Infinite subcountable sets are countably infinite). Let Y be infinite and suppose $i : Y \to \omega$ is injective. Then

$$Y \cong \omega. \tag{10}$$

Proof. The range

$$I := \{n \in \omega : \exists y \in Y\ (i(y) = n)\}$$

exists by Separation. The map $i : Y \to I$ is a bijection, so I is infinite.

By Proposition A.3, I has a least element. Define the increasing enumeration $e : \omega \to I$ by the local recursion

$$e(0) = \min I,$$

$$e(k+1) = \min\{m \in I : e(k) < m\}. \tag{11}$$

The successor set is nonempty. If no member of I exceeded $e(k)$, then

$$I \subseteq e(k) + 1$$

would be finite by Lemma 2.8, contrary to the infinitude of I. The step in (11) depends only on the current value, and every finite history exists uniquely by induction. Lemma 2.6 therefore gives the graph of e.

The map e is onto I. Since it is strictly increasing, induction gives $e(k) \geq k$. If $m \in I$ were omitted, then $e(m+1) > m$, so there would be a least j with $m < e(j)$. One has $j > 0$ because $e(0) = \min I \leq m$. Minimality of j gives $e(j-1) \leq m$, and this inequality is strict because m is omitted from the range of e. Thus

$$e(j-1) < m < e(j),$$

contradicting the definition of $e(j)$ as the least member of I above $e(j-1)$. Thus e is a bijection $\omega \to I$. Composing it with the inverse of $i : Y \to I$ gives a bijection $\omega \to Y$. □

Lemma 2.12 (Finite choice). Let $m \in \omega$, let Y be a set, and let $R \subseteq m \times Y$. If every fiber

$$R_i := \{y \in Y : (i, y) \in R\}$$

is nonempty, then there is a function $c : m \to Y$ with $c(i) \in R_i$ for every $i < m$.

Proof. Use induction on m. The empty function handles $m = 0$. For $m+1$, apply the induction hypothesis to $R \restriction (m \times Y)$, instantiate the nonemptiness of the last fiber once to obtain $y \in R_m$, and extend the resulting function by the pair (m, y). Pairing and Union form the extension. □

3. A countable benchmark and the Dedekind-infinite alternative

Lemma 3.1 (Choice-free monotone subsequence). Let $(X, <)$ be a linear order and let $a : \omega \to X$ be injective. There is an internal strictly increasing or strictly decreasing subsequence

$$b : \omega \longrightarrow X. \tag{12}$$

Proof. Call n a **peak** if

$$a(m) < a(n) \qquad \text{for every } m > n.$$

The peak set $P_{\text{peak}} \subseteq \omega$ exists by Separation.

If P_{peak} is unbounded, Proposition A.3 gives its least element p_0 and, at each stage, the least peak greater than p_k. The finite histories of this local recursion exist uniquely, so Lemma 2.6 applies. For $i < j$, peakness of p_i gives $a(p_j) < a(p_i)$; hence $b(i) := a(p_i)$ is strictly decreasing.

Suppose P_{peak} is bounded. If it is empty, put $N = 0$. Otherwise choose M with $P_{\text{peak}} \subseteq M + 1$. Lemma 2.8 makes P_{peak} finite and supplies its greatest element; put

$$N = \max(P_{\text{peak}}) + 1.$$

Set $n_0 = N$. Since n_k is not a peak, there is $m > n_k$ such that

$$\neg(a(m) < a(n_k)).$$

Injectivity gives $a(m) \neq a(n_k)$, hence $a(n_k) < a(m)$. Let n_{k+1} be the least such m. Again the local step has unique finite histories, so Lemma 2.6 applies. Then $b(k) := a(n_k)$ is strictly increasing. □

Lemma 3.2 (A monotone sequence has DLO complement). Let $L = (X, <)$ be a DLO, let $b : \omega \to X$ be strictly monotone, and put

$$B := b[\omega], \qquad C := X \setminus B. \tag{13}$$

Then C, with the inherited order, is a nonempty DLO without endpoints.

Proof. Assume first that b is increasing.

Density between $b(0)$ and $b(1)$ gives a point outside B, so $C \neq \emptyset$.

Let $x < y$ lie in C. Choose z with $x < z < y$. If $z \in C$, stop. Otherwise $z = b(n)$.

- If $b(n+1) < y$, choose a point strictly between $b(n)$ and $b(n+1)$. No point of the increasing sequence lies in that interval.
- If $y \le b(n+1)$, choose a point strictly between $b(n)$ and y. Again no sequence point lies there.

Thus C is dense.

To find a point of C below x, choose $z < x$. If $z \in C$, stop. If $z = b(n)$ and $n > 0$, choose a point between $b(n-1)$ and $b(n)$. If $n = 0$, choose a point below $b(0)$. The proof of a point above x is analogous, using the interval between $b(n)$ and $b(n+1)$ when necessary.

The decreasing case follows by applying the increasing argument to the reverse order. ☐

Definition 3.3 (Benchmark-small rays and signatures). Let B be a countably infinite set and let $L = (Y, <)$ be a DLO.

A set is B**-small** if it injects into B, and B**-large** otherwise. Define

$$\mathsf{L}_B(L) := \{y \in Y : (-\infty, y)_L \hookrightarrow B\}, \tag{14}$$

and

$$\mathsf{R}_B(L) := \{y \in Y : (y, \infty)_L \hookrightarrow B\}. \tag{15}$$

For a set A, put

$$\mathrm{bc}_B(A) = \begin{cases} 0, & A = \emptyset, \\ 1, & A \neq \emptyset \text{ and } A \hookrightarrow B, \\ 2, & A \not\hookrightarrow B. \end{cases}$$

The benchmark signature is

$$\mathrm{Sig}_B(L) := (\mathrm{bc}_B(\mathsf{L}_B(L)), \mathrm{bc}_B(\mathsf{R}_B(L))). \tag{16}$$

Lemma 3.4 (Elementary countable-benchmark facts). Let B be countably infinite.

(1) Every internally finite disjoint union of B-small sets is B-small.
(2) B can be partitioned into any prescribed positive finite number of countably infinite pieces.
(3) Every set bijective with B carries a DLO without endpoints.
(4) If Y is infinite and $Y \hookrightarrow B$, then $Y \cong B$.

Proof. Fix a bijection $\beta : B \to \omega$.

For item 1, let $(A_i)_{i<m}$ be a coded pairwise disjoint family and assume $A_i \hookrightarrow B$ for each $i < m$. Put $U = \bigcup_{i<m} A_i$. Every injection $A_i \to B$ is an element of the fixed set $\mathcal{P}(U \times B)$. Apply Lemma 2.12 to choose injections $f_i : A_i \to B$ for all $i < m$. If $m = 0$, the claim is trivial. If $m > 0$, define

$$F(a) = \beta^{-1}(m \cdot \beta(f_i(a)) + i) \qquad (a \in A_i). \tag{17}$$

Division with remainder by m shows that F is injective.

For item 2, for $i < m$ put

$$B_i = \left\{x \in B : \exists q \in \omega \left(\beta(x) = mq + i\right)\right\}.$$

The division algorithm gives a partition $B = \bigsqcup_{i<m} B_i$, and $q \mapsto \beta^{-1}(mq+i)$ is a bijection $\omega \to B_i$.

Item 3 follows by transporting the explicit countable DLO of Lemma B.4. Item 4 follows from Lemma 2.11. ☐

Proposition 3.5 (Invariance of benchmark signatures). Let $f : L \to M$ be an order isomorphism.

(1) For a fixed benchmark B,

$$f[\mathsf{L}_B(L)] = \mathsf{L}_B(M),$$

$$f[\mathsf{R}_B(L)] = \mathsf{R}_B(M),$$

and

$$\mathrm{Sig}_B(L) = \mathrm{Sig}_B(M). \tag{18}$$

(2) If $B \cong B'$, then for every order L,

$$\mathsf{L}_B(L) = \mathsf{L}_{B'}(L), \qquad \mathsf{R}_B(L) = \mathsf{R}_{B'}(L),$$

and the two benchmark signatures agree.

The isomorphism f is not required to preserve the benchmark subset.

Proof. Corresponding rays are bijective under f, so one injects into B exactly when the other does. This proves item 1. If $B \cong B'$, injectability into B and injectability into B' are equivalent, proving item 2. ☐

Lemma 3.6 (Countable-ray decomposition). Let C be a DLO disjoint from a countably infinite set B, and assume

$$C \not\hookrightarrow B. \tag{19}$$

Put

$$P := \mathsf{L}_B(C), \qquad Q := \mathsf{R}_B(C), \qquad G := C \setminus (P \cup Q). \tag{20}$$

Then:

(1) P is an initial segment and Q is a final segment;
(2) $P \cap Q = \emptyset$;
(3) if $P \not\hookrightarrow B$, every $p \in P$ has a B-small left ray in P and a B-large right ray in P;
(4) if $Q \not\hookrightarrow B$, every $q \in Q$ has a B-large left ray in Q and a B-small right ray in Q;
(5) if $P \hookrightarrow B$ and $Q \hookrightarrow B$, then G is a nonempty DLO and every point of G has two rays in G which do not inject into $B \sqcup P \sqcup Q$.

Proof. Ray inclusion makes P initial and Q final.

If $x \in P \cap Q$, both rays of x inject into B. Lemma 3.4 gives an injection

$$C = (-\infty, x)_C \sqcup \{x\} \sqcup (x, \infty)_C \hookrightarrow B,$$

contrary to (19).

Suppose $P \not\hookrightarrow B$. For $p \in P$, its left ray in P is its left ray in C and is B-small. If its right ray in P were also B-small, then P would inject into B. The assertion for Q is dual.

Finally suppose $P, Q \hookrightarrow B$. If $G = \emptyset$, then $C = P \sqcup Q$ injects into B. Thus $G \neq \emptyset$. Convexity is immediate. A first point of G would have a left ray contained in P, and a last point would have a right ray contained in Q, contrary to the definitions of P and Q. Hence G is a DLO.

Put

$$\widehat{B} := B \sqcup P \sqcup Q.$$

The set $\widehat{B}$ contains B and is infinite. Lemma 3.4(1) gives $\widehat{B} \hookrightarrow B$, so Lemma 3.4(4) gives $\widehat{B} \cong B$.

If a ray in G injected into $\widehat{B}$, it would inject into B by Proposition 3.5(2). Adding the appropriate B-small side block would make the corresponding ray in C B-small, placing the point in P or Q. $\square$

Corollary 3.7 (Endpoint facts for the ray pieces). Under the hypotheses of Lemma 3.6:

(1) P has no least point and Q has no greatest point whenever they are nonempty;
(2) if $P \not\hookrightarrow B$, then P has no greatest point and hence is a DLO;
(3) if $Q \not\hookrightarrow B$, then Q has no least point and hence is a DLO;
(4) G is convex and is internally dense whenever it has at least two points; it may be empty or have endpoints.

Proof. A nonempty initial segment of a no-left-endpoint order has no least point, and the dual statement holds for Q.

If P had a greatest point p, then $(p, \infty)_P = \emptyset$ would be B-small, contradicting Lemma 3.6(3). The proof for Q is dual. Convex subsets of a dense order are internally dense. □

Lemma 3.8 (Small/large calculus). Let B be countably infinite.

(1) An internally finite union of B-small sets is B-small.
(2) If $A \not\hookrightarrow B$ and $A \subseteq R$, then $R \not\hookrightarrow B$.

Proof. For item 1, let $(A_i)_{i<m}$ be a coded finite family of B-small sets. Define by Separation

$$A_i' := \{a \in A_i : \forall j < i \ (a \notin A_j)\}.$$

Then the A_i' are pairwise disjoint, each is B-small by restriction, and

$$\bigcup_{i<m} A_i = \bigsqcup_{i<m} A_i'.$$

Apply Lemma 3.4(1). Item 2 follows by restricting any hypothetical injection $R \to B$ to A. □

Theorem 3.9 (Countable benchmark four-type theorem). Let B be countably infinite, let C be a disjoint DLO, and suppose $C \not\hookrightarrow B$. Then the carrier

$$X := B \sqcup C$$

supports at least four pairwise nonisomorphic DLOs.

Proof. Use the decomposition $C = P + G + Q$ from Lemma 3.6. Empty blocks are deleted before applying Lemma 2.7.

Case 1: $P \not\hookrightarrow B$ and $Q \not\hookrightarrow B$.

Partition $B = B_{\mathrm{L}} \sqcup B_{\mathrm{R}}$ into two countably infinite sets and place DLOs $E_{\mathrm{L}}, E_{\mathrm{R}}$ on them. Define

$$L_{22} := E_{\mathrm{L}} + P + G + Q + E_{\mathrm{R}},$$

$$L_{11} := E_{\mathrm{L}} + Q + G + P + E_{\mathrm{R}},$$

$$L_{21} := E_{\mathrm{L}} + P + G + Q^* + E_{\mathrm{R}},$$

and

$$L_{12} := E_{\mathrm{L}} + P^* + G + Q + E_{\mathrm{R}}. \tag{21}$$

Corollary 3.7 verifies every boundary in the reduced block list, including the case $G = \emptyset$. The exact ray loci are:

Order	induced order on left-small locus	induced order on right-small locus	signature
L_{22}	$E_{\mathrm{L}} + P$	$Q + E_{\mathrm{R}}$	$(2, 2)$
L_{11}	E_{L}	E_{R}	$(1, 1)$
L_{21}	$E_{\mathrm{L}} + P$	E_{R}	$(2, 1)$
L_{12}	E_{L}	$Q + E_{\mathrm{R}}$	$(1, 2)$

For instance, in L_{11} every point of Q has a large left ray within Q, and every point of P has a large right ray within P. Points farther to the right or left contain one of these large blocks in the corresponding ray. Thus the only small left and right loci are E_{L} and E_{R}. For L_{21}, the right ray inside Q^* is the left ray in Q and is therefore large; hence the right-small locus is exactly E_{R}. Dually, the left ray inside P^* is the right ray in P and is large, so the left-small locus of L_{12} is exactly E_{L}. The remaining entries follow from Lemmas 3.6 and 3.8.

Case 2: $P \not\hookrightarrow B$ and $Q \hookrightarrow B$.

Put

$$\widetilde{B} := B \sqcup Q, \qquad K := C \setminus Q = P + G.$$

The set $\widetilde{B}$ is infinite and injects into B, so Lemma 3.4(4) gives $\widetilde{B} \cong B$.

The order K has no least point because it contains the nonempty initial DLO P. It has no greatest point: if $u = \max K$, then $(u, \infty)_C \subseteq Q \hookrightarrow B$, so $u \in Q$, contrary to $u \in K$. Thus K is a DLO.

We compute its ray loci. If $p \in P$, then

$$(-\infty, p)_K = (-\infty, p)_C \hookrightarrow B \hookrightarrow \widetilde{B}.$$

If $x \in K \setminus P$, then $P \subsetneq (-\infty, x)_K$, and $P \not\hookrightarrow \widetilde{B}$ because $\widetilde{B} \cong B$. Hence

$$\mathsf{L}_{\widetilde{B}}(K) = P. \tag{22}$$

If $(x, \infty)_K \hookrightarrow \widetilde{B}$, then Proposition 3.5(2) gives $(x, \infty)_K \hookrightarrow B$. Since $Q \hookrightarrow B$, Lemma 3.4(1) would give

$$(x, \infty)_C = (x, \infty)_K \sqcup Q \hookrightarrow B,$$

forcing $x \in Q$, a contradiction. Therefore

$$\mathsf{R}_{\widetilde{B}}(K) = \emptyset. \tag{23}$$

Independently place a DLO E on all of $\widetilde{B}$. For the two-sided constructions, partition $\widetilde{B}$ into two countably infinite sets and place independent DLO relations E_0, E_1 on them. Define

$$M_{20} := E + K, \qquad M_{21} := E_0 + K + E_1,$$

$$M_{02} := K^* + E, \qquad M_{12} := E_0 + K^* + E_1. \tag{24}$$

Throughout this case the loci may be computed with respect to either B or $\widetilde{B}$: Proposition 3.5(2) and $\widetilde{B} \cong B$ make the two notions of smallness identical. The table records the induced orders on the loci.

The exact ray loci are:

Order	induced order on left-small locus	induced order on right-small locus	signature
M_{20}	$E + P$	$\emptyset$	$(2, 0)$
M_{21}	$E_0 + P$	E_1	$(2, 1)$
M_{02}	$\emptyset$	$P^* + E$	$(0, 2)$
M_{12}	E_0	$P^* + E_1$	$(1, 2)$

These equalities follow directly from (22)–(23), Lemma 3.8, and reversal.

Case 3: $P \hookrightarrow B$ **and** $Q \not\hookrightarrow B$**.**

Apply Case 2 to the reverse order.

Case 4: $P \hookrightarrow B$ and $Q \hookrightarrow B$.

Put

$$\widehat{B} := B \sqcup P \sqcup Q.$$

As in Lemma 3.6, $\widehat{B} \cong B$. Lemma 3.6 gives a nonempty DLO G in which every point has two $\widehat{B}$-large rays.

Independently place a DLO E on all of $\widehat{B}$. For the two-sided construction, partition $\widehat{B}$ into two countably infinite sets and place independent DLO relations E_0, E_1 on them. Choose $g \in G$ and put

$$G^- := (-\infty, g)_G, \qquad G^+ := (g, \infty)_G.$$

Define

$$N_{10} := E + G, \qquad N_{01} := G + E,$$

$$N_{11} := E_0 + G + E_1, \qquad N_{00} := G^- + E + \{g\} + G^+. \tag{25}$$

Proposition 3.5(2) and $\widehat{B} \cong B$ again make B-smallness and $\widehat{B}$-smallness identical. The table records the induced orders on the loci.

The exact ray loci are:

Order	induced order on left-small locus	induced order on right-small locus	signature
N_{10}	E	$\emptyset$	$(1,0)$
N_{01}	$\emptyset$	E	$(0,1)$
N_{11}	E_0	E_1	$(1,1)$
N_{00}	$\emptyset$	$\emptyset$	$(0,0)$

For N_{00}, consider the four possible locations of a point. A point in G^- retains its original large left ray in G, while its right ray contains the original large right ray. A point in E has G^- to its left and G^+ to its right. The point g has rays G^- and G^+. The argument for a point in G^+ is dual. Lemma 3.8 therefore gives empty small-ray loci on both sides.

In every case, Proposition 3.5 distinguishes the four orders. □

Lemma 3.10 (Finite subsets of DLO regions). Let $(X, <)$ be a DLO, and let J be one of the following nonempty regions:

- X;

- an open interval (a, b);
- a left ray $(-\infty, a)$;
- a right ray (a, ∞).

For every $k \in \omega$, there is a strictly increasing injection $k \to J$.

Proof. Use induction on k. The empty function handles $k = 0$. For $k = 1$, choose any point of J.

Suppose $k > 0$ and $f : k \to J$ is strictly increasing. Let $x = f(0)$. If $J = (a, b)$ or $J = (a, \infty)$, density gives $a < y < x$. If $J = (-\infty, a)$ or $J = X$, the absence of a least point gives $y < x$. In every case $y \in J$. Define $f^+ : k + 1 \to J$ by $f^+(0) = y$ and $f^+(i + 1) = f(i)$. $\square$

Lemma 3.11 (At-most-countable DLO uniqueness). Let $X \hookrightarrow \omega$. If X carries a DLO, then any two DLOs on X are isomorphic.

Proof. A DLO carrier is infinite, so Lemma 2.11 gives a bijection

$$e : \omega \longrightarrow X.$$

Let $<_0$ and $<_1$ be DLO relations on X. A **finite partial isomorphism** is a finite injective function $p \subseteq X \times X$ satisfying

$$x <_0 x' \quad \Longleftrightarrow \quad p(x) <_1 p(x')$$

for all $x, x' \in \mathrm{dom}(p)$.

For a finite partial isomorphism p and $a \notin \mathrm{dom}(p)$, define

$$U_p(a) := \left\{ y \in X \setminus \mathrm{ran}(p) : \forall (x, z) \in p \begin{bmatrix} x <_0 a \Longleftrightarrow z <_1 y, \\ a <_0 x \Longleftrightarrow y <_1 z \end{bmatrix} \right\}. \tag{26}$$

Because p is finite, choose $r \in \omega$ and a bijection $u : r \to \mathrm{dom}(p)$. Lemma 2.9 gives the unique $<_0$-increasing enumeration

$$a_0 <_0 a_1 <_0 \cdots <_0 a_{r-1}$$

of the domain. Since p is a partial isomorphism,

$$p(a_0) <_1 p(a_1) <_1 \cdots <_1 p(a_{r-1}),$$

and $p \circ u : r \to \mathrm{ran}(p)$ is a bijection.

The position of a relative to the domain list determines one nonempty open interval, ray, or all of X in the target order; call it $J_p(a)$. The candidate set is exactly

$$U_p(a) = J_p(a) \setminus \operatorname{ran}(p).$$

Lemma 3.10 gives an injection $r+1 \to J_p(a)$. If $U_p(a) = \emptyset$, then $J_p(a) \subseteq \operatorname{ran}(p)$, and composition with the inverse of $p \circ u$ would give an injection $r+1 \to r$, contrary to Corollary 2.10. Hence $U_p(a) \neq \emptyset$.

Let $d(p,a)$ be the element of $U_p(a)$ with least e-index, and put

$$\operatorname{DomExt}(p,a) = p \cup \{(a, d(p,a))\}. \tag{27}$$

If $a \in \operatorname{dom}(p)$, set $\operatorname{DomExt}(p,a) = p$. Define $\operatorname{RanExt}(p,b)$ by applying the same construction to the inverse partial isomorphism and then inverting back.

Now define

$$p_0 = \emptyset,$$

$$p_{2n+1} = \operatorname{DomExt}(p_{2n}, e(n)),$$

$$p_{2n+2} = \operatorname{RanExt}(p_{2n+1}, e(n)). \tag{28}$$

Let $\Psi(n,p,p')$ assert that, for the unique k with either $n = 2k$ or $n = 2k+1$,

$$p' = \begin{cases} \operatorname{DomExt}(p, e(k)), & n = 2k, \\ \operatorname{RanExt}(p, e(k)), & n = 2k+1. \end{cases}$$

Division with remainder by 2 is available from Proposition B.2, so this is a formula on the fixed state space $\mathcal{P}(X \times X)$. Each step is uniquely prescribed and preserves finite partial isomorphism. Induction gives a unique finite history of every length, so Lemma 2.6 produces the recursion graph in

$$\omega \times \mathcal{P}(X \times X).$$

The sequence is increasing under inclusion. Define

$$f := \{(x,y) \in X \times X : \exists n < \omega \; ((x,y) \in p_n)\}\,. \tag{29}$$

Because the p_n form a chain, f is a function and is injective. Every $e(n)$ enters the domain by stage $2n+1$ and the range by stage $2n+2$, so f is total and onto. If $x, x' \in X$, both occur together in the domain of some p_n; order preservation of p_n gives order preservation of f. Thus f is an isomorphism.□

Theorem 3.12 (Dedekind-infinite spectrum alternative). Let X carry a DLO and suppose $\omega \hookrightarrow X$. Then

$$X \hookrightarrow \omega \quad \text{or} \quad s(X) \geq 4. \tag{30}$$

In the first alternative, $s(X) = 1$.

Proof. Choose an injection $\omega \to X$. Lemma 3.1 gives a strictly monotone subsequence with range $B \cong \omega$. Put $C = X \setminus B$. By Lemma 3.2, C is a DLO.

If $C \hookrightarrow B$, then the finite union $B \sqcup C = X$ injects into ω by Lemma 3.4(1). Lemma 3.11 gives $s(X) = 1$.

If $C \not\hookrightarrow B$, Theorem 3.9 gives $s(X) \geq 4$. □

4. Exact-three flank localization

Lemma 4.1 (A self-dual exact-three type exists). Assume $s(X) = 3$, and choose representatives R_0, R_1, R_2 as in Lemma 2.5. Reversal induces an involution

$$\sigma : \{0, 1, 2\} \longrightarrow \{0, 1, 2\}$$

by

$$R_i^* \cong R_{\sigma(i)}.$$

The involution σ has a fixed point. Hence one of the three DLO types is self-dual.

Proof. Exactness gives existence and uniqueness of $\sigma(i)$. Reversing twice gives $\sigma^2 = \mathrm{id}$. Every involution on a three-element set has a fixed point. □

Theorem 4.2 (Lindenbaum's order theorem). This classical result is often called Lindenbaum's theorem; see Ervin 2017, §2.3. A full proof is included because the weak base theory matters here.

Let P, Q be linear orders. Suppose:

(1) P is isomorphic to an initial segment of Q;
(2) Q is isomorphic to a final segment of P.

Then

$$P \cong Q. \tag{31}$$

Proof. Let $f : P \to Q$ have initial-segment range I, and let $g : Q \to P$ have final-segment range J. Put

$$P_0 := P \setminus J, \qquad \theta := g \circ f.$$

Define

$$\mathcal{C} := \{D \in \mathcal{P}(P) : D \text{ is an initial segment},\ P_0 \subseteq D,\ \theta[D] \subseteq D\}\,. \tag{32}$$

This is a nonempty set because $P \in \mathcal{C}$. Define its intersection by Separation on P:

$$C_\theta := \{x \in P : \forall D \in \mathcal{C}\ (x \in D)\}. \tag{33}$$

An intersection of initial segments is an initial segment. Moreover, $P_0 \subseteq C_\theta$. If $D \in \mathcal{C}$, then $C_\theta \subseteq D$ and hence

$$\theta[C_\theta] \subseteq \theta[D] \subseteq D.$$

Here the image is the Separation-defined set

$$\theta[C_\theta] = \{y \in P : \exists x \in C_\theta\ ((x, y) \in \theta)\}.$$

Therefore $\theta[C_\theta] \subseteq C_\theta$.

The set

$$P_0 \cup \theta[C_\theta]$$

is an initial segment. Suppose $y < \theta(c)$ for some $c \in C_\theta$. If $y \in P_0$, stop. Otherwise $y \in J$, so $y = g(q)$ for some $q \in Q$. From

$$g(q) < g(f(c))$$

we obtain $q < f(c)$. Since $I = \operatorname{ran}(f)$ is an initial segment and $f(c) \in I$, there is $p < c$ with $q = f(p)$. The set C_θ is initial, so $p \in C_\theta$, and

$$y = \theta(p) \in \theta[C_\theta].$$

The set $P_0 \cup \theta[C_\theta]$ contains P_0 and is θ-closed because

$$\theta[P_0 \cup \theta[C_\theta]] \subseteq \theta[C_\theta].$$

Minimality of C_θ therefore gives

$$C_\theta = P_0 \cup \theta[C_\theta]. \tag{34}$$

Define $h : P \to Q$ by

$$h(x) = \begin{cases} f(x), & x \in C_\theta, \\ g^{-1}(x), & x \notin C_\theta. \end{cases}$$

The branch ranges are disjoint: if $f(c) = g^{-1}(x)$ with $c \in C_\theta$ and $x \notin C_\theta$, then

$$x = \theta(c) \in C_\theta.$$

For surjectivity, take $y \in Q$ and put $x = g(y) \in J$. If $x \notin C_\theta$, then $y = g^{-1}(x)$. If $x \in C_\theta$, then $x \notin P_0$, so (34) gives $x = \theta(c)$ for some $c \in C_\theta$, and $y = f(c)$.

Order preservation holds on each branch. In the mixed case $x \in C_\theta < x' \notin C_\theta$, suppose

$$g^{-1}(x') \le f(x).$$

Because I is an initial segment, $g^{-1}(x') = f(z)$ for some $z \le x$. Then

$$x' = \theta(z) \in C_\theta,$$

a contradiction. Thus h is an order isomorphism. □

Corollary 4.3 (Common-host self-duality criterion). If A and A^* are both isomorphic to initial segments of one linear order, or both to final segments of one linear order, then

$$A \cong A^*. \tag{35}$$

Proof. The two actual segments in the common host are comparable by inclusion. Suppose, for example, that A is isomorphic to an initial segment of A^*. Reversing this embedding makes A^* isomorphic to a final segment of A. Theorem 4.2, with $(P, Q) = (A, A^*)$, gives $A \cong A^*$. The other containment and the final-segment version are symmetric. □

Remark 4.4 (Why the final-segment hypothesis is essential). The initial/final asymmetry in Theorem 4.2 is essential. Both-way initial-segment embeddings do not suffice: $\mathbb{Q}$ is an initial segment of $\mathbb{Q} + \mathbf{1}$, while $\mathbb{Q} + \mathbf{1}$ is isomorphic to a proper initial segment of $\mathbb{Q}$, but the two orders are not isomorphic.

Theorem 4.5 (Symmetric flank self-duality). Assume $s(X) = 3$, and fix representatives R_0, R_1, R_2 and the reversal involution

$$\sigma : \{0,1,2\} \longrightarrow \{0,1,2\}$$

from Lemma 4.1.

Let $L^\circ = (X, \lhd)$ be a DLO with an involutive order reversal

$$k : L^\circ \longrightarrow L^\circ.$$

Suppose

$$X = A \sqcup M \sqcup k[A]$$

is a convex partition and

$$L^\circ = A + M + k[A]. \tag{36}$$

Assume A and $k[A]$ are nonempty DLOs without endpoints and $k[M] = M$. Then

$$A \cong A^*. \tag{37}$$

Proof. A self-dual DLO is represented by a fixed point of σ. Indeed, if $U \cong R_i$ and $U \cong U^*$, then

$$R_i \cong U \cong U^* \cong R_i^* \cong R_{\sigma(i)}.$$

The representatives are pairwise nonisomorphic, so $\sigma(i) = i$.

An involution on three points has one of two forms.

One fixed point and one transposed pair.

Let i_* be the unique fixed point. On the same carrier define

$$Q := A^* + M + k[A]^*.$$

After empty blocks are removed, Lemma 2.7 shows that Q is a DLO. If M is nonempty, the boundary $A^* \mid M$ is dense because A^* has no greatest point, and $M \mid k[A]^*$ is dense because $k[A]^*$ has no least point. A nonsingleton M is convex in L° and hence internally dense. If M is empty, the new boundary $A^* \mid k[A]^*$ satisfies the same endpoint condition.

The map k reverses Q blockwise. For example, for $x, y \in A$,

$$x <_{A^*} y \iff y <_A x \iff k(x) <_{k[A]} k(y) \iff k(y) <_{k[A]^*} k(x).$$

Thus both L° and Q are self-dual. The observation above gives

$$L^\circ \cong R_{i_*} \cong Q.$$

Under an isomorphism $L^\circ \to Q$, the image of the initial block A is an initial segment of Q of type A, while the literal first block of Q is an initial segment of type A^*. Corollary 4.3 gives $A \cong A^*$.

All three points fixed.

Define

$$R := A + M + k[A]^*, \qquad S := A^* + M + k[A].$$

The same explicit reduced-boundary check using Lemma 2.7 shows that R and S are DLOs: in each order the left flank has no greatest point, the right flank has no least point, and any nonsingleton middle block is convex and dense. The map k is an anti-isomorphism $R \to S$, and hence

$$R^* \cong S.$$

Every DLO on X is self-dual in this case. If $U \cong R_i$, then

$$U^* \cong R_i^* \cong R_{\sigma(i)} = R_i \cong U.$$

Therefore

$$R \cong R^* \cong S.$$

Under an isomorphism $R \to S$, the image of the initial block A and the literal initial block A^* of S are initial segments of one host. Corollary 4.3 gives $A \cong A^*$. $\square$

Proposition 4.6 (Geometry of an involutive reversal). Let $L = (X, <)$ be a DLO and let h be an involutive order reversal. Put

$$H^- := \{x : x < h(x)\},$$

$$H^+ := \{x : h(x) < x\},$$

and

$$F_h := \{x : h(x) = x\}.$$

Then:

(1) F_h is empty or a singleton;
(2) H^- and H^+ are nonempty convex DLOs without endpoints;
(3) $X = H^- + F_h + H^+$;
(4) h anti-isomorphically exchanges the two halves.

Proof. An order reversal has at most one fixed point. The two strict inequality sets are respectively initial and final, are convex, and are exchanged by h.

They are nonempty. Indeed, h cannot fix every point: if $a < b$, order reversal would give $h(b) < h(a)$, contradicting $h(a) = a$ and $h(b) = b$. Choose a nonfixed point x. If $x < h(x)$, then

$$x \in H^-, \qquad h(x) \in H^+,$$

because $h^2 = \mathrm{id}$. The other inequality is symmetric.

We verify that H^- has no greatest point; the other endpoint statements follow by reversal or initiality. Take $x \in H^-$, so $x < h(x)$. Choose

$$x < z_1 < z_2 < h(x).$$

At most one of z_1, z_2 is fixed by h, so choose a nonfixed z among them.

- If $z < h(z)$, then $z \in H^-$ and $x < z$.
- If $h(z) < z$, reversing $z < h(x)$ gives $x < h(z)$, and $h(z) \in H^-$.

Thus x is not greatest. Density is inherited from convexity. □

Corollary 4.7 (Exact-three reflection halves are self-dual). Assume $s(X) = 3$. Let $L = (X, <)$ be a self-dual DLO and let h be an involutive reversal. Then the two reflection halves in Proposition 4.6 are self-dual.

Proof. Apply Theorem 4.5 to

$$X = H^- + F_h + H^+.$$

The two flank blocks are exchanged by h, and the middle is h-invariant. Thus H^- is self-dual. Since $h : H^- \to H^+$ is an anti-isomorphism, the same conclusion follows for H^+. □

Lemma 4.8 (Rigidity consequences). Let L be rigid.

(1) Every convex suborder of L is rigid.

(2) If a convex suborder is self-dual, it has a unique reversal, and that reversal is involutive.

Proof. Let C be a convex suborder and let f be an automorphism of C. Every point outside C lies uniformly below all of C or uniformly above all of C; otherwise convexity would place it in C. Hence extending f by the identity outside C preserves all mixed comparisons and gives an automorphism of L. Rigidity of L therefore implies rigidity of C.

Two reversals of one rigid order differ by an automorphism. The square of a reversal is an automorphism. Thus a self-dual rigid convex suborder has a unique involutive reversal. ☐

5. The rigid dyadic reflection tree

Throughout this section, assume $s(X) = 3$ and let $L = (X, <)$ be a rigid self-dual representative with involutive reversal h. Let $H := H^-$ be its left reflection half.

Lemma 5.1 (Geometry of one reflection split). Let I be a nonempty convex rigid self-dual DLO and let r be its unique reversal. Put

$$I_0 := \{x \in I : x < r(x)\},$$

$$I_1 := \{x \in I : r(x) < x\}. \tag{38}$$

Then:

(1) I_0 and I_1 are nonempty proper convex DLOs without endpoints;
(2) r anti-isomorphically exchanges them;
(3) the fixed-point set $F_r := \{x \in I : r(x) = x\}$ is empty or a singleton;
(4) one has the disjoint convex decomposition $I = I_0 \sqcup F_r \sqcup I_1$.

Proof. By Lemma 4.8, the unique reversal is involutive:

$$r^2 = \mathrm{id}_I.$$

The fixed-point assertion is standard for an order reversal. The two strict inequality sets are respectively initial and final, are convex, and are exchanged by r.

They are nonempty. As in Proposition 4.6, r cannot fix every point. If x is nonfixed and $x < r(x)$, then $x \in I_0$ and $r(x) \in I_1$; the other case is symmetric. The displayed decomposition follows from trichotomy.

To show that I_0 has no greatest point, take $x \in I_0$ and choose

$$x < z_1 < z_2 < r(x).$$

At most one of z_1, z_2 is fixed by r, so choose a nonfixed z.

- If $z < r(z)$, then $z \in I_0$ and $x < z$.
- If $r(z) < z$, reversing $z < r(x)$ gives $x < r(z)$, and $r(z) \in I_0$.

The remaining endpoint statements are symmetric or follow from initiality and finality. Density is inherited from convexity. □

Lemma 5.2 (Sharp singleton-placement lemma). Let $q \geq 1$. Let $C_0, \dots, C_{q-1}$ be nonempty DLO blocks without endpoints and let $z_0, \dots, z_{t-1}$ be singleton blocks.

If $t \leq q$, the order

$$z_0 + C_0 + z_1 + C_1 + \cdots + z_{t-1} + C_{t-1} + C_t + \cdots + C_{q-1} \tag{39}$$

with the initial alternating part omitted when $t = 0$, has no greatest point and contains no adjacent singleton blocks.

If $t = q + 1$, no ordering of the q DLO blocks and t singleton blocks can simultaneously avoid adjacent singletons and avoid a singleton final block.

Proof. The displayed arrangement proves the first assertion. For the second, q nonsingleton blocks provide at most q separators between singleton blocks. Accommodating $q+1$ singleton blocks without adjacency forces the alternating pattern to begin and end with a singleton. □

The second assertion is not used later; it records that the numerical bound in the localization argument is sharp.

Lemma 5.3 (Labeled finite localization makes every child self-dual). Let $n < \omega$. Suppose that:

(1) the half H is partitioned into cells $(I_s)_{s \in 2^n}$ and a center set $P_{<n}$;
(2) every I_s is a nonempty convex rigid self-dual DLO;
(3) there is an injective tag map

$$\tau : P_{<n} \longrightarrow 2^{<n}.$$

Split every cell by its unique reversal. Tag the possible new center of I_s by the word s. Let D be one of the resulting level-$(n+1)$ child cells. Then

$$D \cong D^*. \tag{40}$$

Proof. After all splits, the child cells are indexed by the binary words of length $n+1$. The old center tags lie in $2^{<n}$ and the new center tags lie in

2^n. Old centers lie outside all level-n cells, while a new center lies inside its parent cell; new centers from distinct parents are distinct. Hence the combined tag map is injective into $2^{<n+1}$.

Let d index the target cell D and put $a = v_{n+1}(d)$, where v_{n+1} is the bijection of Lemma B.3. Define

$$\rho(s) = \begin{cases} v_{n+1}(s), & v_{n+1}(s) < a, \\ v_{n+1}(s) - 1, & a < v_{n+1}(s). \end{cases}$$

Then

$$\rho : \{s \in 2^{n+1} : s \neq d\} \longrightarrow q, \qquad q := 2^{n+1} - 1, \tag{41}$$

is a bijection. Thus the remaining child cells have a canonical enumeration $C_0, \dots, C_{q-1}$.

The normalized bounded code c_{n+1} of Lemma B.3, composed with the combined center tag, injects the total center set into q. Lemma B.3 gives the explicit bijection

$$2^{<n+1} \cong 2^{n+1} - 1 = q.$$

Thus the center bound is sharp at the level of available tags. Lemma 2.8 gives some $t \leq q$ and an enumeration $z_0, \dots, z_{t-1}$ of all center points. The second assertion of Lemma 5.2 explains why one additional center would make the required placement impossible. Lemma 5.2 gives a block order W_D on $H \setminus D$ in which every center point is followed by a DLO cell and the final block is a DLO.

Give D and $h[D]$ their orders inherited from L. Give W_D the specified internally finite block-sum order, whose graph exists by Proposition B.5. Give $h[W_D]$ the mirror order satisfying

$$u <_{h[W_D]} v \quad \Longleftrightarrow \quad h(v) <_{W_D} h(u). \tag{42}$$

Put

$$M_D := W_D + F_h + h[W_D]$$

and define the same-carrier order

$$L_D := D + M_D + h[D]. \tag{43}$$

Empty blocks are omitted. The five possible pieces $D, W_D, F_h, h[W_D], h[D]$ partition X exactly.

The following table verifies every boundary in the reduced block list.

Boundary	Range	Reason for density
$D \mid z_0$	$t > 0$	D has no greatest point
$D \mid C_0$	$t = 0$	D has no greatest point
$z_i \mid C_i$	$0 \leq i < t$	C_i has no least point
$C_i \mid z_{i+1}$	$0 \leq i < t-1$	C_i has no greatest point
$C_j \mid C_{j+1}$	$0 \leq j < q-1$ if $t = 0$; $t-1 \leq j < q-1$ if $0 < t < q$; none if $t = q$	C_j has no greatest point
$W_D \mid F_h$	$F_h \neq \emptyset$	W_D has no greatest point
$F_h \mid h[W_D]$	$F_h \neq \emptyset$	$h[W_D]$ has no least point
$W_D \mid h[W_D]$	$F_h = \emptyset$	W_D has no greatest point and $h[W_D]$ has no least point
internal mirror boundaries	all corresponding indices	the duals of the preceding internal cases
$h[W_D] \mid h[D]$	always	$h[D]$ has no least point

The first block D has no least point and the last block $h[D]$ has no greatest point. Thus every reduced boundary, including the cases $t = 0$ and $F_h = \emptyset$, satisfies Lemma 2.7. Hence L_D is a DLO.

The map h reverses L_D, exchanges the flanks D and $h[D]$, and satisfies

$$h[M_D] = M_D.$$

Thus Theorem 4.5 applies to $X = D + M_D + h[D]$ and gives $D \cong D^*$. $\square$

Definition 5.4 (Complete dyadic level states). Put

$$U := 2^{<\omega} \times \mathcal{P}(H) \times \mathcal{P}(H \times H) \times \mathcal{P}(H). \tag{44}$$

A record $(s, I, r, F) \in U$ consists of a node, a cell, a reversal graph, and its fixed-point set.

For $n < \omega$, a subset $\mathcal{S} \subseteq U$ is a **valid level-n state** if:

(1) for every $s \in 2^{\leq n}$, exactly one record (s, I_s, r_s, F_s) belongs to $\mathcal{S}$, and no record with longer node belongs to $\mathcal{S}$;

(2) $I_\emptyset = H$;
(3) r_s is a reversal of I_s, every reversal of I_s equals r_s, and

$$F_s = \{x \in I_s : r_s(x) = x\};$$

(4) for $|s| < n$,

$$I_{s0} = \{x \in I_s : x < r_s(x)\},$$

$$I_{s1} = \{x \in I_s : r_s(x) < x\};$$

(5) every I_s is nonempty, convex in L, rigid, self-dual, and a DLO;
(6) for every $k \leq n$, the following partition and bound hold simultaneously:

$$H = \left(\bigsqcup_{|s|=k} I_s \right) \sqcup P_{<k}, \tag{45}$$

where

$$P_{<k} := \{x \in H : \exists s\ (|s| < k \text{ and } x \in F_s)\},$$

and

$$|P_{<k}| \leq 2^k - 1.$$

Write $\mathrm{Valid}(n, \mathcal{S})$ for the conjunction of these clauses.

Theorem 5.5 (Actualization of the dyadic tree). For every $n < \omega$, there is a unique valid level-n state $\mathcal{S}_n$. The states are coherent under restriction, and the complete dyadic tree

$$\mathcal{T} := \big\{u \in U : \exists n \in \omega\, \exists \mathcal{S} \in \mathcal{P}(U)\, \big(\mathrm{Valid}(n, \mathcal{S}) \wedge u \in \mathcal{S}\big)\big\} \tag{46}$$

is a set in Z_{sep}.

Proof. We use induction on n.

For $n = 0$, Corollary 4.7 makes H self-dual and Lemma 4.8 makes it rigid. Its reversal and fixed-point set are unique, so $\mathcal{S}_0$ exists and is unique.

Assume $\mathcal{S}_n$ exists uniquely. Every level-n cell is convex in L, rigid, and self-dual.

Every old center has a unique node tag. If $x \in F_s$, then x lies in neither child of s and hence in no descendant cell. If s and t are incomparable, their cells lie in disjoint branches of a common level partition. Thus one point cannot belong to both F_s and F_t, and

$$P_{<n} \longrightarrow 2^{<n}, \qquad x \longmapsto \text{the unique } s \text{ with } x \in F_s,$$

is injective.

Lemma 5.1 determines the two geometric children of every level-n cell. Apply Lemma 5.3 with this tag map; every child is self-dual. Each child is convex in its parent and hence in L, and is rigid by Lemma 4.8. Its reversal and fixed-point set are therefore unique.

Define the successor state explicitly. For $u \in U$, let $\mathrm{Old}_{\mathcal{S}_n}(u)$ mean $u \in \mathcal{S}_n$. Let $\mathrm{New}_{\mathcal{S}_n}(u)$ mean that

$$u = (s, I, r, F), \qquad s = t^\frown\varepsilon, \qquad |t| = n, \qquad \varepsilon \in 2,$$

there is a parent record $(t, I_t, r_t, F_t) \in \mathcal{S}_n$, and:

- if $\varepsilon = 0$, then $I = \{x \in I_t : x < r_t(x)\}$;
- if $\varepsilon = 1$, then $I = \{x \in I_t : r_t(x) < x\}$;
- r is the unique reversal of I;
- $F = \{x \in I : r(x) = x\}$.

Put

$$\mathcal{S}_{n+1} = \left\{ u \in U : \mathrm{Old}_{\mathcal{S}_n}(u) \vee \mathrm{New}_{\mathcal{S}_n}(u) \right\}. \tag{47}$$

This is a Separation-defined subset of the fixed record space U.

Tag old centers by their old nodes and a possible new center of I_s by $s \in 2^n$. This gives an injection

$$P_{<n+1} \longrightarrow 2^{<n+1}.$$

The bijection c_{n+1} of Lemma B.3 maps $2^{<n+1}$ onto $2^{n+1} - 1$, proving

$$|P_{<n+1}| \leq 2^{n+1} - 1.$$

The children and accumulated centers give (45), so (47) is the unique valid level-$(n+1)$ state.

If $m > n$, the restriction of any valid level-m state to nodes of length at most n is a valid level-n state and therefore equals $\mathcal{S}_n$. Hence the states are coherent.

The predicate $\mathrm{Valid}(n, \mathcal{S})$ is a formula on the fixed set $\omega \times \mathcal{P}(U)$. Separation gives

$$\mathcal{G} := \{(n, \mathcal{S}) \in \omega \times \mathcal{P}(U) : \mathrm{Valid}(n, \mathcal{S})\}.$$

The definition (46) is then Separation on U and forms no Replacement-generated range. $\square$

Lemma 5.6 (The center set is at most countable). Define

$$Z_H := \{x \in H : \exists (s, I, r, F) \in \mathcal{T} \ (x \in F)\}. \tag{48}$$

Then

$$Z_H \hookrightarrow 2^{<\omega} \hookrightarrow \omega. \tag{49}$$

Proof. A center point in F_s belongs to neither child of s and therefore to no descendant cell. Center points at incomparable nodes lie in disjoint cells. Hence every $x \in Z_H$ belongs to F_s for a unique node s. If two centers have the same node tag s, they both lie in the singleton F_s and are equal. Thus the graph $x \mapsto s$ is an injection obtained by Separation from $H \times 2^{<\omega}$. Lemma B.3 supplies an injection $2^{<\omega} \hookrightarrow \omega$. $\square$

Theorem 5.7 (An off-center point generates an injective sequence). Let $x \in H \setminus Z_H$. For every n, let s_n be the unique word of length n whose cell contains x, and put

$$y_n := r_{s_n}(x). \tag{50}$$

Then $n \mapsto y_n$ is injective.

Proof. The level partition and $x \notin Z_H$ give the unique cell I_{s_n}. Since x is not fixed by r_{s_n}, the points x and y_n lie in opposite children of I_{s_n}. The next chosen cell $I_{s_{n+1}}$ is the child containing x, so

$$y_n \notin I_{s_{n+1}}.$$

If $m > n$, then

$$y_m \in I_{s_m} \subseteq I_{s_{n+1}}.$$

Thus $y_m \neq y_n$. By Theorem 5.5 the valid level states are coherent; together with Definition 5.4(1), this implies that every node $s \in 2^{<\omega}$ has exactly one record in $\mathcal{T}$.

The graph of the sequence is the Separation-defined set

$$G_x := \left\{ (n, y) \in \omega \times X : \exists s, I, r, F \begin{bmatrix} (s, I, r, F) \in \mathcal{T}, \\ |s| = n,\ x \in I,\ (x, y) \in r \end{bmatrix} \right\}. \tag{51}$$

The coherence and uniqueness just noted make G_x a function, and the preceding argument makes it injective. □

Corollary 5.8 (Rigid carrier dichotomy). Under the standing hypotheses of this section,

$$X \hookrightarrow \omega \quad \text{or} \quad \omega \hookrightarrow X. \tag{52}$$

Proof. If $H \hookrightarrow \omega$, then $h[H] \hookrightarrow \omega$ and F_h is empty or a singleton. Lemma 3.4(1), with benchmark ω, therefore gives

$$X = H + F_h + h[H] \hookrightarrow \omega.$$

If $H \not\hookrightarrow \omega$, Lemma 5.6 implies $H \neq Z_H$. Choose one point in $H \setminus Z_H$ and apply Theorem 5.7. □

6. Exact-three exclusion

Lemma 6.1 (A nontrivial increasing automorphism gives a countable orbit). Let g be a nonidentity increasing automorphism of a linear order on X. Then

$$\omega \hookrightarrow X.$$

Proof. Choose x with $g(x) \neq x$. Replacing g by g^{-1} if necessary, assume

$$x < g(x).$$

Define $f(0) = x$ and $f(n+1) = g(f(n))$. The local step is unique and every finite history exists by induction, so Lemma 2.6 gives the graph of f by Separation on $\omega \times X$. Applying the increasing map g inductively gives

$$f(n) < f(n+1)$$

for every n, so f is injective. □

Theorem 6.2 (Exact-three carrier dichotomy). If $s(X) = 3$, then

$$X \hookrightarrow \omega \quad \text{or} \quad \omega \hookrightarrow X.$$

Proof. By Lemma 4.1, choose a self-dual representative $L = (X, <)$ and an internal reversal h.

If L has a nontrivial increasing automorphism, Lemma 6.1 gives $\omega \hookrightarrow X$.

Otherwise L is rigid. Then h^2 is an automorphism, so h is involutive. Corollary 5.8 gives the stated dichotomy. $\square$

Theorem 6.3 (No exact-three DLO spectrum). Z_{sep} proves

$$\neg\exists X\,(s(X) = 3). \tag{53}$$

Proof. Assume $s(X) = 3$. Theorem 6.2 gives

$$X \hookrightarrow \omega \quad \text{or} \quad \omega \hookrightarrow X.$$

In the first case, Lemma 3.11 gives $s(X) = 1$. In the second, Theorem 3.12 gives either $X \hookrightarrow \omega$ or $s(X) \geq 4$; the first of these again gives $s(X) = 1$ by Lemma 3.11. Each alternative contradicts $s(X) = 3$ by Definition 2.4. $\square$

Corollary 6.4 (ZF consequence). ZF proves that no set X satisfies $s(X) = 3$ in the sense of Definition 2.4.

Proof. ZF contains every axiom of Z_{sep}. $\square$

7. Scope and further questions

For each standard finite $n \geq 2$, let

$$(\mathrm{FS})_n$$

denote the metatheoretic statement

$$Z_{\text{sep}} \vdash \neg\exists X\ \mathrm{Spec}_{=n}(X). \tag{54}$$

Theorem 6.3 proves $(\mathrm{FS})_3$. It does not establish the full scheme $(\mathrm{FS})_n$ for all standard finite $n \geq 2$.

Theorem 3.12 shows that a Dedekind-infinite DLO carrier is either at most countable or carries at least four pairwise nonisomorphic DLOs. Consequently, the present argument excludes exact two on Dedekind-infinite carriers. The present paper does not decide exact two on a Dedekind-finite carrier, and it does not classify spectra of standard finite size at least four.

The proof of Theorem 4.5 uses exact three only through the reversal involution on the three chosen representatives: it has either one fixed point or fixes all three points. For exact two, reversal may exchange the two types and have no fixed point, so Lemma 4.1 supplies no self-dual representative. For five or more types, reversal may have several fixed points together with transposed pairs; then the comparison orders in Theorem 4.5 need not belong to one forced self-dual type. Even when a larger finite spectrum has a favorable reversal pattern, Theorem 3.12 yields only the lower bound $s(X) \geq 4$, which does not contradict $s(X) = n$ for $n \geq 4$. These are the structural points at which the present argument stops. Possible extensions are deliberately left outside this paper so that the exact-three proof has a minimal verification surface.

Appendix A. Basic set formation in Z_{sep}

Proposition A.1 (Construction of ω and induction). The theory Z_{sep} proves the existence of a set ω satisfying the usual induction scheme.

Proof. By Infinity, fix an inductive set I. Separation on I gives

$$0 := \{x \in I : x \neq x\} = \emptyset.$$

For a set x, write $x^+ = x \cup \{x\}$. For $J \subseteq I$, let $\mathrm{Ind}_I(J)$ mean

$$0 \in J \quad \text{and} \quad \forall x \in J \ (x^+ \in J).$$

Define by Separation on I

$$\omega := \left\{x \in I : \forall J \in \mathcal{P}(I) \left(\mathrm{Ind}_I(J) \Rightarrow x \in J\right)\right\}. \tag{55}$$

The set ω is inductive and is contained in every inductive subset of I. If a formula $\varphi(n, \vec{p})$ satisfies $\varphi(0, \vec{p})$ and

$$\forall n \in \omega \left(\varphi(n, \vec{p}) \Rightarrow \varphi(n^+, \vec{p})\right),$$

then

$$J_\varphi := \{n \in \omega : \varphi(n, \vec{p})\}$$

is inductive. Minimality of ω gives $J_\varphi = \omega$. This is the induction scheme used throughout the paper. □

Proposition A.2 (Finite ordinals and successor arithmetic). The following hold in Z_{sep}.

(1) Every $n \in \omega$ is a transitive ordinal and is well-ordered by membership.
(2) Any $m, n \in \omega$ are comparable: exactly one of $m \in n$, $m = n$, or $n \in m$ holds.
(3) The successor map is injective on ω.

(4) Every nonzero $n \in \omega$ is the successor of a unique member of ω.

Proof. A simultaneous induction on n proves that n is transitive and that membership linearly well-orders it. At the successor step, $n^+ = n \cup \{n\}$ adds one new greatest element to the ordinal n; transitivity and the well-order property are preserved. Comparability of finite ordinals follows by the same induction. If $m^+ = n^+$, comparability rules out $m \in n$ and $n \in m$, so $m = n$. Finally, if $n \neq 0$, the greatest element of the finite ordinal n is its unique predecessor. These finite-ordinal arguments use induction only; no Foundation axiom is required. See Jech 2003 for the standard development. □

Proposition A.3 (Least-element principle for ω). Every nonempty subset of ω has a unique least element.

Proof. Let $S \subseteq \omega$ be nonempty and instantiate one $a \in S$. The set $S \cap (a+1)$ is a nonempty subset of the finite ordinal $a+1$. Lemma 2.8 gives its increasing enumeration; the first value is least in $S \cap (a+1)$ and therefore least in all of S. Uniqueness follows from linearity. □

Proposition A.4 (Ordered pairs and Cartesian products). Ordered pairs, Cartesian products, and finite iterated products are sets in Z_{sep}.

Proof. Use the Kuratowski pair

$$\langle a, b\rangle = \{\{a\}, \{a, b\}\}.$$

If $a \in A$ and $b \in B$, then $\langle a, b\rangle \in \mathcal{P}(\mathcal{P}(A \cup B))$. Hence

$$A \times B = \{p \in \mathcal{P}(\mathcal{P}(A \cup B)) : \exists a \in A\, \exists b \in B\, (p = \langle a, b\rangle)\}$$

exists by Separation. Finite iterated products are obtained by repeating this construction. □

Appendix B. Weak-set-theoretic coding details

Lemma B.1 (Finite functions in fixed power sets). For sets A, B, every finite function from a finite ordinal into A, and every finite partial function from A to B, is coded inside a fixed power set constructed from $\omega \times A$ or $A \times B$.

Proof. A function is a set of ordered pairs. Finite functions with domain an initial segment of ω are subsets of $\omega \times A$, and finite partial functions from A to B are subsets of $A \times B$. Proposition A.4 supplies the products, Power Set supplies the ambient coding sets, and Separation expresses functionality, finiteness of the domain, and the desired finite conditions. □

Proposition B.2 (Finite arithmetic on ω). The graphs of addition, multiplication, exponentiation, and division with remainder on ω are sets in Z_{sep}.

Proof. Define each graph inside a fixed finite power of ω. For example, (a, n, c) belongs to the addition graph iff there is a finite history of length $n + 1$, beginning with a, ending with c, and taking successors at each step. Lemmas B.1 and 2.6 and Separation give the graph. Multiplication and exponentiation are treated by the recursions

$$a \cdot 0 = 0, \qquad a \cdot (n + 1) = a \cdot n + a,$$

$$a^0 = 1, \qquad a^{n+1} = a^n \cdot a.$$

For each recursion, induction supplies a unique finite history of every length, so Lemma 2.6 applies. Induction also gives the usual order and cancellation laws. For every positive m and $r \in \omega$, induction gives unique $q \in \omega$ and $i < m$ with

$$r = mq + i.$$

These are the only arithmetic facts used below. □

Lemma B.3 (Explicit coding and counting of finite binary words). For every $n < \omega$, the valuation map

$$v_n(s) := \sum_{i<n} s(i)2^i \tag{56}$$

is a bijection from the set of binary words of length n onto the finite ordinal 2^n.

There is also an injection

$$c : 2^{<\omega} \longrightarrow \omega \tag{57}$$

such that, for every $k < \omega$, there is a bijection

$$c_k : 2^{<k} \longrightarrow 2^k - 1 \tag{58}$$

satisfying

$$c(s) = c_k(s) + 1$$

for every $s \in 2^{<k}$.

Proof. Use induction on n.

For $n = 0$, the empty word is the only word and has value 0.

Assume v_n is a bijection onto 2^n. A word s of length $n+1$ is uniquely determined by $s \restriction n$ and its last bit.

- If $s(n) = 0$, then $v_{n+1}(s) = v_n(s \restriction n) \in [0, 2^n)$.
- If $s(n) = 1$, then $v_{n+1}(s) = 2^n + v_n(s \restriction n) \in [2^n, 2^{n+1})$.

The induction hypothesis gives bijections onto the two intervals.

For a word s of length n, put

$$c(s) = 2^n + v_n(s).$$

Proposition B.2 supplies the arithmetic graphs, and Separation gives the graph of c. Words of different lengths lie in disjoint intervals $[2^n, 2^{n+1})$, while v_n is injective at a fixed length.

If $|s| < k$, then $1 \leq c(s) < 2^k$. Define

$$c_k(s) := c(s) - 1$$

on $2^{<k}$. Conversely, every $r < 2^k - 1$ has $r+1$ in a unique interval $[2^n, 2^{n+1})$ with $n < k$, and the bijection v_n supplies the unique word s with $c(s) = r+1$. Thus c_k is a bijection onto $2^k - 1$ and satisfies $c(s) = c_k(s) + 1$. □

Lemma B.4 (An explicit countable DLO). There is a DLO relation on ω whose graph is a set in Z_{sep}.

Proof. Let D be the set of canonical signed dyadic codes

$$D := \{(0,0,0)\} \cup \{(\varepsilon, m, n) \in 2 \times (\omega \setminus \{0\}) \times \omega : n = 0 \text{ or } m \text{ is odd}\}.$$

Interpret $(0,0,0)$ as 0, $(0,m,n)$ as $m/2^n$, and $(1,m,n)$ as $-m/2^n$. Compare positive codes by cross multiplication, negative codes in the reverse way, and put every negative code below 0 below every positive code. Proposition B.2 supplies the arithmetic relations, so Separation on $D \times D$ gives the order graph.

The represented dyadic rationals are dense. Between two represented values, their arithmetic mean is dyadic. If its numerator is zero, use the distinguished code $(0,0,0)$. Otherwise divide the numerator by 2 and decrease the exponent while the numerator is even and the exponent is positive. This takes at most the original exponent many steps and yields the canonical code. They have no endpoints because adding or subtracting 1 remains dyadic.

For an explicit coding into ω, put

$$\pi(a, b) := 2^a(2b + 1) - 1$$

and

$$\Pi(\varepsilon, m, n) := \pi(\varepsilon, \pi(m, n)).$$

The uniqueness of the highest power of 2 dividing a positive natural number makes π, and hence Π, injective. Thus $D \hookrightarrow \omega$. The map $m \mapsto (0, m+1, 0)$ injects ω into D. If D were bijective with a finite ordinal r, restriction to $r+1$ would contradict Corollary 2.10(2). Hence D is infinite, and Lemma 2.11 gives $D \cong \omega$. Transport the dyadic order along this bijection. $\square$

Proposition B.5 (Internally finite block sums). Let $J \in \omega$. Let $\mathfrak{B}$ be the graph of a function with domain J whose value at j is a block B_j, and let $\mathfrak{R}$ be the graph of a function with domain J whose value at j is a relation

$$R_j \subseteq B_j \times B_j.$$

Assume that the blocks are pairwise disjoint and that every (B_j, R_j) is a linear order.

The range of the block family is the Separation-defined set

$$\operatorname{ran}(\mathfrak{B}) := \left\{ B \in \bigcup\bigcup \mathfrak{B} : \exists j \in J\,(\langle j, B\rangle \in \mathfrak{B}) \right\}.$$

Put

$$Y := \bigcup \operatorname{ran}(\mathfrak{B}).$$

Define $x \prec y$ iff there are $i, j \in J$, blocks B_i, B_j, and a relation R_i such that

$$\langle i, B_i\rangle \in \mathfrak{B}, \qquad \langle j, B_j\rangle \in \mathfrak{B}, \qquad x \in B_i, \qquad y \in B_j,$$

and either $i < j$, or

$$i = j, \qquad \langle i, R_i\rangle \in \mathfrak{R}, \qquad \langle x, y\rangle \in R_i.$$

Then $\prec \subseteq Y \times Y$ is a set in Z_{sep}.

Proof. The range exists by Separation from $\bigcup\bigcup \mathfrak{B}$, and Y exists by Union. The displayed condition is one first-order formula on the fixed set $Y \times Y$ with set parameters $\mathfrak{B}$ and $\mathfrak{R}$. Separation therefore gives the graph of $\prec$. $\square$

Proposition B.6 (Back-and-forth without Replacement). The recursion in Lemma 3.11 is formalizable in Z_{sep}.

Proof. The state space is $\mathcal{P}(X \times X)$, while the recursion graph lives in

$$\omega \times \mathcal{P}(X \times X).$$

The candidate set (26), the least-index choice, and the domain/range extensions are all Separation-defined from the current state. Lemma 3.10, Lemma 2.9, and Corollary 2.10(2) give existence of an unused candidate and the required finite-order enumerations. The unique step formula displayed in Lemma 3.11 has unique finite histories, so Lemma 2.6 gives the graph. The final isomorphism is defined directly by Separation on $X \times X$ as in (29); no Replacement-generated range is formed. □

Proposition B.7 (Complete-level tree coding). The construction in Theorem 5.5 is formalizable in Z_{sep}.

Proof. The record space U is the fixed set in (44); Proposition A.4 supplies all products. A valid level state is a subset of U, so all candidate states lie in $\mathcal{P}(U)$. Every clause of Definition 5.4 is first-order over $X, L, H, n, \mathcal{S}$:

- the finite node domain is expressed using the word-length relation on $2^{<\omega}$;
- "r is the unique reversal of I" is the conjunction that r is a reversal and every reversal equals r;
- rigidity and self-duality quantify over subsets of $H \times H$;
- the fixed-point set is Separation-defined;
- the finite partition and center bound use Lemma B.3 and Definition 2.2;
- the successor state is the Separation-defined set (47).

Induction proves existence and uniqueness of a valid state for every n. Separation on $\omega \times \mathcal{P}(U)$ produces the valid-state graph, and Separation on U produces (46). □

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AI-use disclosure

AI systems were used extensively in an iterative, human-directed workflow. OpenAI ChatGPT was used for exploratory proof development, counterexample search, drafting, and virtual review; Anthropic Claude Code was used for Lean formalization, proof repair, and validation; and OpenAI Codex was used for mathematical and formalization audits, release review, and editorial checks. AI assistance was also used to translate and polish English prose. The author set the research objectives, decided which candidate arguments to retain or reject, reviewed and approved the final manuscript and formalization, and takes responsibility for all mathematical claims and any remaining errors. The AI systems are tools, not authors. The separately cited Sol transcript was produced in an external run conducted by Glazer and served as a public source and catalyst; it was not one of the author's AI sessions.

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