FOURTEEN POINTS IN THREE-SPACE

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Abstract. We show that every finite subscheme of length at most fourteen in complex affine three-space is a limit of reduced subschemes. Equivalently, the Hilbert schemes of twelve, thirteen and fourteen points in $\mathbb{A}^3$ are irreducible. The proof uses two ways of smoothing a finite scheme: placing it on a surface with a rational double point, and splitting off a single point. Which of the two applies is decided mainly by the quadrics that vanish on the tangent cone. For one Hilbert function, $(1, 3, 6, 4)$, we use a computer-assisted theorem of Douvropoulos, Jelisiejew, Nødland and Teitler on very compressed algebras.

1. Introduction

A collection of d distinct points in complex affine space can be moved so that some of the points come together. The limits of such motions are finite subschemes of length d. All finite subschemes of $\mathbb{A}^n$ of length d are parametrized by the Hilbert scheme $\mathrm{Hilb}_d(\mathbb{A}^n)$, and those that arise as limits of d distinct points are called *smoothable*. The smoothable subschemes form an irreducible component of $\mathrm{Hilb}_d(\mathbb{A}^n)$ of dimension nd, so the Hilbert scheme is irreducible exactly when every finite subscheme of length d is smoothable.

For surfaces this is always the case, by a theorem of Fogarty [10]:

> *Every finite subscheme of a smooth surface is a limit of reduced subschemes.*

In higher dimensions it fails. Iarrobino showed [13, 14]:

> *If $n \geq 3$ and $d \geq 78$, then the Hilbert scheme of d points in $\mathbb{A}^n$ is reducible.*

When $n \geq 4$, reducibility already starts at $d = 8$ [15, 2]. When $n = 3$, the Hilbert scheme of d points is irreducible for $d \leq 7$ by Mazzola [19], for $d = 8$ by Cartwright, Erman, Velasco and Viray [2], for $d = 9$ and 10 by Henni and Jardim [12], building on work of Šivic [24], and for $d = 11$ by Douvropoulos, Jelisiejew, Nødland and Teitler [6]. Thus the smallest d for which $\mathrm{Hilb}_d(\mathbb{A}^3)$ is reducible lies between 12 and 78 [16, Problem V].

We now state our main result.

Theorem 1.1. *Every finite subscheme of $\mathbb{A}^3$ of length at most fourteen is smoothable. In particular, the Hilbert schemes $\mathrm{Hilb}_{12}(\mathbb{A}^3)$, $\mathrm{Hilb}_{13}(\mathbb{A}^3)$ and $\mathrm{Hilb}_{14}(\mathbb{A}^3)$ are irreducible, of dimensions 36, 39 and 42.*

Consequently, the smallest d for which $\mathrm{Hilb}_d(\mathbb{A}^3)$ is reducible lies between 15 and 78; the sequels [20, 21, 22] extend Theorem 1.1 to lengths at most

twenty-one, which raises the lower bound to 22. The question is local, so Theorem 1.1 holds for any smooth irreducible quasi-projective threefold in place of $\mathbb{A}^3$, see [6, Section 1].

Every finite scheme is a disjoint union of local ones, and it suffices to consider a local subscheme Z supported at the origin, say $Z = V(I)$. The *tangent cone* of Z is the subscheme of $\mathbb{A}^3$ defined by the lowest-degree homogeneous components of the elements of I. The proof of Theorem 1.1 is organized by the quadrics that vanish on the tangent cone, and it uses two ways of smoothing Z.

The first is to find a surface through Z whose Hilbert schemes of points are irreducible. By a theorem of Zheng [25], see also [5], this is the case for surfaces with rational double points. A quadric that vanishes on the tangent cone and is not the square of a linear form produces such a surface through a scheme isomorphic to Z, and one obtains a criterion that holds in every length.

Theorem 1.2. *Let $Z \subset \mathbb{A}^3$ be a finite subscheme supported at the origin. If the tangent cone of Z lies on a quadric that is not the square of a linear form, or on a nonzero quadric and on eight linearly independent cubics, then Z is smoothable.*

In particular, Z is smoothable whenever its tangent cone lies on two linearly independent quadrics, because a pencil of quadrics cannot consist of squares. A quadric that is not a square gives Z an equation of type A; when the quadric is a square, the cubics in the second case give an equation of type D; see Section 4. Equations of types E_6, E_7 and E_8 can appear when all the relevant cubics are cubes.

The second way is to split off a point. Let a, b, c be polynomial coordinates centered at the origin, for instance linear coordinates, and let $L = (b, c)$ be the ideal of the curve $b = c = 0$. If the ideal quotient $(I : L)$ is not contained in $I + L$, then a flat family deforms Z into the disjoint union of a single point, which leaves the origin along the curve, and a scheme of smaller length (Proposition 5.3). In its simplest form this is the ray family of Casnati, Jelisiejew and Notari [3], which was used in a similar way in [6].

Both ways are combined with the fact that a limit of smoothable schemes is smoothable. For some Hilbert functions, every algebra is the limit of a one-parameter family whose general member is smoothed in one of the two ways.

For at most fourteen points, the choice between the two is guided by the number of independent quadrics on the tangent cone. When Z has embedding dimension three, this number is $6 - H_A(2)$, where H_A is the Hilbert function of the local algebra $A = \mathcal{O}(Z)$ (Section 3):

$H_A(2)$	quadrics	how Z is smoothed
≤ 4	≥ 2	on a surface with a singularity of type A_k
5	1	on a surface with a singularity of type A_k, D_k or E_k, as a limit of such schemes, or by splitting off a point
6	0	by splitting off a point, or as a limit of such schemes, except for $(1,3,6,4)$

In the middle row, when the quadric is the square of a linear form, the cubic terms of the equations of Z decide between the two ways (Theorem 8.9). In the last row, the algebras with Hilbert function $(1,3,6,k)$, $k \leq 3$, are limits of algebras with a ray; for $k = 3$ the direction of the ray is found on a plane cubic curve (Lemma 6.4). The Hilbert functions $(1,3,6,3,1)$ and $(1,3,6,2,2)$ are treated with Macaulay's inverse systems, which describe an ideal by the polynomials that it annihilates. For $(1,3,6,3,1)$, the ideal of a suitable line is always a ray; in the main case this follows from a trace identity for the second derivatives of a quartic form (Lemma 8.5). A general algebra with Hilbert function $(1,3,6,4)$ has no ray (Remark 6.8). These algebras are very compressed, and for them we use the theorem of Douvropoulos, Jelisiejew, Nødland and Teitler [6, Proposition 1.2], whose proof uses a computer calculation.

Section 2 recalls the Hilbert scheme and the basic properties of smoothability. Section 3 introduces tangent cones, Hilbert functions and inverse systems of local algebras. Section 4 explains how equations of type A, D and E arise, and proves Theorem 1.2. Section 5 constructs the families that split off a point. Sections 6, 7 and 8 prove Theorem 1.1 for at most twelve points, for thirteen and for fourteen points. The proof is by induction on the length, starting from a single point; in particular, it does not use the results for lengths at most eleven quoted above.

2. Finite schemes and their smoothings

2.1. Throughout, we work over the complex numbers. We write $S = \mathbb{C}[x,y,z]$, $\mathfrak{m} = (x,y,z)$ for the ideal of the origin, S_k for the space of forms of degree k in S, and $S_{\leq k} = S_0 \oplus \cdots \oplus S_k$. When we write $V(I) = \{0\}$, we mean that the zero scheme of I is supported at the origin.

A *finite subscheme* Z of $\mathbb{A}^n$ is the zero scheme $V(I)$ of an ideal $I \subseteq \mathbb{C}[x_1,\dots,x_n]$ of finite colength. Its *length* is $\ell(Z) = \dim_{\mathbb{C}} \mathbb{C}[x_1,\dots,x_n]/I$, and its coordinate algebra $\mathcal{O}(Z) = \mathbb{C}[x_1,\dots,x_n]/I$ is a finite $\mathbb{C}$-algebra. The subscheme Z is *reduced* if I is radical, that is, if Z consists of $\ell(Z)$ distinct points, and it is *local* if it is supported at a single point.

2.2. The *Hilbert scheme of d points* $\mathrm{Hilb}_d(\mathbb{A}^n)$ is the fine moduli space of finite subschemes of $\mathbb{A}^n$ of length d, see [23, Chapter 18]. Its closed points are the ideals of colength d, and a morphism from a scheme T to $\mathrm{Hilb}_d(\mathbb{A}^n)$ is the same as a closed subscheme of $T \times \mathbb{A}^n$ that is finite and flat of degree d over T.

For example, if $\mathcal{I} \subseteq \mathbb{C}[t][x_1, \ldots, x_n]$ is an ideal such that $\mathbb{C}[t][x_1, \ldots, x_n]/\mathcal{I}$ is a free $\mathbb{C}[t]$-module of rank d, then $t_0 \mapsto \mathcal{I}|_{t=t_0}$ is a morphism $\mathbb{A}^1 \to \mathrm{Hilb}_d(\mathbb{A}^n)$. Here $\mathcal{I}|_{t=t_0}$ denotes the image of $\mathcal{I}$ under the substitution $t = t_0$.

The reduced subschemes of length d form an open subset of $\mathrm{Hilb}_d(\mathbb{A}^n)$, isomorphic to the complement of the diagonals in the d-th symmetric power of $\mathbb{A}^n$. It is irreducible of dimension nd.

Definition 2.3. The *smoothable component* $\mathrm{R}_d(\mathbb{A}^n)$ is the closure of the locus of reduced subschemes in $\mathrm{Hilb}_d(\mathbb{A}^n)$. A finite subscheme of length d is *smoothable* if it is a point of $\mathrm{R}_d(\mathbb{A}^n)$.

Thus $\mathrm{Hilb}_d(\mathbb{A}^n)$ is irreducible if and only if every finite subscheme of $\mathbb{A}^n$ of length d is smoothable.

Proposition 2.4. *Smoothability has the following properties.*

(1) (Limits) *Let T be a variety with a morphism $T \to \mathrm{Hilb}_d(\mathbb{A}^n)$. If the subschemes parametrized by a dense subset of T are smoothable, then all subschemes parametrized by T are smoothable.*
(2) (Disjoint unions) *A disjoint union of smoothable finite subschemes is smoothable.*
(3) (Intrinsic) *Whether Z is smoothable depends only on the isomorphism class of the $\mathbb{C}$-algebra $\mathcal{O}(Z)$, and neither on the embedding of Z into $\mathbb{A}^n$ nor on n.*
(4) (Surfaces) *Let $X \subseteq \mathbb{A}^n$ be a closed surface such that $\mathrm{Hilb}_d(X)$ is irreducible. Then every subscheme of X of length d is smoothable.*
(5) (Linear families) *Let $U \subseteq \mathbb{C}$ be a nonempty Zariski open subset, and for $t \in U$ let $I_t \subseteq S$ be an ideal of colength d with $\mathfrak{m}^N \subseteq I_t$, for an N independent of t. Suppose that, modulo $\mathfrak{m}^N$, each I_t is spanned by the values at t of finitely many fixed elements of $S[t]$. If $V(I_t)$ is smoothable for all but finitely many $t \in U$, then it is smoothable for every $t \in U$.*

Proof. For (1), the preimage of $\mathrm{R}_d(\mathbb{A}^n)$ in T is closed and dense. Part (3) is [2, Lemma 4.1]: a finite subscheme Z is smoothable if and only if its algebra $\mathcal{O}(Z)$ is the special fiber of a finite flat family over $\mathbb{C}[[t]]$ with smooth generic fiber. Part (2) follows by taking the disjoint union of such families, that is, the direct product of their coordinate algebras, as remarked after [2, Lemma 4.1]. For (4), the Hilbert scheme $\mathrm{Hilb}_d(X)$ is the closed subscheme of $\mathrm{Hilb}_d(\mathbb{A}^n)$ that parametrizes the subschemes contained in X. The locus of reduced subschemes of X is nonempty and open in $\mathrm{Hilb}_d(X)$, hence dense because $\mathrm{Hilb}_d(X)$ is irreducible, and these subschemes are reduced subschemes of $\mathbb{A}^n$. For (5), the spaces $I_t/\mathfrak{m}^N$ have constant dimension and are spanned by vectors that depend polynomially on t, so they form a subbundle of the trivial bundle over U with fiber $S/\mathfrak{m}^N$. Since every fiber is an ideal, the quotient bundle, with fibers S/I_t, is a finite flat family of subschemes of $\mathbb{A}^3$ over U, that is, a morphism $U \to \mathrm{Hilb}_d(\mathbb{A}^3)$, and (1) applies. □

By (3), it makes sense to call a finite $\mathbb{C}$-algebra *smoothable*, and we will do so.

2.5. The *embedding dimension* of a finite local $\mathbb{C}$-algebra A with maximal ideal $\mathfrak{m}_A$ is $\dim_{\mathbb{C}} \mathfrak{m}_A/\mathfrak{m}_A^2$. It is the smallest n for which A is a quotient of $\mathbb{C}[x_1, \ldots, x_n]$, that is, for which $\operatorname{Spec} A$ embeds into $\mathbb{A}^n$. Algebras of embedding dimension at most two are handled by Fogarty's theorem.

Theorem 2.6 (Fogarty [10, Proposition 2.3 and Theorem 2.4])**.** *For every d, the Hilbert scheme $\operatorname{Hilb}_d(\mathbb{A}^2)$ is irreducible. Hence every finite local $\mathbb{C}$-algebra of embedding dimension at most two is smoothable.*

Proposition 2.7. *Suppose that every finite subscheme of $\mathbb{A}^3$ of length less than d is smoothable. If every local algebra S/I of length d with $I \subseteq \mathfrak{m}^2$ is smoothable, then every finite subscheme of $\mathbb{A}^3$ of length d is smoothable.*

Proof. A finite subscheme is the disjoint union of its local components, and each of them has length less than d unless the subscheme is local. By Proposition 2.4(2) and (3), we may therefore assume that $Z = V(I)$ is local and supported at the origin, so that $I \subseteq \mathfrak{m}$. The embedding dimension of S/I is $\dim_{\mathbb{C}} \mathfrak{m}/(\mathfrak{m}^2 + I)$. If $I \not\subseteq \mathfrak{m}^2$, it is at most two, and Theorem 2.6 applies. □

3. Tangent cones and inverse systems

3.1. Let $I \subseteq S$ be an ideal of finite colength with $V(I) = \{0\}$, and let $A = S/I$. By the Nullstellensatz, $\mathfrak{m}^N \subseteq I$ for some N. The *leading form* f^* of a nonzero polynomial f is its nonzero homogeneous component of lowest degree, and this degree is the *order* of f. The leading forms of the elements of I span a homogeneous ideal I^*. The *tangent cone* of $Z = V(I)$ is the subscheme of $\mathbb{A}^3$ defined by I^*, and we say that the tangent cone *lies on* a form q if $q \in I^*$. The *Hilbert function* of A is

$$H_A(k) = \dim_{\mathbb{C}} S_k/I_k^*, \qquad k \geq 0.$$

The algebra S/I^* is the associated graded algebra of A with respect to its maximal ideal, and H_A is its Hilbert function.

Lemma 3.2. (1) *The sum of the values of H_A is the length of A, and $H_A(1)$ is the embedding dimension of A.*
(2) *If $H_A(k) = 0$, then $\mathfrak{m}^k \subseteq I$. In particular, $H_A(j) = 0$ for all $j \geq k$.*
(3) *If $H_A(k) = 1$ for some $k \geq 1$, then $H_A(k+1) \leq 1$.*

Proof. A form $h \in S_k$ maps to zero in $\mathfrak{m}^k A/\mathfrak{m}^{k+1}A$ exactly when $h \in I + \mathfrak{m}^{k+1}$, that is, when h is zero or the leading form of an element of I. Hence $\mathfrak{m}^k A/\mathfrak{m}^{k+1}A \cong S_k/I_k^*$, and (1) follows. If $H_A(k) = 0$, then $\mathfrak{m}^k A = \mathfrak{m}^{k+1}A$, so $\mathfrak{m}^k A = 0$ by Nakayama's lemma. For (3), let $G = S/I^*$ and let $u = \ell_1 \cdots \ell_k$ be a monomial whose class spans G_k. For a linear form v, the class of $v\ell_2 \cdots \ell_k$ is a multiple $\lambda(v)u$ in G_k, so $vu = \ell_1(v\ell_2 \cdots \ell_k) = \lambda(v)\ell_1 u$ in G. Hence $G_{k+1} = S_1 u$ is spanned by $\ell_1 u$. □

3.3. We use formal coordinates. Let $\widehat{S} = \mathbb{C}[[x, y, z]]$ with maximal ideal $\widehat{\mathfrak{m}}$. Since $\mathfrak{m}^N \subseteq I$, the inclusion $S \subset \widehat{S}$ induces an isomorphism $S/I \cong \widehat{S}/I\widehat{S}$, and $\widehat{\mathfrak{m}}^N \subseteq I\widehat{S}$. A *formal change of coordinates* is an automorphism φ of the $\mathbb{C}$-algebra $\widehat{S}$. It preserves $\widehat{\mathfrak{m}}$, and the ideal $I_\varphi = \varphi(I\widehat{S}) \cap S$ contains $\mathfrak{m}^N$ and satisfies

$$S/I_\varphi \cong \widehat{S}/\varphi(I\widehat{S}) \cong \widehat{S}/I\widehat{S} \cong S/I.$$

By Proposition 2.4(3), S/I is smoothable if and only if S/I_φ is. When we say that $I\widehat{S}$ contains a series *in suitable coordinates*, we mean that $\varphi(I\widehat{S})$ contains it for some φ. Since $\widehat{\mathfrak{m}}^N \subseteq I\widehat{S}$, we may add to any element of $I\widehat{S}$ any element of $\widehat{\mathfrak{m}}^N$, in any coordinates.

3.4. For some Hilbert functions it is convenient to describe an ideal by the polynomials that it annihilates. Let $T = \mathbb{C}[X, Y, Z]$, and let S act on T by differentiation, so that x, y and z act as $\partial/\partial X$, $\partial/\partial Y$ and $\partial/\partial Z$. We write $g \circ F$ for the action of $g \in S$ on $F \in T$, and T_k, $T_{\leq k}$ and $T_{<k} = T_{\leq k-1}$ for the spaces of forms of degree k and of polynomials of degree at most k and less than k. For subsets $U \subseteq S$ and $V \subseteq T$, we write $U \circ V$ for the span of the elements $g \circ F$ with $g \in U$ and $F \in V$. The pairing

$$\langle g, F\rangle = (g \circ F)(0)$$

is perfect between $S/\mathfrak{m}^N$ and $T_{<N}$, because $\langle x^\alpha, X^\beta\rangle = \alpha!$ if $\alpha = \beta$ and 0 otherwise. For linear forms we have $v \circ P = v(p)$, where $p \in \mathbb{C}^3$ is the vector of coefficients of $P = p_1X + p_2Y + p_3Z$, and more generally $g \circ P^k = k!\, g(p)$ for $g \in S_k$. The *inverse system* of an ideal $I \subseteq S$ is

$$M = \{F \in T : \ g \circ F = 0 \ \text{ for all } g \in I\}.$$

For nonzero $F \in T$, the nonzero homogeneous component of highest degree is the *top form* of F.

Lemma 3.5 (Macaulay [18])**.** *Let $I \subseteq S$ be an ideal with $\mathfrak{m}^N \subseteq I$, let $A = S/I$, and let M be its inverse system.*

1. *M is a subspace of $T_{<N}$ of dimension $\dim_{\mathbb{C}} A$ that is closed under differentiation, and $I = \{g \in S : \langle g, F\rangle = 0$ for all $F \in M\}$.*
2. *Conversely, if $M \subseteq T_{<N}$ is a subspace closed under differentiation, then $\{g \in S : g \circ M = 0\}$ is an ideal containing $\mathfrak{m}^N$ whose inverse system is M.*
3. *For every k, $H_A(k)$ is the dimension of the space of top forms of degree k of elements of M, together with 0.*

Proof. For $g, h \in S$ and $F \in T$ we have $\langle h, g \circ F\rangle = \langle hg, F\rangle$. If $J \subseteq S$ is an ideal, it follows that $g \circ F = 0$ for all $g \in J$ exactly when $\langle g, F\rangle = 0$ for all $g \in J$, since the values $\langle h, g \circ F\rangle$ are the Taylor coefficients of $g \circ F$ up to nonzero factors. As $\mathfrak{m}^N \subseteq I$ annihilates only polynomials of degree less than N, the space M is the orthogonal complement of $I/\mathfrak{m}^N$ in $T_{<N}$, which proves the first and the last claim of (1). If $F \in M$ and $v \in S_1$, then $\langle g, v \circ F\rangle = \langle gv, F\rangle = 0$ for $g \in I$, so M is closed under differentiation. For

(2), the same identity shows that the orthogonal complement J of M in S is an ideal, which contains $\mathfrak{m}^N$ and equals $\{g : g \circ M = 0\}$. By the argument for (1), the inverse system of J is the orthogonal complement of $J/\mathfrak{m}^N$ in $T_{<N}$, which is M. For (3), $M \cap T_{\leq k}$ is the inverse system of $I + \mathfrak{m}^{k+1}$, so its dimension is $\dim_{\mathbb{C}} A/\mathfrak{m}^{k+1}A = \sum_{j \leq k} H_A(j)$ by the proof of Lemma 3.2. Passing to top forms identifies $(M \cap T_{\leq k})/(M \cap T_{\leq k-1})$ with the space in (3). □

By Lemma 3.5(3), if $H_A(k) = \dim S_k$ for $k \leq 2$, then $T_{\leq 2} \subseteq M$. We refer to [9] and [8, Section 21.2] for the general theory of inverse systems.

4. Surfaces with rational double points

4.1. The *ADE polynomials* are

$$A_k = x^2 + y^2 + z^{k+1} \quad (k \geq 1), \qquad D_k = x^2 + y^2 z + z^{k-1} \quad (k \geq 4),$$
$$E_6 = x^2 + y^3 + z^4, \qquad E_7 = x^2 + y^3 + yz^3, \qquad E_8 = x^2 + y^3 + z^5.$$

For each ADE polynomial f, the surface $X_f = V(f) \subset \mathbb{A}^3$ is normal and smooth away from the origin, where it has a rational double point [7]. Moreover, X_f is isomorphic to the quotient $\mathbb{C}^2/\Gamma$ of the plane by a finite subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{C})$: the invariants of Γ satisfy a single weighted homogeneous relation, found by Klein, which becomes f after rescaling the variables [17].

Theorem 4.2 (Zheng)**.** *For every finite subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{C})$ and every d, the Hilbert scheme $\mathrm{Hilb}_d(\mathbb{C}^2/\Gamma)$ is irreducible.*

Zheng's theorem [25, Theorem 1.1] asserts more generally that $\mathrm{Hilb}_d(X)$ is irreducible for every quasi-projective normal surface X with at worst rational double points. For the quotients $\mathbb{C}^2/\Gamma$, Craw, Gammelgaard, Gyenge and Szendrői proposed a second proof, which identifies the underlying reduced Hilbert schemes with quiver varieties [4]. Its Lemma 4.3 has a gap, found by Yehao Zhou; Craw and Yamagishi explain this and give a complete proof of the irreducibility that is independent of Zheng's work [5, Proposition 5.1, Theorem 5.5 and Remark 5.6]. We use only the irreducibility.

We say that I *contains an equation of type A, D or E* if, in suitable coordinates, $I\widehat{S}$ contains uf for a unit $u \in \widehat{S}$ and an ADE polynomial f of that type.

Corollary 4.3. *If I contains an equation of type A, D or E, then S/I is smoothable.*

Proof. Choose φ with $uf \in \varphi(I\widehat{S})$. Then $f \in \varphi(I\widehat{S}) \cap S = I_\varphi$, so $V(I_\varphi)$ is a subscheme of X_f. By Theorem 4.2 and Proposition 2.4(4), the algebra S/I_φ is smoothable, and so is $S/I \cong S/I_\varphi$. □

4.4. The next proposition finds such equations. The *rank* of a quadric is the rank of its symmetric 3×3 matrix; a nonzero quadric has rank one exactly when it is the square of a linear form. If $f \in S$ has leading form x^2, its *transverse cubic* $\gamma_f \in \mathbb{C}[y,z]_3$ is the cubic part of $f(0,y,z)$. We regard $0 = 0^3$ as a cube, so a binary cubic that is not a cube has at least two distinct linear factors.

Proposition 4.5. *Let $I \subseteq S$ be an ideal of finite colength with $V(I) = \{0\}$, and let $f \in I$ be of order two.*

(a) *If the rank of f^* is at least two, then I contains an equation of type A.*
(b) *If $f^* = x^2$ and γ_f is not the cube of a linear form, then I contains an equation of type D.*
(c) *If $f^* = x^2$, γ_f is a nonzero cube, and $\mathfrak{m}^5 \subseteq I$, then I contains an equation of type E_6, E_7 or E_8.*

Proof. Fix $N \geq 4$ with $\mathfrak{m}^N \subseteq I$. By the splitting lemma [11, Section I.2], there are formal coordinates in which

$$f = x^2 + y^2 + z^2, \qquad f = x^2 + y^2 + g(z), \qquad \text{or} \qquad f = x^2 + g(y,z),$$

according as the rank of f^* is three, two or one, where $g = 0$ or g has order at least three. In the last case, the change of coordinates fixes y and z and replaces x by $x - \xi$, up to a unit factor, where $\xi \in (y,z)^2$ solves $\partial f/\partial x = 0$. Thus $g(y,z) = f(\xi,y,z)$, and the cubic part of g is γ_f.

(a) If the rank is three, f is the polynomial A_1. If the rank is two, we add z^N to f in case $g = 0$. Then $g = z^k v$ with $k \geq 3$ and $v \in \mathbb{C}[[z]]$ a unit, and in the coordinate $z' = zv^{1/k}$ we have $f = x^2 + y^2 + z'^k$, which is A_{k-1}.

(b) By Hensel's lemma,[1] g factors according to a factorization of γ_f into coprime factors. If γ_f has three distinct linear factors, then $g = g_1 g_2 g_3$ with power series g_i of order one whose linear parts are pairwise independent. Using $u = g_1$ and $v = g_2$ as coordinates in the (y,z)-plane, we can write $g_3 = \alpha u + \beta v$ with units α, β. In the coordinates $x' = (\alpha\beta)^{1/2} x$, $u' = \alpha u$, $v' = \beta v$ we obtain $\alpha\beta f = x'^2 + u'v'(u' + v')$, which becomes D_4 after a linear change of u' and v' and a rescaling.

Otherwise $\gamma_f = \ell m^2$ with independent linear forms ℓ and m, and $g = g_1 g_2$ with leading forms $g_1^* = \ell$ and $g_2^* = m^2$. We take $z = g_1$ and a coordinate y with linear part m. The splitting lemma, applied to g_2 in the variable y, changes y only and brings g_2 to the form $y^2 + \beta(z)$, where $\beta = 0$ or β has order at least three. Thus $f = x^2 + zy^2 + z\beta(z)$. If $\beta = 0$, we add z^N and obtain D_{N+1}. Otherwise $\beta = z^{k-2}\varepsilon$ with $k \geq 5$ and $\varepsilon \in \mathbb{C}[[z]]$ a unit; in the coordinates $x' = \varepsilon^{-1/2} x$ and $y' = \varepsilon^{-1/2} y$ we have $f = \varepsilon\,(x'^2 + zy'^2 + z^{k-1})$, which is εD_k.

[1] Every factorization of the leading form of a power series in two variables into coprime factors lifts to a factorization of the series. One lifts degree by degree, using that two coprime binary forms of degrees p and q generate all binary forms of degree at least $p+q-1$.

(c) Here $f = x^2 + g(y,z)$ with $\gamma_f = \ell^3$ for a nonzero linear form ℓ, and we may take $y = \ell$. Modulo $\widehat{\mathfrak{m}}^5$, which lies in $I\widehat{S}$, we have $g \equiv y^3 + \alpha z^4 + \beta y z^3 + y^2 h$ for a quadric h, and in the coordinate $y' = y + h/3$ this becomes $g \equiv y'^3 + \alpha z^4 + \beta y' z^3$. If $\alpha \neq 0$, the coordinate $z' = z + \beta y/(4\alpha)$ turns $\alpha z^4 + \beta y z^3$ into $\alpha z'^4$ plus $y^2 h'$ for a quadric h', which the change $y \mapsto y + h'/3$ removes as before. So $I\widehat{S}$ contains $x^2 + y^3 + \alpha z^4$ if $\alpha \neq 0$, it contains $x^2 + y^3 + \beta y z^3$ if $\alpha = 0 \neq \beta$, and it contains $x^2 + y^3$, hence $x^2 + y^3 + z^5$, if $\alpha = \beta = 0$. After rescaling z, these are E_6, E_7 and E_8. □

4.6. We need two facts about small linear systems and one observation about transverse cubics.

Lemma 4.7. (1) *Every two-dimensional space of quadrics contains a quadric of rank at least two.*

(2) *Every affine plane in $\mathbb{C}[y,z]_3$ contains a binary cubic that is not a cube.*

Proof. (1) Otherwise the space would be spanned by squares ℓ_1^2 and ℓ_2^2 of independent linear forms, and $\ell_1^2 + \ell_2^2$ has rank two. (2) The cubes ℓ^3 form a closed cone $K \subset \mathbb{C}[y,z]_3$, the affine cone over the twisted cubic. Suppose that $\gamma + U \subseteq K$ for a two-dimensional linear space U. For $u \in U$, the points $s\gamma + u$ lie in K for every $s \neq 0$, because K is a cone; hence also for $s = 0$, because K is closed. Thus $U \subseteq K$. But if ℓ_1^3 and ℓ_2^3 are independent elements of U, then

$$\ell_1^3 + \ell_2^3 = (\ell_1 + \ell_2)(\ell_1 + \omega\ell_2)(\ell_1 + \omega^2\ell_2), \qquad \omega = e^{2\pi i/3},$$

has three distinct linear factors, a contradiction. □

Lemma 4.8. *Let $f \in I$ have leading form x^2, and let $\pi\colon S_3 \to \mathbb{C}[y,z]_3$ be the substitution $x = 0$. The transverse cubics of the elements of I with leading form x^2 form the affine space $\gamma_f + \pi(I_3^*)$, and*

$$\dim \pi(I_3^*) \;\geq\; \dim I_3^* - 6.$$

Proof. If $f' \in I$ has leading form x^2, then $f' - f$ is an element of I of order at least three, whose cubic part lies in I_3^*; conversely, every element of I_3^* is the cubic part of such an element. Since the transverse cubic of $f + h$ is $\gamma_f + \pi(h_3)$, where h_3 is the cubic part of h, the first claim follows. The kernel of π is xS_2, which has dimension six. □

4.9. We can now prove the surface criterion.

Proof of Theorem 1.2. Write $Z = V(I)$ with $V(I) = \{0\}$. If $I \not\subseteq \mathfrak{m}^2$, then Z has embedding dimension at most two and is smoothable by Theorem 2.6. So let $I \subseteq \mathfrak{m}^2$. If the tangent cone lies on a quadric q that is not a square, that is, of rank at least two, then $q = f^*$ for some $f \in I$, and Proposition 4.5(a) and Corollary 4.3 show that Z is smoothable. By Lemma 4.7(1), it remains to consider the case $I_2^* = \mathbb{C}x^2$ and $\dim I_3^* \geq 8$, after a linear change of coordinates. Choose $f \in I$ with $f^* = x^2$. By Lemma 4.8, the transverse

cubics of such elements fill an affine space of dimension at least two, and by Lemma 4.7(2) one of them is not a cube. We conclude by Proposition 4.5(b) and Corollary 4.3. □

Remark 4.10. In terms of the Hilbert function, Theorem 1.2 applies to every local algebra A of embedding dimension three with $H_A(2) \leq 4$, and to every such algebra with $H_A(2) = 5$ and $H_A(3) \leq 2$, since $\dim I_2^* = 6 - H_A(2)$ and $\dim I_3^* = 10 - H_A(3)$. There is no bound on the length.

4.11. When the tangent cone lies on a single square, one cubic equation fewer suffices if $\mathfrak{m}^5 \subseteq I$.

Corollary 4.12. *Let $I \subseteq S$ be an ideal of finite colength with $V(I) = \{0\}$, $I_2^* = \mathbb{C}x^2$, $\dim I_3^* \geq 7$ and $\mathfrak{m}^5 \subseteq I$. Then S/I is smoothable.*

Proof. By Lemma 4.8, the transverse cubics of the elements of I with leading form x^2 fill an affine space of dimension at least one, which contains a nonzero element. If one of them is not a cube, Proposition 4.5(b) applies; otherwise one of them is a nonzero cube, and Proposition 4.5(c) applies. In both cases S/I is smoothable by Corollary 4.3. □

5. Splitting off a point

5.1. We say that $a, b, c \in S$ are *polynomial coordinates centered at the origin* if $S = \mathbb{C}[a, b, c]$ and a, b, c vanish at the origin; for example, a, $b - \beta a^2$, $c - \gamma a^2$ for linear coordinates a, b, c. For ideals I and L of S, the *ideal quotient* is $(I : L) = \{u \in S : uL \subseteq I\}$.

Definition 5.2. Let $I \subseteq S$ be an ideal of finite colength with $V(I) = \{0\}$. A *ray* of I is an ideal $L = (b, c)$, for polynomial coordinates a, b, c centered at the origin, such that $(I : L) \not\subseteq I + L$.

The curve $V(L)$ is the path along which a point will leave $V(I)$. For example, (b, c) is a ray if $a^e b \in I$, $a^e c \in I$ and $a^e \notin I + (b, c)$ for some $e \geq 1$, since then a^e lies in $(I : L)$ but not in $I + L$.

Proposition 5.3 (Splitting off a point)**.** *Let $I \subseteq S$ be an ideal of colength d with $V(I) = \{0\}$, and let $L = (b, c)$ be a ray of I, for polynomial coordinates a, b, c centered at the origin. Then there is an ideal $\mathcal{I} \subseteq S[t]$ such that $S[t]/\mathcal{I}$ is a free $\mathbb{C}[t]$-module of rank d, $\mathcal{I}|_{t=0} = I$, and for every $t_0 \neq 0$ the subscheme $V(\mathcal{I}|_{t=t_0})$ is the disjoint union of the reduced point $(a, b, c) = (t_0, 0, 0)$ and a subscheme of length $d - 1$ supported at the origin.*

Proof. We have $S/L = \mathbb{C}[a]$. The images of I and $(I : L)$ in $\mathbb{C}[a]$ are ideals (a^r) and (a^s), and $s < r$ because $(I : L) \not\subseteq I + L$. Choose $F \in I$ with image a^r, and $u \in (I : L)$ with image a^{r-1}; for instance, $u = a^{r-1-s}u'$ for an element $u' \in (I : L)$ with image a^s. Let $J = I \cap L$. Every element of I is congruent modulo J to a multiple of F by a polynomial in a, so $I = J + (F)$.

As $\mathbb{C}[a]$-modules, $S/J = \mathbb{C}[a] \oplus M$ with $M = L/J$, and M is finite-dimensional because it embeds into S/I. Both F and u kill M, since FL and uL lie in $I \cap L = J$. Set

$$F_t = F - tu \in S[t] \qquad \text{and} \qquad \mathcal{I} = JS[t] + (F_t).$$

Then F_t kills $M[t]$, and $F_t \equiv a^r - ta^{r-1}$ modulo L. So the ideal generated by F_t in $(S/J)[t] = \mathbb{C}[a,t] \oplus M[t]$ consists of the pairs $((a^r - ta^{r-1})p,\ \mu(p))$ with $p \in \mathbb{C}[a,t]$, for a $\mathbb{C}[a,t]$-linear map μ to $M[t]$. Division by the monic polynomial $a^r - ta^{r-1}$ brings every class to a normal form (h, m) with $\deg_a h < r$ and $m \in M[t]$. The normal form is unique, because a pair $\big((a^r - ta^{r-1})p,\ \mu(p)\big)$ whose first entry has a-degree less than r has $p = 0$, and so it is zero. So $S[t]/\mathcal{I}$ is a free $\mathbb{C}[t]$-module with basis $1, a, \dots, a^{r-1}$ together with a basis of M. At $t = 0$ we recover $J + (F) = I$, so the rank is d.

Let $t_0 \neq 0$ and $I_{t_0} = \mathcal{I}|_{t=t_0} = J + (F_{t_0})$. Since $V(J) = \{0\} \cup V(L)$ and F_{t_0} restricts to $a^{r-1}(a - t_0)$ on the curve $V(L)$, the subscheme $V(I_{t_0})$ is supported at the origin and at $p = (t_0, 0, 0)$. Near p the ideal I is the unit ideal, so $J = L$ locally at p, and $I_{t_0} = (b, c, a - t_0)$ locally at p. Thus p is a reduced point, and the part at the origin has length $d - 1$. □

Corollary 5.4. *If, in addition, every finite subscheme of $\mathbb{A}^3$ of length $d-1$ is smoothable, then S/I is smoothable.*

Proof. The family of Proposition 5.3 is a morphism $\mathbb{A}^1 \to \mathrm{Hilb}_d(\mathbb{A}^3)$. For $t_0 \neq 0$ it parametrizes a disjoint union of a point and a subscheme of length $d - 1$, which is smoothable by Proposition 2.4(2). Hence S/I is smoothable by Proposition 2.4(1). □

Remark 5.5. For rays witnessed by a pure power, $a^e \in (I : L) \setminus (I + L)$, Proposition 5.3 is a ray family in the sense of [3, Section 5], in the form of [6, Proposition 3.7 and Corollary 3.8]. The general form is used in Lemmas 6.4 and 7.3 and, through Lemma 5.6 below, in Section 8.

In terms of inverse systems (Section 3.4), rays along lines are detected as follows. For a nonzero $P \in T_1$ with coefficient vector p, let $N_P = \{v \in S_1 : v(p) = 0\}$, and let L_P be the ideal generated by N_P, the ideal of the line through 0 and p. For an inverse system M, the space $L_P \circ M$ is defined as in Section 3.4.

Lemma 5.6. *Let $I \subseteq S$ be an ideal of finite colength with $V(I) = \{0\}$, and let M be its inverse system. If some power P^k of a nonzero $P \in T_1$ lies in M but not in $L_P \circ M$, then L_P is a ray of I.*

Proof. Choose N with $\mathfrak{m}^N \subseteq I$. The subspace $L_P \circ M$ of $T_{<N}$ does not contain P^k, so there is $g \in S$ with $\langle g, L_P \circ M\rangle = 0$ and $\langle g, P^k\rangle \neq 0$. For $\ell \in L_P$ and $F \in M$ we have $\langle g\ell, F\rangle = \langle g, \ell \circ F\rangle = 0$, so $gL_P \subseteq I$ by Lemma 3.5(1), that is, $g \in (I : L_P)$. If $g = i + \ell$ with $i \in I$ and $\ell \in L_P$, then $\langle g, P^k\rangle = \langle i, P^k\rangle + \langle \ell, P^k\rangle = 0$, because $P^k \in M$ and $\ell \circ P^k = 0$. Hence

$g \notin I + L_P$. Finally, $L_P = (b, c)$ for a basis b, c of N_P, and a, b, c are linear coordinates for any linear form a with $a(p) \neq 0$. So L_P is a ray of I. □

6. Twelve points

6.1. Let $A = S/I$ be a local algebra of length $d \leq 12$ with $I \subseteq \mathfrak{m}^2$. By Lemma 3.2, the Hilbert functions that are not entirely covered by Theorem 1.2 are $(1, 3, 6)$ and $(1, 3, 6, 1)$ for $d \leq 11$, and

$$(1, 3, 5, 3), \quad (1, 3, 6, 2), \quad (1, 3, 6, 1, 1)$$

for $d = 12$. For $(1, 3, 5, 3)$ only the algebras whose tangent cone lies on the square of a linear form need attention, and Proposition 4.5 treats them. The following two lemmas produce rays for the others, for $(1, 3, 6)$, $(1, 3, 6, 1)$ and $(1, 3, 6, 2)$ after passing to the general member of a family.

Lemma 6.2. *Let $A = S/I$ be a local algebra supported at the origin with Hilbert function $(1, 3, 6, 1, \ldots, 1)$, with last nonzero value in degree $s \geq 4$. Then there are polynomial coordinates a, b, c centered at the origin such that $a^2 b \in I$, $a^2 c \in I$, and $a^2 \notin I + (b, c)$. In particular, (b, c) is a ray of I.*

Proof. Let $G = S/I^*$. Its components $G_3, \ldots, G_s$ are lines, and $G_4 = G_1 G_3$. Hence multiplication defines a nonzero linear form on G_1 with values in $\operatorname{Hom}(G_3, G_4) \cong \mathbb{C}$, and we can choose linear coordinates a, b, c with $bG_3 = cG_3 = 0$. Then $bG_k = cG_k = 0$ for all $k \geq 3$, since $G_k = G_1^{k-3} G_3$. Every monomial of degree $k \geq 4$ other than a^k is a multiple of b or c by a monomial of degree $k-1$, so it vanishes in G. Therefore G_k is spanned by a^k for $4 \leq k \leq s$, and G_3 is spanned by a^3. Writing $a^2 b = \kappa a^3$ in G_3 and multiplying by a, we get $0 = a^3 b = \kappa a^4$, so $\kappa = 0$. Thus $a^2 b$ and, in the same way, $a^2 c$ vanish in $G_3 = \mathfrak{m}^3 A / \mathfrak{m}^4 A$.

Since $H_A(s + 1) = 0$, we have $\mathfrak{m}^{s+1} A = 0$ by Lemma 3.2, so $\mathfrak{m}^4 A$ is spanned by $a^4, \ldots, a^s$. Hence $a^2 b = a^2 \beta$ and $a^2 c = a^2 \gamma$ in A for polynomials $\beta, \gamma \in (a^2)$ in a alone, and the polynomial coordinates a, $b' = b - \beta$, $c' = c - \gamma$ satisfy $a^2 b', a^2 c' \in I$. Finally, $I \subseteq \mathfrak{m}^3$ because $H_A(2) = 6$, and the quotient map $S \to S/(b', c') = \mathbb{C}[a]$ sends b and c into (a^2), hence $\mathfrak{m}^3$ into (a^3). So $a^2 \notin I + (b', c')$. □

Lemma 6.2 is a special case of the long-tail criterion [6, Proposition 4.2].

6.3. An algebra with $H_A(2) = 6$ and $H_A(4) = 0$ is determined by its cubic equations. When they have codimension at most three in S_3, the algebra is a limit of algebras with a ray.

Lemma 6.4. *Let $A = S/I$ be a local algebra supported at the origin with $H_A(2) = 6$ and $H_A(4) = 0$, and let $k = H_A(3)$. Then $I = V + \mathfrak{m}^4$ for the subspace $V = I \cap S_3$ of codimension k in S_3, and conversely every subspace of codimension k gives such an algebra. If $k \leq 3$ and every finite subscheme of $\mathbb{A}^3$ of length $9 + k$ is smoothable, then A is smoothable.*

Proof. Since $H_A(2) = 6$ and $H_A(4) = 0$, we have $I \subseteq \mathfrak{m}^3$ and $\mathfrak{m}^4 \subseteq I$ by Lemma 3.2. Every $f \in I$ is the sum of its cubic part and an element of $\mathfrak{m}^4$, so $I = V + \mathfrak{m}^4$, and $\dim V = 10 - k$. The converse is clear.

We first find linear coordinates a, b, c and a nonzero quadric q with $qb, qc \in V$. For a linear form u, let $\mu_u \colon S_2 \to S_3/V$ be the map $w \mapsto uw + V$. If $k \leq 2$, any linear coordinates work, because $(\mu_b, \mu_c) \colon S_2 \to (S_3/V)^2$ maps a space of dimension six to one of dimension $2k \leq 4$, and so has a nonzero kernel. Let $k = 3$. In fixed bases, (μ_x, μ_z) and $(\mu_{y-\lambda x}, \mu_z) = (\mu_y - \lambda\mu_x, \mu_z)$ are 6×6 matrices, and the determinant of the second is a polynomial in λ of degree at most three whose coefficient of λ^3 is minus the determinant of the first. So either (μ_x, μ_z) has a nonzero kernel, and we take $(a, b, c) = (y, x, z)$, or $(\mu_{y-\lambda x}, \mu_z)$ has a nonzero kernel for a root λ of this polynomial, and we take $(a, b, c) = (x, y - \lambda x, z)$.

Let p be the point with coordinates $(a, b, c) = (1, 0, 0)$, so that the quotient map $S \to S/(b, c) = \mathbb{C}[a]$ sends a form h of degree j to $h(p)a^j$. The cubics qb and qc are linearly independent, and we write $V = V' \oplus \mathbb{C}qb \oplus \mathbb{C}qc$. For $t \in \mathbb{C}$ put $q_t = q + ta^2$ and

$$I_t = V' + \mathbb{C}q_t b + \mathbb{C}q_t c + \mathfrak{m}^4,$$

so that $I_0 = I$. Let $U \subseteq \mathbb{C}$ be the set of t for which $V' + \mathbb{C}q_t b + \mathbb{C}q_t c$ has dimension $10 - k$; it is the complement of a finite set and contains 0. For $t \in U$, the space I_t is an ideal of colength $10 + k$, and $q_t \in (I_t : (b, c))$. The quotient map sends $I_t \subseteq \mathfrak{m}^3$ into (a^3) and q_t to $(q(p) + t)a^2$. Hence (b, c) is a ray of I_t for $t \neq -q(p)$, and S/I_t is smoothable by Corollary 5.4. By Proposition 2.4(5), $A = S/I_0$ is smoothable. □

For $k = 3$, the points $p = (1, s, t)$ that admit such a quadric, with b, c spanning the linear forms vanishing at p, are the zeros of the determinant of (μ_{y-sx}, μ_{z-tx}). No row of μ_x occurs twice in a nonzero term of its expansion, so this is a polynomial of degree at most three in s and t: the points lie on a plane curve of degree at most three, unless they fill the plane. On the line $t = 0$ it is the cubic polynomial in $\lambda = s$ used in the proof.

Example 6.5. The ideal

$$I = (x^2y,\ x^2z,\ xy^2,\ y^2z,\ xz^2,\ yz^2,\ x^3 - y^3,\ y^3 - z^3) + \mathfrak{m}^4$$

has Hilbert function $(1, 3, 6, 2)$; the cubics in its inverse system form the Hesse pencil, spanned by $X^3 + Y^3 + Z^3$ and XYZ. Here $q = x^2$ satisfies $qy, qz \in I$, and $q(p) = 1$ for $p = (1, 0, 0)$, so (y, z) is already a ray of I. In the proof of Proposition 5.3 we may take $r = 3$, $F = x^3 - y^3$ and $u = x^2$, so the point $(t, 0, 0)$ leaves the origin along the family $(I \cap (y, z)) + (x^3 - tx^2 - y^3)$.

6.6. We now prove the theorem for at most twelve points.

Theorem 6.7. *Every finite subscheme of $\mathbb{A}^3$ of length at most twelve is smoothable.*

Proof. We argue by induction on the length $d \leq 12$; a subscheme of length one is a reduced point. By Proposition 2.7, it suffices to show that every local algebra $A = S/I$ of length d with $I \subseteq \mathfrak{m}^2$ is smoothable, provided that every finite subscheme of $\mathbb{A}^3$ of length less than d is smoothable. By Lemma 3.2, the Hilbert function of A satisfies $H_A(0) = 1$ and $H_A(1) = 3$, so $d \geq 4$, and $\sum_{k\geq 2} H_A(k) = d - 4 \leq 8$, and it has no internal zeros.

If $H_A(2) \leq 4$, Theorem 1.2 applies.

If $H_A(2) = 5$, then $\sum_{k\geq 3} H_A(k) \leq 3$. If $H_A(3) \leq 2$, Theorem 1.2 applies. Otherwise $H_A(3) = 3$ and $H_A(4) = 0$, so $\mathfrak{m}^5 \subseteq \mathfrak{m}^4 \subseteq I$. The tangent cone lies on a unique quadric q up to scalar. If q has rank at least two, Theorem 1.2 applies; otherwise $q = x^2$ in suitable coordinates, and Corollary 4.12 applies.

If $H_A(2) = 6$, then $\sum_{k\geq 3} H_A(k) \leq 2$, so either $H_A(4) = 0$ or the Hilbert function is $(1, 3, 6, 1, 1)$. In the first case $k = H_A(3) \leq 2$ and $9 + k = d - 1$, so A is smoothable by Lemma 6.4. In the second case it is smoothable by Lemma 6.2 and Corollary 5.4. □

Remark 6.8. An algebra S/I is *very compressed* if $\mathfrak{m}^{s+1} \subseteq I \subseteq \mathfrak{m}^s$ for some s. Douvropoulos, Jelisiejew, Nødland and Teitler prove, with a computer calculation, that every very compressed algebra S/I of length at most 95 is smoothable [6, Proposition 1.2]. In the proof of Theorem 1.1 we use this only for the Hilbert function $(1, 3, 6, 4)$. There rays do not suffice: for a general six-dimensional $V \subseteq S_3$, the ideal $I = V + \mathfrak{m}^4$ has no ray. Indeed, let $u \in (I : L) \setminus (I + L)$ for a ray $L = (b, c)$, and let N be the plane spanned by the linear parts of b and c. Since $uL \subseteq \mathfrak{m}^3$, we have $u \in \mathfrak{m}^2$. If the quadratic part q of u is nonzero, then $qN \subseteq V$; but the triples $(N, [q], V)$ with $qN \subseteq V$ form a family of dimension $2 + 5 + 16 < 24 = \dim \mathrm{Gr}(6, S_3)$. If $q = 0$, then the cubics in V vanish at the point where N vanishes, since otherwise the image of I in $S/L \cong \mathbb{C}[a]$ would contain that of $\mathfrak{m}^3 \ni u$; a general V has no such point, because the pairs (p, V) with V in the hyperplane of cubics vanishing at p form a family of dimension $2 + 18 < 24$.

7. Thirteen points

7.1. Let $A = S/I$ be a local algebra of length thirteen with $I \subseteq \mathfrak{m}^2$. By Lemma 3.2, the Hilbert functions that are not entirely covered by Theorem 1.2 are

$$(1, 3, 5, 3, 1), \quad (1, 3, 5, 4), \quad (1, 3, 6, 3), \quad (1, 3, 6, 2, 1), \quad (1, 3, 6, 1, 1, 1),$$

and for the first two only the algebras whose tangent cone lies on the square of a linear form need attention. Proposition 4.5 treats the first, Lemma 6.2 the last, and Lemma 6.4 the third. The following two lemmas treat the others.

Lemma 7.2. *Every local algebra with Hilbert function* $(1, 3, 5, k)$ *is smoothable.*

Proof. Let $A = S/I$ be such an algebra, supported at the origin. Then $\mathfrak{m}^4 \subseteq I$ by Lemma 3.2, the tangent cone lies on a unique quadric q up to

scalar, and $K = I_3^*$ is a subspace of S_3 of dimension $10-k$ containing qS_1. If $f \in I$ has leading form q and cubic part g, then $q+g \in I$, and

$$I = \mathbb{C}(q+g) + K + \mathfrak{m}^4,$$

since $K \subseteq I$ and every element of I is congruent modulo $K+\mathfrak{m}^4$ to a multiple of $q+g$.

Write $K = qS_1 \oplus K'$ and choose a quadric q' of rank three. For $t \in \mathbb{C}$ put $q_t = q + tq'$ and

$$I_t = \mathbb{C}(q_t+g) + q_tS_1 + K' + \mathfrak{m}^4,$$

so that $I_0 = I$. Let $U \subseteq \mathbb{C}$ be the set of t for which $q_tS_1 + K'$ has dimension $10-k$; it is the complement of a finite set and contains 0. For $t \in U$, the space I_t is an ideal of colength $9+k$, since $S_1(q_t+g) \subseteq q_tS_1 + \mathfrak{m}^4$ and $S_1K' \subseteq \mathfrak{m}^4$. For all but finitely many t, the quadric q_t has rank three, and S/I_t is smoothable by Theorem 1.2. By Proposition 2.4(5), $A = S/I_0$ is smoothable. □

For $k=3$, this gives a second proof of the case $(1,3,5,3)$ of Theorem 6.7.

Lemma 7.3. *Let $A = S/I$ be a local algebra supported at the origin with Hilbert function $(1,3,6,2,1)$. Then I has a ray.*

Proof. Let $G = S/I^*$. Multiplication defines a bilinear map $G_1 \times G_3 \to G_4 \cong \mathbb{C}$, hence a linear map $\psi\colon G_1 \to G_3^\vee$. It is nonzero because $G_4 = G_1G_3$, and its rank is one or two. Recall that $\mathfrak{m}^5 \subseteq I$ and $I \subseteq \mathfrak{m}^3$.

Suppose first that ψ has rank one, and choose linear coordinates a,b,c with $\psi(b) = \psi(c) = 0$, so that $bG_3 = cG_3 = 0$. As in the proof of Lemma 6.2, every monomial of degree four other than a^4 vanishes in G, so G_4 is spanned by a^4. In particular a^3b and a^3c vanish in $G_4 = \mathfrak{m}^4A/\mathfrak{m}^5A$, and since $\mathfrak{m}^5 \subseteq I$, they lie in I. If a^3 were congruent modulo I to an element of (b,c), then a^3 would lie in (b,c) modulo I_3^*, and a^4 would vanish in G_4. Hence $a^3 \notin I+(b,c)$, and (b,c) is a ray.

Suppose now that ψ has rank two. Choose c spanning its kernel and complete it to linear coordinates a,b,c. Then ψ induces a perfect pairing between $G_1/\mathbb{C}c$ and G_3, and $cG_3 = 0$. As $\mathfrak{m}^5 \subseteq I$, the latter gives $cS_3 \subseteq I$. For a linear form ℓ, the element $\ell\cdot c^2a = c\cdot \ell ca$ therefore lies in I, so the class of c^2a in G_3 pairs to zero with G_1 and vanishes; similarly for c^2b. Thus c^2a and c^2b lie in $\mathfrak{m}^4A = G_4$. Since $\psi(a)$ and $\psi(b)$ are linearly independent, the map $S_3 \to G_4 \times G_4$, $h \mapsto (ha, hb)$, is surjective, so there is $h \in S_3$ with $ha = -c^2a$ and $hb = -c^2b$ in A. The element $u = c^2 + h$ then satisfies $ua, ub \in I$. It is not in $I+(a,b)$, because the quotient map $S \to S/(a,b) = \mathbb{C}[c]$ sends $I \subseteq \mathfrak{m}^3$ into (c^3) and u to c^2 plus a multiple of c^3. So (a,b) is a ray. □

In the second case c^2 itself need not lie in $(I : (a,b))$; the correction h makes the argument work for every algebra with this Hilbert function.

7.4. We now prove the theorem for thirteen points.

Theorem 7.5. *Every finite subscheme of $\mathbb{A}^3$ of length thirteen is smoothable.*

Proof. By Theorem 6.7 and Proposition 2.7, it suffices to show that every local algebra $A = S/I$ of length thirteen with $I \subseteq \mathfrak{m}^2$ is smoothable. Its Hilbert function starts with $(1, 3)$, has no internal zeros, and satisfies $\sum_{k\geq 2} H_A(k) = 9$.

If $H_A(2) \leq 4$, Theorem 1.2 applies.

If $H_A(2) = 5$, then $\sum_{k\geq 3} H_A(k) = 4$. If $H_A(3) \leq 2$, Theorem 1.2 applies, and if $H_A(3) = 4$, the Hilbert function is $(1, 3, 5, 4)$ and Lemma 7.2 applies. Otherwise $H_A(3) = 3$, $H_A(4) = 1$, and $\mathfrak{m}^5 \subseteq I$. If the quadric on the tangent cone has rank at least two, Theorem 1.2 applies; otherwise it is x^2 in suitable coordinates, and Corollary 4.12 applies.

If $H_A(2) = 6$, then $\sum_{k\geq 3} H_A(k) = 3$, and by Lemma 3.2(3) the Hilbert function is $(1, 3, 6, 3)$, $(1, 3, 6, 2, 1)$ or $(1, 3, 6, 1, 1, 1)$. In the last two cases, Lemmas 7.3 and 6.2 give a ray, and A is smoothable by Corollary 5.4 and Theorem 6.7. In the first case, A is smoothable by Lemma 6.4 and Theorem 6.7. □

8. Fourteen points

8.1. Let $A = S/I$ be a local algebra of length fourteen with $I \subseteq \mathfrak{m}^2$. By Lemma 3.2, the Hilbert functions that are not entirely covered by Theorem 1.2 are

$$(1, 3, 5, 5),\quad (1, 3, 5, 4, 1),\quad (1, 3, 5, 3, 2),\quad (1, 3, 5, 3, 1, 1),$$
$$(1, 3, 6, 4),\quad (1, 3, 6, 3, 1),\quad (1, 3, 6, 2, 2),\quad (1, 3, 6, 2, 1, 1),\quad (1, 3, 6, 1, 1, 1, 1).$$

Lemma 7.2 treats the first, Lemma 6.2 the last, and [6, Proposition 1.2] the algebras with Hilbert function $(1, 3, 6, 4)$, which are very compressed. For the others we begin with tails of ones that start in degree four.

Lemma 8.2. *Let $A = S/I$ be a local algebra supported at the origin with $H_A(4) = H_A(5) = 1$. Then there are linear coordinates a, b, c and polynomials $\beta, \beta' \in (a^2)$ in a alone such that $a^3(b - \beta)$ and $a^3(c - \beta')$ lie in I, and all elements of I_2^* and of I_3^* vanish at the point p with coordinates $(a, b, c) = (1, 0, 0)$. If $I \subseteq \mathfrak{m}^3$, then $(b - \beta, c - \beta')$ is a ray of I.*

Proof. Let $G = S/I^*$, and let s be the last degree with $H_A(s) \neq 0$. By Lemma 3.2, $G_4, \ldots, G_s$ are lines and $s \geq 5$. As in the proof of Lemma 6.2, multiplication defines a nonzero linear map $G_1 \to \operatorname{Hom}(G_4, G_5) \cong \mathbb{C}$, and we choose linear coordinates a, b, c with $bG_4 = cG_4 = 0$. Then $bG_k = cG_k = 0$ for all $k \geq 4$, and every monomial of degree $k \geq 5$ other than a^k vanishes in G. So G_k is spanned by a^k for $5 \leq k \leq s$, and G_4 is spanned by a^4, since $a^5 \neq 0$. Writing $a^3b = \kappa a^4$ in G_4 and multiplying by a, we get $0 = a^4b = \kappa a^5$, so $\kappa = 0$. Thus a^3b and, in the same way, a^3c vanish in $G_4 = \mathfrak{m}^4A/\mathfrak{m}^5A$. They lie in $\mathfrak{m}^5A$, which is spanned by $a^5, \ldots, a^s$ because $\mathfrak{m}^{s+1}A = 0$, and this gives β and β'.

A form vanishes at p exactly when its coefficient of a^k is zero. If a quadric in I_2^* did not vanish at p, then a^2 would lie in $bG_1 + cG_1$, and $a^5 = a^3 \cdot a^2$ in $a^3bG_1 + a^3cG_1 = 0$. If a cubic in I_3^* did not vanish at p, then a^3 would lie in $bG_2 + cG_2$, and $a^5 = a^2 \cdot a^3$ in $bG_4 + cG_4 = 0$. Both contradict $a^5 \neq 0$.

Finally, the quotient map $S \to S/(b-\beta, c-\beta') = \mathbb{C}[a]$ sends b and c into (a^2), and hence it sends a form h of degree k to $h(p)a^k$ modulo (a^{k+1}). If $I \subseteq \mathfrak{m}^3$, it follows that I is sent into (a^4). Then $a^3 \notin I + (b-\beta, c-\beta')$, while a^3 lies in $(I : (b-\beta, c-\beta'))$. □

This covers $(1, 3, 6, 2, 1, 1)$. When the tangent cone lies on a quadric, the same ray works under a condition on transverse cubics.

Lemma 8.3. *Let $A = S/I$ be a local algebra supported at the origin with Hilbert function $(1, 3, 5, 4, 1)$, or $(1, 3, 5, 3, 1, \ldots, 1)$ with $H_A(5) = 1$, whose tangent cone lies on x^2. Suppose that the transverse cubics of the elements of I with leading form x^2 are all zero in the first case, and all cubes in the second. Then I has a ray.*

Proof. We first find $f \in I$ with leading form x^2 and $\gamma_f = 0$. In the first case every such f works. In the second, the transverse cubics form an affine space $\gamma_f + U$ with $U = \pi(I_3^*)$ of dimension at least $7 - 6 = 1$, by Lemma 4.8. As in the proof of Lemma 4.7(2), U consists of cubes, and by Lemma 4.7(2) itself U contains no plane. So $U = \mathbb{C}\ell^3$ for a linear form ℓ. Moreover $\gamma_f \in U$. Otherwise $\gamma_f = \ell_1^3$ with ℓ_1 not proportional to ℓ, and the transverse cubic $\ell_1^3 + \ell^3$ would have three distinct linear factors, as in the proof of Lemma 4.7(2). Subtracting from f an element of I of order three whose cubic part is sent to γ_f by π, we obtain $\gamma_f = 0$.

Next we find linear coordinates a, b_1, c_1 and polynomials $\beta, \beta' \in (a^2)$ in a alone such that, with $b = b_1 - \beta$, $c = c_1 - \beta'$ and p the point with coordinates $(a, b_1, c_1) = (1, 0, 0)$,

$$a^3b,\ a^3c \in I, \qquad x(p) = 0, \qquad \text{and all elements of } I_3^* \text{ vanish at } p.$$

For $(1, 3, 5, 3, 1, \ldots, 1)$, Lemma 8.2 provides them, since $x^2 \in I_2^*$ vanishes at p. For $(1, 3, 5, 4, 1)$, we have $U = 0$, so $I_3^* \subseteq xS_2$, with equality because both have dimension six. Then $xS_3 \subseteq I_4^*$, and since $\dim I_4^* = 14$, the forms in I_4^* that do not involve x form a hyperplane in $\mathbb{C}[y, z]_4$. As fourth powers of linear forms span $\mathbb{C}[y, z]_4$, there is a linear form $a \in \mathbb{C}[y, z]$ with $a^4 \notin I_4^*$. The linear map $\mathbb{C}[y, z]_1 \to S_4/I_4^* \cong \mathbb{C}$, $v \mapsto a^3v$, is nonzero at a; let c_1 span its kernel, and let $b_1 = x$ and $\beta = \beta' = 0$. Since $\mathfrak{m}^5 \subseteq I$, every element of I_4^* lies in I, so $a^3b, a^3c \in I$. The point p lies on the plane $x = 0$, and $I_3^* = xS_2$ vanishes at p.

Finally, as in the proof of Lemma 8.2, the quotient map $S \to S/(b, c) = \mathbb{C}[a]$ sends a form h of degree k to $h(p)a^k$ modulo (a^{k+1}). Elements of I of order at least three are therefore sent into (a^4). An element of I of order two is a multiple of f plus an element of order at least three. Write $f = x^2 + f_3 +$(terms of order at least four). Since $x(p) = 0$, the image of x lies in (a^2), and

$f_3(p) = \gamma_f(p) = 0$ because p lies on the plane $x = 0$. So f is sent into (a^4), and hence so is I. Thus a^3 lies in $(I : (b, c))$ but not in $I + (b, c)$. □

8.4. For the Hilbert functions $(1, 3, 6, 3, 1)$ and $(1, 3, 6, 2, 2)$ we use inverse systems, with the notation introduced in Section 3.4 and before Lemma 5.6. For a nonzero $P \in T_1$ with coefficient vector p, the space $W_P = N_P S_1$ consists of the quadrics that vanish at p. Since inverse systems are closed under differentiation, $L_P \circ M = N_P \circ M$ for every inverse system M.

Lemma 8.5. *Let $A = S/I$ be a local algebra supported at the origin with Hilbert function $(1, 3, 6, 3, 1)$. Then there is a nonzero $P \in T_1$ for which L_P is a ray of I.*

Proof. Let M be the inverse system of I. Since $\mathfrak{m}^5 \subseteq I$, we have $M \subseteq T_{\leq 4}$ by Lemma 3.5(1). By Lemma 3.5(3), M contains $T_{\leq 2}$, and $M \cap T_{\leq 3}$ has dimension 13. Hence $M \cap T_{\leq 3} = T_{\leq 2} \oplus D$ for the three-dimensional space $D = M \cap T_3$, and

$$M = T_{\leq 2} \oplus D \oplus \mathbb{C}\Phi, \qquad \Phi = F + G,$$

for a nonzero quartic F and a cubic G. Since $v \circ \Phi \in M$ has degree at most three for $v \in S_1$, the space $S_1 \circ F$ of first derivatives of F lies in D. Its dimension r is the number of essential variables of F, and $1 \leq r \leq 3$. We write $\operatorname{Ann}(F)_1 = \{v \in S_1 : v \circ F = 0\}$.

We first record a criterion for rays. For every P, the space $N_P \circ M$ is spanned by $T_{\leq 1}$, $N_P \circ D$, and the elements $v \circ \Phi = v \circ F + v \circ G$ with $v \in N_P$. If $N_P \cap \operatorname{Ann}(F)_1 = 0$, then $v \circ \Phi$ has the nonzero cubic part $v \circ F$ for $v \neq 0$, so the elements of $N_P \circ M$ of degree at most two are those of $T_{\leq 1} + N_P \circ D$. As $P^2 \in M$, Lemma 5.6 gives

$$N_P \cap \operatorname{Ann}(F)_1 = 0 \quad \text{and} \quad P^2 \notin N_P \circ D \quad \Longrightarrow \quad L_P \text{ is a ray of } I. \tag{1}$$

Three essential variables. Let $r = 3$. Then $\operatorname{Ann}(F)_1 = 0$, $D = S_1 \circ F$, and $N_P \circ D = W_P \circ F$. If $S_2 \circ F \neq T_2$, we choose P with $P^2 \notin S_2 \circ F$, which is possible because the squares of linear forms span T_2, and (1) applies. Otherwise $w \mapsto w \circ F$ is an isomorphism $S_2 \to T_2$. Let w_P be the quadric with $w_P \circ F = P^2$. Since W_P consists of the quadrics that vanish at p, it suffices to find P with $w_P(p) \neq 0$. Choose a basis $w_1, \ldots, w_6$ of S_2, and let B be the *catalecticant matrix* of F, the symmetric matrix with entries $B_{ij} = (w_i w_j) \circ F$. It is invertible, because $w \mapsto w \circ F$ is an isomorphism and the pairing between S_2 and T_2 is perfect. Writing $w_P = \sum_j c_j w_j$ and applying w_i to $w_P \circ F = P^2$, we get $\sum_j B_{ij} c_j = 2 w_i(p)$. Hence $w_P(p) = 2\Psi(p)$ for the quartic

$$\Psi = \sum_{i,j} (B^{-1})_{ij}\, w_i w_j \in S_4,$$

and Ψ does not vanish identically, because

$$\Psi \circ F = \sum_{i,j} (B^{-1})_{ij} B_{ij} = \operatorname{tr}(B^{-1} B) = 6. \tag{2}$$

So $w_P(p) \neq 0$ for general P, and (1) applies.

Two essential variables. Let $r = 2$. After a linear change of coordinates, $\operatorname{Ann}(F)_1 = \mathbb{C}z$, so that $F \in \mathbb{C}[X, Y]$. Write $D = S_1 \circ F \oplus \mathbb{C}E$ for a cubic E, let $E' = z \circ E$, and let P be such that $z(p) \neq 0$. Then $N_P \cap \operatorname{Ann}(F)_1 = 0$, and $N_P \circ D = W_P \circ F + N_P \circ E$, where $W_P \circ F \subseteq \mathbb{C}[X, Y]_2$, the kernel of $z\circ$ on T_2. So if $P^2 \in N_P \circ D$, then $z \circ P^2 = 2z(p)P$ lies in $N_P \circ E'$, that is, $P = v \circ E'$ for some $v \in N_P$. If the map $\psi\colon S_1 \to T_1$, $v \mapsto v \circ E'$, is an isomorphism, this says that $\psi^{-1}(P) \circ P = 0$; and $P \mapsto \psi^{-1}(P) \circ P$ is a nondegenerate quadratic form, because ψ comes from the nondegenerate symmetric bilinear form $(v, v') \mapsto (vv') \circ E'$. Otherwise it says that P lies in the proper subspace $\psi(S_1)$. In both cases (1) applies to general P.

One essential variable. Let $r = 1$. Then F is annihilated by a plane of linear forms, so F is a multiple of Q^4 for some $Q \in T_1$, and $Q^3 \in S_1 \circ F \subseteq D$. For $P = Q$, every element of N_P annihilates F, so $N_P \circ M$ is spanned by $T_{\leq 1}$, $N_P \circ D$ and $N_P \circ G$, and lies in $T_{\leq 2}$. Thus $P^3 \in M$ does not lie in $L_P \circ M$, and L_P is a ray by Lemma 5.6. □

For the Hilbert function $(1, 3, 6, 2, 2)$ we need the following description of pairs of quartic forms with few first derivatives.

Lemma 8.6. *Let $W \subseteq T_4$ and $D \subseteq T_3$ be two-dimensional spaces with $S_1 \circ W \subseteq D$. Then, after a linear change of coordinates, there is a nonzero quadric $q \in \mathbb{C}[x, y]_2$ such that W and D consist of all forms in $\mathbb{C}[X, Y]$ of degree four and three, respectively, that are annihilated by q.*

Proof. The forms whose first derivatives span a line are the fourth powers of linear forms. If every element of W were such a power, then W would be spanned by Q_1^4 and Q_2^4 for independent $Q_1, Q_2 \in T_1$, but $Q_1^4 + Q_2^4$ is not a fourth power. So some $F \in W$ satisfies $\dim S_1 \circ F = 2$, and then $S_1 \circ F = D$. After a linear change of coordinates $z \circ F = 0$, so $F \in \mathbb{C}[X, Y]$ and $z \circ D = S_1 \circ (z \circ F) = 0$. For $F' \in W$, the first derivatives of $z \circ F'$ lie in $z \circ D = 0$, so $z \circ F' = 0$. Thus W and D lie in $\mathbb{C}[X, Y]$.

The binary cubics $h \in \mathbb{C}[x, y]_3$ with $h \circ D = 0$ form a plane K, and for $v \in \mathbb{C}[x, y]_1$ and $h \in K$ we have $(vh) \circ W = h \circ (v \circ W) = 0$. So the four products of x and y with a basis h_1, h_2 of K lie in the three-dimensional space of binary quartics annihilating W, and there is a relation $v_1h_1 = v_2h_2$ with $v_1, v_2 \in \mathbb{C}[x, y]_1$ not both zero. Since h_1 and h_2 are independent, so are v_1 and v_2, and $h_1 = v_2q$, $h_2 = v_1q$ for a quadric q. Then $(vq) \circ D = 0$ for all $v \in \mathbb{C}[x, y]_1$, so $q \circ D = 0$, and for $F' \in W$ the first derivatives of $q \circ F'$ lie in $q \circ D = 0$, so $q \circ W = 0$. Since $q\circ$ maps $\mathbb{C}[X, Y]_4$ onto $\mathbb{C}[X, Y]_2$ and $\mathbb{C}[X, Y]_3$ onto $\mathbb{C}[X, Y]_1$, being dual to multiplication by q, its kernels in these degrees are two-dimensional, and they are W and D. □

Lemma 8.7. *If every finite subscheme of $\mathbb{A}^3$ of length thirteen is smoothable, then every local algebra with Hilbert function $(1, 3, 6, 2, 2)$ is smoothable.*

Proof. Let $A = S/I$ be such an algebra, supported at the origin, with inverse system M. As in the proof of Lemma 8.5, $M \subseteq T_{\leq 4}$ and $M =$

$T_{\leq 2} \oplus D \oplus \operatorname{span}(\Phi_1, \Phi_2)$, where $D = M \cap T_3$ has dimension two and $\Phi_i = F_i + G_i$ for linearly independent quartics F_1, F_2 and cubics G_1, G_2. The space $W = \operatorname{span}(F_1, F_2)$ satisfies $S_1 \circ W \subseteq D$, and we use the coordinates and the quadric q of Lemma 8.6.

Suppose first that $q = \ell_1\ell_2$ with independent linear forms ℓ_1, ℓ_2, and choose nonzero $Q_1, Q_2 \in \mathbb{C}[X, Y]_1$ with $\ell_1 \circ Q_1 = \ell_2 \circ Q_2 = 0$. Then $q \circ Q_i^4 = 0$, so $W = \operatorname{span}(Q_1^4, Q_2^4)$ and $D = \operatorname{span}(Q_1^3, Q_2^3)$, and we may take $F_1 = Q_1^4$ and $F_2 = Q_2^4$. Let $P = Q_1$. Every $v \in N_P$ annihilates Q_1^4 and sends Q_2^4 to a multiple of Q_2^3, so the cubic part of every element of $N_P \circ M$ is a multiple of Q_2^3. Hence $P^3 \in D \subseteq M$ does not lie in $L_P \circ M$, and L_P is a ray by Lemma 5.6. So A is smoothable by Corollary 5.4.

Suppose now that q is a square. In suitable coordinates $q = y^2$, so $W = \operatorname{span}(X^4, X^3Y)$ and $D = \operatorname{span}(X^3, X^2Y)$, and we may take $F_1 = X^4$ and $F_2 = X^3Y$. For $t \in \mathbb{C}$ let

$$F_1(t) = X^4 + 6tX^2Y^2 + t^2Y^4, \qquad F_2(t) = X^3Y + tXY^3,$$

and let D_t be the span of $D_1(t) = X^3 + 3tXY^2$ and $D_2(t) = 3X^2Y + tY^3$. These four forms are annihilated by $q_t = y^2 - tx^2$, and

$$x \circ F_1(t) = 4D_1(t), \quad y \circ F_1(t) = 4tD_2(t),$$
$$x \circ F_2(t) = D_2(t), \quad y \circ F_2(t) = D_1(t),$$

while their derivatives by z vanish. Since the first derivatives of G_1 and G_2 lie in $T_{\leq 2}$, the space

$$M_t = T_{\leq 2} \oplus D_t \oplus \operatorname{span}\left(F_1(t) + G_1,\ F_2(t) + G_2\right)$$

is a subspace of dimension fourteen closed under differentiation, with $M_0 = M$. By Lemma 3.5, it is the inverse system of an ideal I_t with Hilbert function $(1, 3, 6, 2, 2)$. We extend the pairing $\mathbb{C}[t]$-bilinearly to $S[t] \times T[t]$, and let $m_1(t), \ldots, m_{14}(t) \in T[t]$ be the fourteen polynomials spanning M_t above. The fourteen monomials $1, x, y, z$, the six quadratic monomials, x^3, x^2y, x^4 and x^3y pair with $m_1(t), \ldots, m_{14}(t)$ by a block triangular matrix. Its diagonal blocks have determinants 8, 36 and 144, independently of t, and the cubics G_i enter only above the diagonal. The derivatives of each $m_j(t)$ are $\mathbb{C}[t]$-linear combinations of $m_1(t), \ldots, m_{14}(t)$. Hence

$$\mathcal{I} = \{g \in S[t] : \langle g, m_j(t)\rangle = 0 \text{ for } j = 1, \ldots, 14\}$$

is an ideal such that $S[t]/\mathcal{I}$ is free with the classes of these monomials as a basis, and $\mathcal{I}|_{t=t_0} = I_{t_0}$ for every t_0. Thus the ideals I_t form a flat family over $\mathbb{A}^1$, as in Section 2.2. For $t \neq 0$ the quadric $q_t = (y - \sqrt{t}\,x)(y + \sqrt{t}\,x)$ has independent factors, so S/I_t is smoothable by the first case. By Proposition 2.4(1), $A = S/I_0$ is smoothable. □

8.8. We now prove the theorem for fourteen points.

Theorem 8.9. *Every finite subscheme of $\mathbb{A}^3$ of length fourteen is smoothable.*

Proof. By Theorem 7.5 and Proposition 2.7, it suffices to show that every local algebra $A = S/I$ of length fourteen with $I \subseteq \mathfrak{m}^2$ is smoothable. Its Hilbert function starts with $(1, 3)$, has no internal zeros, and satisfies $\sum_{k\geq 2} H_A(k) = 10$. By Theorem 7.5 and Corollary 5.4, A is smoothable as soon as I has a ray.

If $H_A(2) \leq 4$, Theorem 1.2 applies.

If $H_A(2) = 5$, then $\sum_{k\geq 3} H_A(k) = 5$. If $H_A(3) \leq 2$, Theorem 1.2 applies, and if $H_A(3) = 5$, Lemma 7.2 applies. Otherwise the Hilbert function is $(1, 3, 5, 4, 1)$, $(1, 3, 5, 3, 2)$ or $(1, 3, 5, 3, 1, 1)$. If the quadric on the tangent cone has rank at least two, Theorem 1.2 applies, so we may assume that it is x^2. For $(1, 3, 5, 3, 2)$, Corollary 4.12 applies. For the other two, if one of the transverse cubics of the elements of I with leading form x^2 is not a cube, Proposition 4.5(b) applies; for $(1, 3, 5, 4, 1)$ we have $\mathfrak{m}^5 \subseteq I$, and if one of them is a nonzero cube, Proposition 4.5(c) applies. In these cases A is smoothable by Corollary 4.3. What remains are the algebras with Hilbert function $(1, 3, 5, 4, 1)$ whose transverse cubics are all zero, and those with Hilbert function $(1, 3, 5, 3, 1, 1)$ whose transverse cubics are all cubes. They have rays by Lemma 8.3.

If $H_A(2) = 6$, then $I \subseteq \mathfrak{m}^3$ and $\sum_{k\geq 3} H_A(k) = 4$, so by Lemma 3.2(3) the Hilbert function is $(1, 3, 6, 4)$, $(1, 3, 6, 3, 1)$, $(1, 3, 6, 2, 2)$, $(1, 3, 6, 2, 1, 1)$ or $(1, 3, 6, 1, 1, 1, 1)$. In the first case $I = V + \mathfrak{m}^4$ for a six-dimensional subspace $V \subseteq S_3$ by Lemma 6.4, so A is very compressed and smoothable by [6, Proposition 1.2] (Remark 6.8). In the second, fourth and fifth cases I has a ray, by Lemmas 8.5, 8.2 and 6.2. In the third case A is smoothable by Lemma 8.7 and Theorem 7.5. □

Theorems 6.7, 7.5 and 8.9 together prove Theorem 1.1.

Remark 8.10. By Macaulay's theorem [1, Theorem 4.2.10], the results above, together with Theorem 8.9 and Corollary 5.4, cover all local algebras of length fifteen and embedding dimension three, except possibly some of those with Hilbert functions

$$(1, 3, 5, 5, 1), \quad (1, 3, 5, 4, 2), \quad (1, 3, 5, 4, 1, 1), \quad (1, 3, 5, 3, 2, 1),$$
$$(1, 3, 6, 4, 1), \quad (1, 3, 6, 3, 2), \quad (1, 3, 6, 2, 2, 1).$$

For example, for $(1, 3, 5, 3, 1, 1, 1)$, Theorem 1.2 and Proposition 4.5(b) leave the algebras whose transverse cubics are all cubes, and they have rays by Lemma 8.3. Similarly, $(1, 3, 5, 3, 3)$ is covered as $(1, 3, 5, 3, 2)$ in the proof of Theorem 8.9, while $(1, 3, 6, 2, 3)$ violates Macaulay's bound. The seven remaining Hilbert functions are treated in [20].

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by Claude agents, is available at `https://www.math.uni-konstanz.de/~michalek/lean.html`.

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