

An Expansion on Paraconsistent Micro-Geometry: A Non-Standard Approach to Subgroup-Stratified Vitali Measures via the Logic of da Costa

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Abstract

The existence of so-called pathological sets with respect to certain contexts of measure theory has often demanded a treatment that better satisfies our intuition or provides greater insight into their nature. The non-measurable sets of Vitali, in particular, seem to exhibit the intuitive properties of infinitesimals, and indeed, to some extent this can be formalized. While the standard Loeb measure yields real-valued extensions, the underlying hyperfinite constructions naturally suggest that Vitali sets possess infinitesimal inner and outer measures. However, a fully explicit bottom-up construction of these infinitesimal assignments has remained elusive, making it difficult to systematically distinguish between different Vitali sets—a gap that this paper seeks to rectify. Thus, given the top-down guaranteed existence of a non-standard framework for Vitali sets, we provide a formal construction from the bottom up that allows for a stratification of measures reflected as a stratification of functions and operators. It is hoped that this stratified construction will find diverse applications from capturing micro-geometric efficiency to ergodic theory and dynamical systems, and be generally seen as a pleasing addition to the literature.

1 Introduction

The Lebesgue measure is perhaps the most successful means of measuring the size of sets of real numbers known; indeed, under the appropriate axioms (such as the Axiom of Dependent Choice combined with set-theoretic determinacy), all subsets of the reals are Lebesgue measurable. However, under the standard Axiom of Choice (AC) within Zermelo-Fraenkel set theory (ZF), it is provable that non-measurable sets must exist.

The classic construction of a non-measurable set is the Vitali set. This is a choice set formed on the equivalence classes of the reals under an equivalence

relation governed by a dense subgroup of the additive group of reals $(\mathbb{R}, +)$. One such relation is given by $x \sim y$ if and only if $x - y \in \mathbb{Q}$ for $x, y \in [0, 1]$. The resulting Vitali set is non-measurable because it fundamentally breaks two pillars of the Lebesgue measure: σ -additivity and translation invariance.

To see how this happens, consider a Vitali set V and a countable partition of the unit interval formed by its rational translations $V + q_i$, where $q_i \in \mathbb{Q} \cap [0, 1]$. If V were Lebesgue measurable, translation invariance would require every $V + q_i$ to have the exact same measure. If that measure were zero, a countable sum would yield zero, failing to cover $[0, 1]$. If it were strictly positive, the countable σ -additivity sum would diverge to infinity, violating the measure of the unit interval (1). In this precise fashion, the measure of V is forced into an irresolvable contradiction.

While many types of non-measurable sets exist, Vitali sets remain uniquely intuitive. Indeed, it does not take much imagination to consider assigning them an infinitesimal measure within an appropriate structural setting. Although there are now several approaches to working with infinitesimals in the literature, we shall stick to that of Robinson and Luxemburg. Thus we begin by defining our subgroup-stratified internal outer and inner measures in this setting, confirming that their properties align with our core aims. In particular, we show that Vitali sets can be assigned a paraconsistent infinitesimal measure, while the standard parts of our new measures successfully recover the classical Lebesgue measure on standard measurable sets.

2 Definition of Measures

2.1 Defining the Internal Base Measure

Let $\mathbb{H}_\alpha \subset \mathbb{R}$ be a countable dense subgroup with target interval length b . We define a sequence of standard approximating selector sets V_n that asymptote to a Vitali set V derived from the subgroup $\mathbb{H}_\alpha$. Since $\mathbb{H}_\alpha$ is a countable dense subgroup of $\mathbb{R}$, we exhaust it via an increasing sequence of finite subsets:

$$\mathbb{H}_{\alpha,1} \subset \mathbb{H}_{\alpha,2} \subset \cdots \subset \mathbb{H}_\alpha \quad \text{with } 0 \in \mathbb{H}_{\alpha,n}$$

Now within our target base interval $[0, b]$, we define an equivalence relation $x \sim_n y$ if and only if their difference belongs to the finite subset or its generated grid:

$$x \sim_n y \iff x - y \in \mathbb{H}_{\alpha,n}$$

This partitions $[0, b]$ (up to boundary conditions) into a finite collection of coset slices or congruence classes:

$$\mathcal{P}_n = [0, b] / \sim_n$$

Let $N_{\alpha,n} = |\mathcal{P}_n|$ denote the total number of these partition components.

Using a deterministic selection rule (or the Axiom of Choice applied finitely), we choose a single canonical representative point s_k from each equivalence class

in $\mathcal{P}_n$. This yields a finite set of discrete points:

$$S_n = \{s_1, s_2, \dots, s_{N_{\alpha,n}}\} \subset [0, b]$$

Because a raw set of discrete points S_n has Lebesgue measure zero and cannot interact smoothly via interval-based shift conditions, we thicken each point into a small, regular proxy interval of width ϵ_n (where $\epsilon_n \rightarrow 0$ as $n \rightarrow \infty$). We define the set-theoretic construction of V_n as the finite union:

$$V_n = \bigcup_{s_k \in S_n} \left[s_k - \frac{\epsilon_n}{2}, s_k + \frac{\epsilon_n}{2} \right] \cap [0, b]$$

We now define the sequence of component-wise standard shift sets $(H_{\alpha,b}^{(n)})_{n \in \mathbb{N}}$, where each term collects the active translation parameters that map the standard approximating selector set V_n into the interval $[0, b]$ at stage n :

$$H_{\alpha,b}^{(n)} = \{h \in \mathbb{H}_{\alpha,n} \mid 0 \leq h \leq b \text{ and } (V_n + h) \cap [0, b] \neq \emptyset\} \quad (1)$$

Lifting this through the non-standard universe via a free ultrafilter $\mathcal{U}$, we define the internal shift set $H_{\alpha,b}^*$ and its corresponding internal cardinality N_α :

$$H_{\alpha,b}^* = \{h \in {}^*\mathbb{H}_\alpha \mid 0 \leq h \leq b \text{ and } ({}^*V + h) \cap [0, b]^* \neq \emptyset\} \quad (2)$$

$$N_\alpha = |H_{\alpha,b}^*| = [(|H_{\alpha,b}^{(n)}|)_{n \in \mathbb{N}}]_{\mathcal{U}} \quad (3)$$

With this hyperfinite partitioning framework established, we can now define the internal base measure for arbitrary measurable sets via stage-wise approximation. Let $E \subseteq [0, b]$ be an arbitrary Lebesgue measurable set, and recall that our sequence of component-wise standard shift sets $(H_{\alpha,b}^{(n)})_{n \in \mathbb{N}}$ generates a stage- n uniform partition of $[0, b]$ into grid intervals $I_{h,n}$ of width $\Delta x_n = \frac{b}{|H_{\alpha,b}^{(n)}|}$, indexed by $h \in H_{\alpha,b}^{(n)}$.

At each finite stage n , let λ_n denote the standard Lebesgue measure restricted to this uniform grid partition. We define the elementary approximating set E_n via grid-snapping:

$$E_n = \bigcup_{h \in J_{E_n}} I_{h,n}, \quad \text{where } J_{E_n} = \{h \in H_{\alpha,b}^{(n)} \mid I_{h,n} \cap E \neq \emptyset\}$$

with $\lim_{n \rightarrow \infty} \lambda(E_n \Delta E) = 0$. In the non-standard universe, the infinitesimal quantum $\ell_{\alpha,b}$ and the lifted index set J_E are defined via the ultrafilter $\mathcal{U}$ as:

$$\ell_{\alpha,b} = \left[\left(\frac{b}{|H_{\alpha,b}^{(n)}|} \right)_{n \in \mathbb{N}} \right]_{\mathcal{U}}, \quad J_E = [(J_{E_n})_{n \in \mathbb{N}}]_{\mathcal{U}}$$

Theorem 1. *Let $E \subseteq [0, b]$ be a Lebesgue measurable set, and let $(E_n)_{n \in \mathbb{N}}$ be a sequence of elementary grid-approximating sets such that $\lim_{n \rightarrow \infty} \lambda(E_n \Delta E) =$*

0. *Defining the internal base measure $\mu_\alpha(E)$ as the ultraproduct lift of the stage-wise Lebesgue measures:*

$$\mu_\alpha(E) := [(\lambda_n(E_n))_{n \in \mathbb{N}}]_{\mathcal{U}}$$

Then $\mu_\alpha(E)$ factorizes into the product of the lifted internal cardinality and the infinitesimal quantum:

$$\mu_\alpha(E) = |J_E| \cdot \ell_{\alpha,b}$$

Furthermore, the standard part of this hyperfinite measure recovers the classical Lebesgue measure:

$$\text{st}(\mu_\alpha(E)) = \lambda(E)$$

Proof of Theorem 1:

1. **Stage-Wise Finitary Identity:** For each finite stage n , because E_n is a finite union of uniform grid intervals $I_{h,n}$ of width $\frac{b}{|H_{\alpha,b}^{(n)}|}$ and indexed by J_{E_n} , the stage-wise Lebesgue measure satisfies the exact algebraic identity:

$$\lambda_n(E_n) = |J_{E_n}| \cdot \lambda_n(I_{h,n}) = |J_{E_n}| \cdot \frac{b}{|H_{\alpha,b}^{(n)}|}$$

2. **Ultraproduct Lift via the Transfer Principle:** Applying the Transfer Principle across the sequence of stage-wise identities over the ultrafilter $\mathcal{U}$ yields:

$$[(\lambda_n(E_n))_{n \in \mathbb{N}}]_{\mathcal{U}} = \left[\left(|J_{E_n}| \cdot \frac{b}{|H_{\alpha,b}^{(n)}|} \right)_{n \in \mathbb{N}} \right]_{\mathcal{U}}$$

3. **Algebraic Factorization:** Because the ultraproduct preserves algebraic operations element-wise, the right-hand side factorizes into the product of the lifted index cardinality and the infinitesimal mesh width:

$$[(|J_{E_n}|)_{n \in \mathbb{N}}]_{\mathcal{U}} \cdot \left[\left(\frac{b}{|H_{\alpha,b}^{(n)}|} \right)_{n \in \mathbb{N}} \right]_{\mathcal{U}} = |J_E| \cdot \ell_{\alpha,b}$$

Substituting our definition of $\mu_\alpha(E)$ establishes the internal base measure formula:

$$\mu_\alpha(E) = |J_E| \cdot \ell_{\alpha,b}$$

4. **Standard Part Recovery:** Because the symmetric difference vanishes in the limit ($\lim_{n \rightarrow \infty} \lambda(E_n \Delta E) = 0$), the stage-wise measures converge to the classical Lebesgue measure: $\lim_{n \rightarrow \infty} \lambda_n(E_n) = \lambda(E)$. Taking the standard part of the hyperfinite ultraproduct lift consequently yields:

$$\text{st}(\mu_\alpha(E)) = \text{st}([(\lambda_n(E_n))_{n \in \mathbb{N}}]_{\mathcal{U}}) = \lim_{n \rightarrow \infty} \lambda_n(E_n) = \lambda(E) \quad \blacksquare$$

Corollary 1. *The base measure is a non-standard measure on the algebra of internal sets.*

2.2 Defining the Hyperfinite Outer Measure

For any internal set $E \subseteq^* [0, b]$, the subgroup-stratified hyperfinite outer measure is defined as:

$$\mu_\alpha^*(E) = \left[\left(\inf \left\{ \sum_{h \in H_{\alpha,b}^{(n)}} c_{h,n} \left(\frac{b}{|H_{\alpha,b}^{(n)}|} \right) \mid E_n \subseteq \bigcup_{h \in H_{\alpha,b}^{(n)}} I_{h,n} \right\} \right) \right]_{n \in \mathbb{N}} \mathcal{U} \quad (4)$$

where the function

$$c_{h,n} : H_{\alpha,b}^{(n)} \rightarrow \{0, 1\}$$

is defined by

$$c_{h,n} = \begin{cases} 1 & \text{if the grid interval } I_{h,n} \text{ is included in the covering set } E_n \\ 0 & \text{if the grid interval } I_{h,n} \text{ is excluded} \end{cases}$$

for each interval index $h \in H_{\alpha,b}^{(n)}$.

Lemma 1. *The hyperfinite outer measure can also be evaluated as:*

$$\mu_\alpha^*(E) = \inf^* \left\{ \sum_{h \in H_{\alpha,b,j}} \text{len}(I_h) \mid E \subseteq \bigcup_{h \in H_{\alpha,b,j}} I_h \right\} \quad (5)$$

Proof of Lemma 1:

1. **Formulating the Stage-Wise Infimum as a Sequence:** For each finite stage $n \in \mathbb{N}$, let m_n denote the real-valued infimum of the finitary covering set:

$$m_n = \inf \left\{ \sum_{h \in H_{\alpha,b}^{(n)}} c_{h,n} \left(\frac{b}{|H_{\alpha,b}^{(n)}|} \right) \mid E_n \subseteq \bigcup_{h \in H_{\alpha,b}^{(n)}} I_{h,n} \right\}$$

Thus, Definition 1 is simply the equivalence class of this real sequence under the ultrafilter $\mathcal{U}$:

$$\mu_\alpha^*(E) = [(m_n)_{n \in \mathbb{N}}]_{\mathcal{U}}$$

2. **Applying Łos's Theorem to the Infimum Operator:** In non-standard analysis, the ultrapower construction allows us to lift operations defined pointwise on sequences into the hyperreal universe ${}^*\mathbb{R}$. By Łos's Theorem, order-theoretic relations and infima commute with the ultrapower functor. That is, the ultrapower of a sequence of infima is identical to the internal infimum ($\inf^*$) of the star-extended set:

$$[(\inf S_n)_{n \in \mathbb{N}}]_{\mathcal{U}} = \inf^* ([S_n]_{n \in \mathbb{N}})_{\mathcal{U}}$$

where S_n is the set of sums being minimized at stage n , and its ultrapower $[S_n]_{\mathcal{U}}$ is the corresponding internal set in *V .

3. **Translating via the Transfer Principle:** Now, we look at what the ultraproduct of those sets and sums evaluates to natively inside *V via the Transfer Principle:

- *The Partition:* The sequence of finite partitions $(H_{\alpha,b}^{(n)})_{n \in \mathbb{N}}$ lifts directly to the hyperfinite partition $H_{\alpha,b}^*$.
- *The Sums and Lengths:* The stage-wise uniform term $\sum c_{h,n} \left(\frac{b}{|H_{\alpha,b}^{(n)}|} \right)$ transfers identically to the internal sum over the hyperfinite partition:

$$\sum_{h \in H_{\alpha,b,j}} \text{len}(I_h)$$

- *The Covering Constraint:* The stage-wise inclusion $E_n \subseteq \bigcup I_{h,n}$ lifts via transfer to the internal inclusion $E \subseteq \bigcup_{h \in H_{\alpha,b,j}} I_h$.
4. **Final Conclusion:** Substituting these transferred components back into the internal infimum from Step 2 yields:

$$\inf^*([(S_n)_{n \in \mathbb{N}}]_{\mathcal{U}}) = \inf^* \left\{ \sum_{h \in H_{\alpha,b,j}} \text{len}(I_h) \mid E \subseteq \bigcup_{h \in H_{\alpha,b,j}} I_h \right\}$$

This matches Definition 2 precisely. ■

Theorem 2. *Consistency with Properties of Outer Measure:*

1. **Non-Negativity:** $\mu_{\alpha}^*(E) \geq 0$
2. **Null Empty Set:** $\mu_{\alpha}^*(\emptyset) = 0$
3. **Monotonicity:** If $A \subseteq B$, then $\mu_{\alpha}^*(A) \leq \mu_{\alpha}^*(B)$
4. **Hyperfinite-Subadditivity:** $\mu_{\alpha}^* \left(\bigcup_{k \in H_{\alpha,b}^*} A_k \right) \leq \sum_{k \in H_{\alpha,b}^*} \mu_{\alpha}^*(A_k)$

Proof of Theorem 2:

1. **Non-negativity:** Non-negativity follows from the fact that the length $\text{len}(I_h)$ of any interval I_h is greater than or equal to zero, meaning the sum of any collection of such lengths must also be greater than or equal to zero. Furthermore, if $\mathcal{S}$ is the set of all such valid covering sums, zero is a lower bound for every element in $\mathcal{S}$. Consequently, $\inf(\mathcal{S})$ (the greatest lower bound) must be greater than or equal to zero, which implies that $\mu_{\alpha}^*(E) \geq 0$.
2. **Null Measure of the Empty Set:** This property follows from the fact that we can choose a valid covering of $\emptyset$ consisting of a degenerate interval of length 0 (or the empty collection). Because this covering yields a sum of 0, the set of all valid covering sums contains 0. Furthermore, by non-negativity, all covering sums are greater than or equal to 0, meaning 0 is a lower bound. Since 0 is both a lower bound and is achievable by a valid cover, the greatest lower bound ($\inf$) must be 0, giving $\mu_{\alpha}^*(\emptyset) = 0$.

3. **Monotonicity:** Monotonicity follows from the inclusion relation of their respective covering sets. Because $A \subseteq B$, any valid collection of intervals covering B will also successfully cover A . Consequently, the set of valid covering sums for B (let's call it $\mathcal{S}_B$) forms a subset of the covering sums for A ($\mathcal{S}_A$), meaning $\mathcal{S}_B \subseteq \mathcal{S}_A$. Since taking the infimum over a larger superset ($\mathcal{S}_A$) yields a lower or equal bound compared to its subset ($\mathcal{S}_B$), we have $\inf(\mathcal{S}_A) \leq \inf(\mathcal{S}_B)$, and thus $\mu_\alpha^*(A) \leq \mu_\alpha^*(B)$ as required.

4. **Hyperfinite Subadditivity:** We prove this property in three main steps:

- **Step 1:** By the transfer principle, for each set A_k indexed by $H_{\alpha,b}^*$, there exists a covering family of intervals $\mathcal{C}_k$ such that the sum of their lengths is strictly bounded above by the outer measure plus a controlled infinitesimal slack δ_k :

$$\sum_{I \in \mathcal{C}_k} \text{len}(I) < \mu_\alpha^*(A_k) + \delta_k$$

- **Step 2:** For an arbitrary point $x \in E$, since $E = \bigcup_{k \in H_{\alpha,b}^*} A_k$, x must belong to at least one set A_k . Because $\mathcal{C}_k$ is a valid covering of A_k , x is contained in the point-set union of that family, meaning $x \in \bigcup \mathcal{C}_k$. Defining our master collection of intervals as $\mathcal{C} = \bigcup_{k \in H_{\alpha,b}^*} \mathcal{C}_k$, the total covered space is the union of all intervals in $\mathcal{C}$:

$$E \subseteq \bigcup_{k \in H_{\alpha,b}^*} \left(\bigcup \mathcal{C}_k \right) = \bigcup \mathcal{C}$$

- **Step 3:** Because $\mu_\alpha^*(E)$ is the infimum over all possible internal coverings, and $\mathcal{C}$ forms a valid cover for E , we have:

$$\mu_\alpha^* \left(\bigcup_{k \in H_{\alpha,b}^*} A_k \right) \leq \sum_{I \in \mathcal{C}} \text{len}(I) \leq \sum_{k \in H_{\alpha,b}^*} \left(\sum_{I \in \mathcal{C}_k} \text{len}(I) \right)$$

Conclusion: Substituting the δ_k -approximations from Step 1 into the chain gives:

$$\mu_\alpha^* \left(\bigcup_{k \in H_{\alpha,b}^*} A_k \right) < \sum_{k \in H_{\alpha,b}^*} (\mu_\alpha^*(A_k) + \delta_k) = \sum_{k \in H_{\alpha,b}^*} \mu_\alpha^*(A_k) + \sum_{k \in H_{\alpha,b}^*} \delta_k$$

Taking the infimum and accounting for the infinitesimal error bounds, the residual term is controlled, yielding the required subadditivity inequality:

$$\mu_\alpha^* \left(\bigcup_{k \in H_{\alpha,b}^*} A_k \right) \leq \sum_{k \in H_{\alpha,b}^*} \mu_\alpha^*(A_k) \quad \blacksquare$$

2.3 Defining the Hyperfinite Inner Measure

Having defined and established the properties of the internal outer measure, the inner measure is defined via internal hyperfinite packings:

$$\mu_*^\alpha(E) = \sup \left\{ \sum_{h \in K^*} \text{len}(I_h) \mid \bigcup_{h \in K^*} I_h \subseteq E \text{ where } K^* \subseteq H_{\alpha,b}^* \text{ and each } I_h \text{ is an internal interval} \right\} \quad (6)$$

Theorem 3. *Properties of the Inner Measure:* By virtue of being a supremum, the inner measure exhibits dual properties to the outer measure. In particular:

1. **Superadditivity for disjoint sets:**

$$\mu_*^\alpha(A \cup B) \geq \mu_*^\alpha(A) + \mu_*^\alpha(B)$$

2. **Bound against the outer measure:** For any given set E , the inner measure is always less than or equal to the outer measure:

$$\mu_*^\alpha(E) \leq \mu_\alpha^*(E)$$

Proof of Theorem 3:

1. **Superadditivity:** Let μ_*^α be evaluated via its internal packing representation. Consider a pair of disjoint sets A and B , so that $A \cap B = \emptyset$. Choosing arbitrary internal interval packings $S_A \subseteq A$ and $S_B \subseteq B$ (where $S_A = \bigcup_{h \in K_A^*} I_h$ and $S_B = \bigcup_{h \in K_B^*} I_h$ with disjoint index sets $K_A^* \cap K_B^* = \emptyset$), their union $S_A \cup S_B$ forms a valid internal packing contained in $A \cup B$. By the finite additivity of interval lengths over disjoint index sets, we have:

$$\text{len}(S_A \cup S_B) = \text{len}(S_A) + \text{len}(S_B)$$

Therefore, the sum of their lengths belongs to the set of evaluations defining the inner measure of the union:

$$\mu_*^\alpha(A \cup B) = \sup \{ \text{len}(S) \mid S \subseteq A \cup B \} \geq \text{len}(S_A) + \text{len}(S_B)$$

Taking the supremum first over all valid packings $S_A \subseteq A$, and subsequently over $S_B \subseteq B$, completes the derivation:

$$\mu_*^\alpha(A \cup B) \geq \mu_*^\alpha(A) + \mu_*^\alpha(B)$$

2. **Inner $\leq$ Outer Inequality:** Let S be any internal packing such that $S \subseteq E$, and let C be any valid internal covering such that $E \subseteq C$. Within this subgroup-stratified framework, the cover takes the form of a union of internal intervals indexed by the hyperfinite shift set:

$$C = \bigcup_{h \in H_{\alpha,b,j}} I_h$$

By transitivity of subset inclusion, since $S \subseteq E$ and $E \subseteq C$, it follows that $S \subseteq C$. Because both are constructed from internal intervals within the hyperfinite algebra, monotonicity applies:

$$\text{len}(S) \leq \sum_{h \in H_{\alpha, bj}} \text{len}(I_h)$$

Holding the cover C fixed and taking the supremum over all possible internal packings $S \subseteq E$ yields the inner measure on the left-hand side:

$$\mu_*^\alpha(E) \leq \sum_{h \in H_{\alpha, bj}} \text{len}(I_h)$$

Since this inequality holds for every valid internal cover C , we can take the infimum over all such covers on the right-hand side, which evaluates precisely to the outer measure:

$$\mu_*^\alpha(E) \leq \inf \left\{ \sum_{h \in H_{\alpha, bj}} \text{len}(I_h) \mid E \subseteq \bigcup_{h \in H_{\alpha, bj}} I_h \right\} = \mu_\alpha^*(E)$$

Thus, $\mu_*^\alpha(E) \leq \mu_\alpha^*(E)$ as required. ■

3 Paraconsistent Preliminaries: Why Da Costa-Style Paraconsistency Over Dialetheism

When formalizing the non-standard hyperfinite measure of pathological sets such as Vitali selectors, one encounters structural divergences between inner packing (μ_*^α) and outer covering (μ_α^*). A common temptation is to adopt a ****dialetheic**** framework (e.g., Graham Priest's Logic of Paradox), which posits that certain contradictions are semantically or ontologically ***true*** (i.e., truth-value gluts). Under a dialetheic view, asserting both $\mu(*V) = 0$ and $\mu(*V) = \ell_{\alpha, b}$ simultaneously would mean accepting that the set genuinely and truly possesses both measures in the object language.

For rigorous measure theory, however, dialetheism introduces severe foundational vulnerabilities:

1. **Loss of Discriminative Stratification:** Dialetheism treats the contradiction as a static feature of the entity, obliterating the vital distinction between well-behaved Lebesgue measurable sets ($\circ$ -consistent) and pathological micro-grid structures ($\neg\circ$ -inconsistent).
2. **Global Semantic Instability:** By accepting conflicting metrical values as simultaneously true without syntactic controls, dialetheism makes it difficult to maintain standard arithmetic operations and order relations within the hyperreal field $^*\mathbb{R}$.

Conversely, ****da Costa-style C -systems**** provide the precise algebraic and proof-theoretic architecture required for micro-geometry. Rather than claiming contradictions are ontologically true, da Costa logic serves as a consistency-adapting firewall. By employing the unary consistency connective ($\circ$) and its negation markup ($\neg\circ$), the framework formally isolates the local inconsistency of unmeasurable coset selectors.

Crucially, this allows us to:

- Maintain classical, bivalent mathematics everywhere that sets are well-behaved ($\circ[\mu_*^\alpha(*E) = \mu_\alpha^*(E)]$);
- Explicitly mark where local infinitesimal valuations collide with macroscopic topological constraints ($\neg\circ[\mu_*^\alpha(*V) = \mu_\alpha^*(V)]$);
- Restrict the rule of adjunction across this inconsistency boundary, preventing the Principle of Explosion (*ex falso quodlibet*) while safely permitting hyperfinite shift summation ($\sum \mu_\alpha^*(V + h) = b$).

Definition (Purely Core-Housed Curated Vitali Set). Let $\mathbb{H}_\alpha$ be a countable dense subgroup of $\mathbb{R}$, and let $\sim_{\mathbb{H}_\alpha}$ denote the equivalence relation on $\mathbb{R}$ defined by $x \sim_{\mathbb{H}_\alpha} y \iff x - y \in \mathbb{H}_\alpha$. Let $\mathcal{V}$ be the family of all classical Vitali sets formed by selecting a single representative from each equivalence class of $\mathbb{R}/\mathbb{H}_\alpha$ restricted to $[0, b]$. A Vitali set $V \in \mathcal{V}$ is termed **purely α -curated** if its non-standard extension (lift) $*V \in {}^*\mathcal{V}$ collapses entirely onto a fundamental micro-grid interval of infinitesimal length $\ell_{\alpha,b} = \frac{b}{N_\alpha}$:

$$*V = I_{\text{core}}$$

such that its internal inner and outer measures are symmetrically and exactly bounded by the base resolution of the micro-grid without auxiliary null components:

$$\mu_*^\alpha(*V) = \mu_\alpha^*(V) = \frac{b}{N_\alpha}$$

Theorem 4 (Existence of Purely Core-Housed Curated Selectors). Let $\mathbb{H}_\alpha$ be any countable dense subgroup of the additive group of reals $\mathbb{R}$. Within the paraconsistent non-standard framework $(\mathcal{L}, \neg\circ)$, the family $\mathcal{V}$ of classical Vitali sets derived from $\mathbb{R}/\mathbb{H}_\alpha$ restricted to $[0, b]$ admits a non-empty subfamily of purely core-housed α -curated selectors $\mathcal{V}_{\text{curated}} \subseteq \mathcal{V}$ whose subgroup-stratified internal lifts $*V \in {}^*\mathcal{V}_{\text{curated}}$ satisfy $*V = I_{\text{core}}$ and resolve into the valuation pair:

$$\text{Val}(\mu_*^\alpha(*V), \mu_\alpha^*(V)) = \text{Val}\left(\frac{b}{N_\alpha}, \frac{b}{N_\alpha}\right)$$

Proof:

1. **Coset Density in Infinitesimal Intervals:** Let $\pi : [0, b] \rightarrow [0, b]/\mathbb{H}_\alpha$ denote the canonical projection map. Because $\mathbb{H}_\alpha$ is a countable dense subgroup, every equivalence class (coset) $c + \mathbb{H}_\alpha$ is dense in $\mathbb{R}$. Consequently, any fundamental micro-grid interval I_{core} of infinitesimal length

$\ell_{\alpha,b} = \frac{b}{N_\alpha}$ within the non-standard extension $^*[0, b]$ intersects every single coset of $\mathbb{R}/\mathbb{H}_\alpha$.

2. **Targeted Internal Representative Selection:** By leveraging the Axiom of Choice internally within the star extension, we select precisely one representative from each coset such that all selected representatives reside entirely within I_{core} . Setting the internal lift equal to this core interval yields $^*V = I_{\text{core}}$, completely eliminating any external null-set union ($N^* = \emptyset$) and avoiding the atomic floor inflation trap.
3. **Non-Standard Saturation and Internal Lifting:** By $\aleph_1$ -saturation of the hyper-extension, the local micro-grid alignment is perfectly secured. The internal lift *V is identically equal to the core interval I_{core} , ensuring no measure leakage or boundary expansion.
4. **Evaluation of Valuations:**
 - The internal inner measure evaluates to the exact length of the internal interval: $\mu_*^\alpha(^*V) = \text{len}(I_{\text{core}}) = \frac{b}{N_\alpha}$.
 - The internal outer measure is identically bounded by the same interval resolution: $\mu_\alpha^*(^*V) = \text{len}(I_{\text{core}}) = \frac{b}{N_\alpha}$.
5. **Paraconsistent Resolution and Valuation Assignment:** In classical measure theory, assigning these symmetric bounds to an unmeasurable coset selector generates a direct contradiction with macroscopic topological spanning constraints ($\mathbf{Val}(0, b)$), which would trigger deductive explosion under the Transfer Principle. Because Theorem 4 operates within the paraconsistent logical architecture $(\mathcal{L}, \neg\circ)$, this contradiction is quarantined rather than explosive. The system assigns these bounds to the set-valued valuation mapping via the restricted rule of adjunction:

$$\mathbf{Val}(\mu_*^\alpha(^*V), \mu_\alpha^*(^*V)) = \mathbf{Val}\left(\frac{b}{N_\alpha}, \frac{b}{N_\alpha}\right)$$

Thus, under the paraconsistent extension, $\mathcal{V}_{\text{curated}}$ admits purely core-housed selectors for any arbitrary countable dense subgroup $\mathbb{H}_\alpha$, completing the proof. ■

4 Logical Foundations and Paraconsistent Inference Rules

To ensure that the dual-valued valuation $\mathbf{Val}(\mu_*^\alpha(^*V), \mu_\alpha^*(^*V)) = \mathbf{Val}\left(\frac{b}{N_\alpha}, \frac{b}{N_\alpha}\right)$ for purely core-housed curated Vitali sets does not trigger logical collapse, we formulate the underlying paraconsistent deduction rules using a da Costa-style C -system framework.

4.1 1. The Consistency Operator and Localized Strata

Let $\mathcal{L}$ be our paraconsistent logic equipped with a unary consistency connective $\circ$. For any formula A , the expression $\circ A$ denotes that A and its negation $\neg A$ do not generate a destructive contradiction within the given domain, allowing classical laws to govern it.

We stratify our measure space into two distinct logical zones:

1. **Macroscopic and Measurable Strata ($\circ$ -Well-Behaved):** For standard Lebesgue measurable sets $E \subseteq [0, b]$ and their internal lifts $*E$, the inner and outer measures coincide identically. We assign the consistency markup:

$$\circ[\mu_\alpha^*(^*E) = \mu_\alpha^\alpha(^*E)]$$

Consequently, classical rules of countable additivity, uniqueness, and complementation apply without restriction.

2. **Pathological Core-Housed Strata ($\neg\circ$ -Inconsistent):** For purely α -curated Vitali selectors $*V$, the structural gap between inner packing and outer covering introduces local inconsistency. We mark this boundary with the inconsistency assertion:

$$\neg \circ [\mu_\alpha^\alpha(^*V) = \mu_\alpha^*(^*V)]$$

Crucially, under da Costa's semantics, this local inconsistency blocks the Principle of Explosion (*ex falso quodlibet*), preventing trivialization of the broader theory.

4.2 2. Paraconsistent Valuation and Restricted Adjunction

When evaluating the curated Vitali set $*V$, our framework bypasses classical bivalent collapse by deploying a set-valued valuation mapping into a paraconsistent algebra $\mathcal{M}_{\text{para}}$:

$$\mathbf{Val}(\mu_\alpha^\alpha(^*V), \mu_\alpha^*(^*V)) = \mathbf{Val}\left(\frac{b}{N_\alpha}, \frac{b}{N_\alpha}\right)$$

In the object language, this corresponds to asserting two simultaneous bounds derived from independent operational procedures (internal packing vs. internal covering):

$$\begin{aligned} P_1 &:= \left[\mu(^*V) = \frac{b}{N_\alpha} \quad (\text{derived via inner packing}) \right] \\ P_2 &:= \left[\mu(^*V) = \frac{b}{N_\alpha} \quad (\text{derived via outer covering}) \right] \end{aligned}$$

Under standard classical logic, the co-existence of differing relational predicates on non-measurable domains forces $\perp$. Under our paraconsistent calculus $\mathcal{L}$,

the rule of ****adjunction****—which traditionally permits the derivation of arbitrary conjunctions—is restricted when operating across unmarked inconsistency boundaries:

$$\frac{P_1, P_2}{P_1 \wedge P_2} \quad (\text{Restricted under } \neg\circ)$$

Instead, these values are preserved as independent structural components within the product space **Val**, allowing the hyperfinite shift summation over the internal partition $H_{\alpha,b}^*$ to resolve cleanly:

$$\sum_{h \in H_{\alpha,b}^*} \mu_{\alpha}^*(^*V + h) = N_{\alpha} \cdot \frac{b}{N_{\alpha}} = b$$

Thus, the paraconsistent markup operator $\circ$ acts as an algebraic firewall, permitting the coexistence of micro-grid scale evaluations and macroscopic additivity without compromising deductive consistency.

4.3 3. Resolution of the Transfer Principle Apparent Paradox

A critical technical subtlety arises when comparing the hyperfinite valuation of the internal lift $\mu_{\alpha}^*(^*V) = \frac{b}{N_{\alpha}}$ with the macroscopic bounds one might naively associate with the standard set V (i.e., **Val**(0, b)). At first glance, assigning an infinitesimal measure $\frac{b}{N_{\alpha}}$ to *V while classical outer measure bounds would suggest a macroscopic span of b appears to violate the ****Transfer Principle**** (Łoś's Theorem), which mandates that internal statements reflect standard properties elementarily.

This apparent contradiction is resolved by recognizing that the internal lift *V is ****not**** an arbitrary internal set, but a ***subgroup-stratified hyperfinite set*** constrained by the dense index partition $H_{\alpha,b}^*$.

1. **Internal vs. External Quantization:** The Transfer Principle holds strictly for internal formulas within the non-standard universe. The classical non-measurability of V is an ***external*** property.
2. **The Markup Firewall ($\neg\circ$):** When mapping standard set-theoretic inclusions down to the micro-grid, the consistency markup $\neg\circ$ explicitly flags that the transfer mapping undergoes a ***stratified rescaling***. The internal measure does not violate transfer; rather, transfer accurately maps the ***finitized stage-wise approximation*** $\mathbb{H}_{\alpha,n}$ into the hyperfinite limit N_{α} , where the macroscopic span b is precisely partitioned into N_{α} infinitesimal quanta of width $\frac{b}{N_{\alpha}}$.

Thus, the valuation **Val** $\left(\frac{b}{N_{\alpha}}, \frac{b}{N_{\alpha}}\right)$ for *V is entirely consistent with the Transfer Principle because the hyperfinite sum over the internal shift set reconstructs the macroscopic length b (as shown in Example 4), preserving structural fidelity without triggering logical explosion.

5 Examples

Example 1. *Let E be a Lebesgue measurable set. Then:*

$$\mu_\alpha(E) = \mu_\alpha^*(E) = \mu_\alpha^\alpha(E) \quad \text{and} \quad \text{st}(\mu_\alpha(E)) = \lambda(E)$$

where λ is the classical Lebesgue measure.

Proof of Measure Coincidence:

1. **Trivial Inclusion Bounds:** Given an internal covering set $C \supseteq E$, we have by definition:

$$\sum_{h \in J_C} \text{len}(I_h) = |J_C| \cdot \frac{b}{N_\alpha} = \mu_\alpha(C)$$

Since E itself can serve as a valid internal cover ($E \subseteq E$), and the internal outer measure is the infimum over all such covers, it follows that $\mu_\alpha^*(E) \leq \mu_\alpha(E)$. Conversely, because E is also a valid internal subset ($E \subseteq E$) and the internal inner measure is the supremum over such packings, we have $\mu_\alpha^\alpha(E) \geq \mu_\alpha(E)$. Combining these with the universal property that $\mu_\alpha^\alpha(E) \leq \mu_\alpha^*(E)$ forces the entire chain to collapse:

$$\mu_\alpha(E) = \mu_\alpha^*(E) = \mu_\alpha^\alpha(E)$$

2. **ε -Regularity Approximation:** Now, given a Lebesgue measurable set E , by the regularity properties of the Lebesgue measure, for every standard real $\varepsilon > 0$ there exist elementary sets A_ε and B_ε (where A_ε is a union of closed intervals and B_ε is an open elementary set) such that:

$$A_\varepsilon \subseteq E \subseteq B_\varepsilon \quad \text{and} \quad \lambda(B_\varepsilon \setminus A_\varepsilon) < \varepsilon$$

Applying the Transfer Principle, we lift this property directly into the non-standard extension via the star transform, obtaining internal counterparts ${}^*A_\varepsilon$ and ${}^*B_\varepsilon$ in the hyperfinite universe such that:

$${}^*A_\varepsilon \subseteq {}^*E \subseteq {}^*B_\varepsilon$$

Furthermore, because these sets can be bracketed by internal measurable sets belonging to $\mathcal{I}$, we find internal sets $S_\varepsilon, C_\varepsilon \in \mathcal{I}$ such that:

$$S_\varepsilon \subseteq {}^*E \subseteq C_\varepsilon \quad \text{and} \quad \mu_\alpha(C_\varepsilon \setminus S_\varepsilon) < \varepsilon$$

Since $S_\varepsilon \in \mathcal{I}$ and $S_\varepsilon \subseteq {}^*E$, S_ε serves as a valid internal packing for *E , meaning the supremum defining the inner measure satisfies:

$$\mu_\alpha^\alpha({}^*E) \geq \mu_\alpha(S_\varepsilon)$$

Similarly, since $C_\varepsilon \in \mathcal{I}$ and ${}^*E \subseteq C_\varepsilon$, C_ε is a valid internal cover for *E , meaning the infimum defining the outer measure satisfies:

$$\mu_\alpha^*({}^*E) \leq \mu_\alpha(C_\varepsilon)$$

It follows that:

$$\mu_\alpha^*({}^*E) - \mu_\alpha^\alpha({}^*E) \leq \mu_\alpha(C_\varepsilon) - \mu_\alpha(S_\varepsilon) = \mu_\alpha(C_\varepsilon \setminus S_\varepsilon) < \varepsilon$$

Because this holds for every standard positive real ε , and the only non-negative hyperreal smaller than every standard positive real is 0, we must have:

$$\mu_\alpha^*({}^*E) = \mu_\alpha^\alpha({}^*E)$$

3. **Standard Part Recovery:** For the final part, noting that $S_\varepsilon \subseteq {}^*E \subseteq C_\varepsilon$, monotonicity gives:

$$\mu_\alpha(S_\varepsilon) \leq \mu_\alpha^*({}^*E) \leq \mu_\alpha(C_\varepsilon)$$

Because S_ε and C_ε are internal sets derived from the star transform of standard elementary sets A_ε and B_ε , their base measures under the standard part directly equate to classical Lebesgue measures:

$$\text{st}(\mu_\alpha(S_\varepsilon)) = \lambda(A_\varepsilon) \quad \text{and} \quad \text{st}(\mu_\alpha(C_\varepsilon)) = \lambda(B_\varepsilon)$$

Applying the standard part map preserves the order:

$$\lambda(A_\varepsilon) \leq \text{st}(\mu_\alpha^*({}^*E)) \leq \lambda(B_\varepsilon)$$

Taking the limit as $\varepsilon \rightarrow 0$ forces both bounds to converge to $\lambda(E)$, yielding:

$$\text{st}(\mu_\alpha^*({}^*E)) = \lambda(E) \quad \blacksquare$$

Example 2. Let X be a Lebesgue null set consisting of one or more points. Then its internal measure is bounded by or equal to the infinitesimal mesh width $\frac{b}{N_\alpha}$, and its standard part vanishes:

$$\text{st}(\mu_\alpha^*({}^*X)) = 0$$

Proof of Example 2:

1. **Case 1: Isolated Points on the Micro-Grid.** A Lebesgue null set X consisting of a single point (or a finite collection of points aligned with the micro-grid) occupies a single fundamental building block of the subgroup-stratified partition. Let $x_0 \in X$ be represented within the non-standard universe as an internal point falling within a minimal grid cell of the stratified partitioning of $[0, b]^*$.

The internal interval $[0, b]^*$ is distributed uniformly across the internal shift set $H_{\alpha, b}^*$ of cardinality N_α . This divides the macroscopic domain into N_α micro-intervals, each possessing the elementary infinitesimal length:

$$\ell_{\alpha, b} = \frac{b}{N_\alpha}$$

A null set X comprising one or more isolated points aligns with the resolution of the micro-grid, meaning it can be tightly bounded by—or identically occupies—a minimal internal covering element corresponding to a single shift index $h \in H_{\alpha, b}^*$. By the definition of the internal base measure, the measure assigned to this constituent cell is its length:

$$\mu_\alpha(X) = \text{len}(I_h) = \ell_{\alpha, b} = \frac{b}{N_\alpha}$$

Taking the standard part of this evaluation yields $\text{st}\left(\frac{b}{N_\alpha}\right) = 0$ because N_α is an infinite hypernatural number. This ensures that while the internal hyperfinite measure retains a non-zero infinitesimal footprint, its standard shadow correctly recovers classical nullity ($\lambda(X) = 0$).

2. **Case 2: General Lebesgue Null Sets.** Let $X \subseteq [0, b]$ be an arbitrary standard Lebesgue null set ($\lambda(X) = 0$). By the definition of Lebesgue null sets, for every standard real $\varepsilon > 0$, there exists an open elementary cover B_ε (a finite or countable union of open intervals, which can be approximated by internal elementary sets in $\mathcal{I}$) such that:

$$X \subseteq B_\varepsilon \quad \text{and} \quad \lambda(B_\varepsilon) < \varepsilon$$

Applying the Transfer Principle to the star extension, this containment lifts directly to the non-standard universe:

$${}^*X \subseteq {}^*B_\varepsilon$$

where ${}^*B_\varepsilon \in \mathcal{I}$ is an internal covering set. Because the base measure of ${}^*B_\varepsilon$ standard-parts to the classical Lebesgue measure, we have:

$$\text{st}(\mu_\alpha({}^*B_\varepsilon)) = \lambda(B_\varepsilon) < \varepsilon$$

By definition of the internal outer measure, *X is bounded above by the measure of any of its internal covers:

$$\mu_\alpha^*({}^*X) \leq \mu_\alpha({}^*B_\varepsilon)$$

Applying the standard part map to both sides preserves the inequality:

$$\text{st}(\mu_\alpha^*({}^*X)) \leq \text{st}(\mu_\alpha({}^*B_\varepsilon)) < \varepsilon$$

Because this upper bound holds for every standard positive real number $\varepsilon > 0$, the standard part of $\mu_\alpha^*({}^*X)$ must be less than every positive

standard real. In the hyperreal number system, the only non-negative number satisfying this condition is 0:

$$\text{st}(\mu_\alpha^*(^*X)) = 0$$

Consequently, while any non-empty internal set registers a strict hyperreal value on a grid of non-zero resolution step $\frac{b}{N_\alpha}$, its standard part vanishes identically, perfectly aligning with classical nullity ($\lambda(X) = 0$). ■

Example 3 (Hyperfinite Additivity of Curated Selectors). *Let $H_{\alpha,b}^*$ be the internal hyperfinite shift set of cardinality N_α corresponding to the countable dense subgroup $\mathbb{H}_\alpha$ restricted to $[0, b]$. For the purely α -curated core-housed Vitali set $^*V = I_{\text{core}}$, the hyperfinite outer measure of the disjoint union of its translates satisfies:*

$$\mu_\alpha^* \left(\bigsqcup_{h \in H_{\alpha,b}^*} (^*V + h) \right) = \sum_{h \in H_{\alpha,b}^*} \mu_\alpha^*(^*V + h) = b$$

Proof:

1. **Internal Additivity via the Transfer Principle:** Because $^*V = I_{\text{core}}$ is an internal hyperfinite set within the non-standard universe *M , and $H_{\alpha,b}^*$ is an internal index set, the collection $\{^*V + h \mid h \in H_{\alpha,b}^*\}$ forms an internal hyperfinite disjoint family. By Łoś's Theorem (The Transfer Principle), internal finite additivity extends directly to hyperfinite structures on internal sets:

$$\mu_\alpha^* \left(\bigsqcup_{h \in H_{\alpha,b}^*} (^*V + h) \right) = \sum_{h \in H_{\alpha,b}^*} \mu_\alpha^*(^*V + h)$$

2. **Paraconsistent Stratification and Inconsistency Markup:** For each translate $^*V + h$, the local evaluation yields the base micro-grid resolution: $\mu_\alpha^*(^*V + h) = \frac{b}{N_\alpha}$. Because these values originate from unmeasurable coset representatives whose micro-valuations collide with classical macroscopic topological bounds, the system assigns the inconsistency markup:

$$\neg \circ [\mu_\alpha^*(^*V + h)] = \mu_\alpha^*(^*V + h)$$

3. **Restricted Adjunction Across the Inconsistency Boundary:** In classical logic, conjoining these N_α independent internal valuations would invoke unrestricted adjunction, forcing a contradictory global combination that triggers the Principle of Explosion. Within our paraconsistent logic $(\mathcal{L}, \neg \circ)$, this inference is explicitly restricted:

$$\frac{P_{h_1}, P_{h_2}, \dots, P_{h_{N_\alpha}}}{\bigwedge_h P_h} \quad (\text{Restricted under } \neg \circ)$$

This algebraic firewall prevents deductive trivialization while safely preserving the components as independent internal scalar entries.

4. **Hyperfinite Arithmetic Evaluation:** Evaluating the sum of the uniform infinitesimal bounds across the internal cardinality N_α yields the exact macroscopic recovery:

$$\sum_{h \in H_{\alpha, bj}} \frac{b}{N_\alpha} = N_\alpha \cdot \frac{b}{N_\alpha} = b$$

Thus, the disjoint union of the curated core-housed Vitali translates resolves cleanly to b under the protection of the paraconsistent $\neg\circ$ framework, completing the proof. ■

6 Conclusion

We have demonstrated that the standard Lebesgue measure can be extended to a family of non-standard internal inner and outer measures that agree with it in the sense that the standard part of the value of the non-standard measure of a Lebesgue measurable set is its Lebesgue measure. Further, we have shown that these internal inner and outer measures can provide a paraconsistent evaluation for the “measure” of Vitali sets as taking both zero value and infinitesimal value: $\frac{b}{N_\alpha}$.

6.1 Methodology

The theoretical framework for this paper is built upon non-standard analysis, with a focus on measure theory on the hyperreal line. Our approach involves reconstructing the standard Lebesgue measure as a family of internal outer measures and inner measures based on the structure of shift sets in order to allow for a paraconsistent evaluation of Vitali sets, whilst maintaining a close adherence to the Lebesgue measure itself on Lebesgue measurable sets.

6.2 Acknowledgment of AI Assistance

The author has had a longstanding interest in the topic of evaluating Vitali sets spanning over two decades. For this research, large language models (LLMs) were extensively utilized to facilitate and accelerate the exploration of these concepts. Specifically, the following tasks were performed with the aid of AI:

- **Conceptual Exploration and Framework Development:** An AI assistant was used as a conceptual partner to explore the logical consistency of our proposed definitions and theoretical constructs.
- **Parameterization:** Developing the formal structure for the parameterized internal inner and outer measures μ_*^α and μ_α^* .
- **Logical Connections:** Formulating the logical connections between different theoretical objects, such as hyperfinite additivity.

- **Refinement:** Identifying flaws in logical reasoning (paraconsistent or otherwise) and polishing written text.

The intellectual work, including the generation of novel ideas, proofs, and the final interpretation of the results, remains the sole responsibility of the author. The AI's role was strictly that of an advanced tool for conceptual clarification and structural assistance.

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