An Unverified AI-Generated Proof Candidate for Ten-Vector Phase Retrieval in $\mathbb{C}^4$

AI-generated mathematical candidate
Human-curated release package
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Status: unverified AI-generated proof candidate. This document does not claim that the open problem has been solved. It has not undergone independent human mathematical validation or peer review. Checks of Lemma 4.1, the characteristic-class calculation, and prior art are welcome; no endorsement or coauthorship is requested.

Abstract

We record an unverified candidate obstruction to phase retrieval by ten fixed vector intensity measurements in complex dimension four. In the proposed argument, after Parseval normalization, a ten-vector phase-retrieving family would define a smooth embedding of $\mathbb{CP}^3$ into the affine hyperplane $H \cong \mathbb{R}^9$ of normalized measurement outcomes. The rank-three normal bundle ν of this embedding satisfies $p_1(\nu) = -4u^2$. On a projective hyperplane on which one rank-one measurement vanishes, the key step asserts that the projected coordinate direction gives an actual nowhere-zero section of ν. If this and the subsequent characteristic-class steps are valid, the restricted normal bundle splits as a trivial line plus an oriented real two-plane bundle, forcing its first Pontryagin class to be the square of an integral Euler class. On $\mathbb{CP}^2$ this would require an integer k with $k^2 = -4$, a contradiction. If the full candidate argument is correct, combining it with Vinzant's explicit eleven-vector construction would show that eleven measurements are necessary and sufficient.

1 Introduction

For a family $A = (a_j)_{j=1}^m \subset \mathbb{C}^d$, consider the intensity map

$$\mathcal{M}_A(x) = \big(|a_1^* x|^2, \dots, |a_m^* x|^2\big).$$

The family performs *complex phase retrieval* if $\mathcal{M}_A(x) = \mathcal{M}_A(y)$ implies $y = \lambda x$ for some $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. Let $m_{\mathbb{C}}(d)$ denote the least number of measurement vectors required in $\mathbb{C}^d$.

Heinosaari, Mazzarella, and Wolf showed that pure-state informational completeness leads to smooth embeddings of complex projective space and used general nonembedding results to derive lower bounds [4]. The candidate argument below uses additional rank-one coordinate structure: a coordinate-zero divisor is proposed to produce a distinguished section of the actual normal bundle. After discarding any zero measurements, Parseval normalization turns a phase-retrieving vector family into a rank-one pure-state informationally complete POVM. Finkelstein's lower bound of $3d-2$ outcomes for such POVMs therefore gives $m_{\mathbb{C}}(4) \geq 10$ [3, Theorem II]. Vinzant constructed an explicit injective eleven-vector family [7, Theorem 1], and Xu's 2017 review recorded $m_{\mathbb{C}}(4) \in [10, 11]$ and explicitly asked whether ten vectors can suffice [9, Table 1 and Open Question 2.6].

Recent work remains compatible with this interval. Xu's 2025 survey again notes that the minimum is unknown even in dimension four [10, Section 4.2.1]. Li proves that at $N = 4M - 5$ there is a nonempty open set of noninjective frames, a different statement that does not decide the ten-vector question [5, Theorem 1.2]. Wang and Shang likewise singled out dimension four as a possible equality case for Finkelstein's $3d - 2$ bound in the rank-one POVM formulation [8]. We do not claim that this short literature discussion establishes novelty or priority for the obstruction below.

The target claims of the candidate argument are the following.

Candidate Claim 1.1. *No family of ten measurement vectors in $\mathbb{C}^4$ performs complex phase retrieval. Zero vectors and repetitions are allowed.*

Candidate Corollary 1.2. *The minimum number of complex phase-retrieval measurements in dimension four is*

$$m_{\mathbb{C}}(4) = 11.$$

The proposed obstruction is not merely an obstruction to an arbitrary smooth embedding $\mathbb{CP}^3 \hookrightarrow \mathbb{R}^9$. It uses coordinate-zero divisors specific to normalized rank-one measurement maps.

2 The Hermitian kernel criterion

Let $\mathrm{Herm}(d)$ denote the real vector space of Hermitian $d \times d$ matrices. Associate to $A = (a_j)_{j=1}^m$ the real-linear measurement map

$$\mathcal{A}_A : \mathrm{Herm}(d) \longrightarrow \mathbb{R}^m, \qquad \mathcal{A}_A(Q)_j = a_j^* Q a_j.$$

We recall the standard kernel criterion [1, Lemma 9].

Lemma 2.1 (Hermitian kernel criterion)**.** *The family A performs complex phase retrieval if and only if* $\ker \mathcal{A}_A$ *contains no nonzero Hermitian matrix of rank at most two.*

Proof. If phase retrieval fails, there are x, y that are not related by a unit-modulus scalar and for which $Q = xx^* - yy^* \in \ker \mathcal{A}_A$. This matrix is Hermitian and has rank at most two. It is also nonzero. Indeed, if $xx^* = yy^*$, then either $x = y = 0$, or the two rank-one operators have the same one-dimensional range, so $y = cx$ for some $c \in \mathbb{C}^\times$. Taking traces gives $\|y\|^2 = \|x\|^2$, hence $|c| = 1$. Either case contradicts the assumed inequivalence of x and y.

Conversely, let $0 \neq Q \in \ker \mathcal{A}_A$ have rank at most two. If Q is indefinite, the spectral theorem gives orthonormal vectors u, v and positive numbers λ, μ such that

$$Q = \lambda uu^* - \mu vv^*.$$

Thus $Q = xx^* - yy^*$ for $x = \sqrt{\lambda}\, u$ and $y = \sqrt{\mu}\, v$. The kernel equations give equal intensities for x and y, while their orthogonality makes them inequivalent modulo phase.

It remains to treat the semidefinite case. Replacing Q by $-Q$ if necessary, assume $Q \geq 0$. For every j,

$$0 = a_j^* Q a_j = \|Q^{1/2} a_j\|^2,$$

so $a_j \in \ker Q = (\mathrm{range}\, Q)^\perp$. Choose $0 \neq z \in \mathrm{range}\, Q$. Then $\mathcal{M}_A(z) = 0 = \mathcal{M}_A(0)$, so phase retrieval fails. $\square$

3 Parseval normalization and the projective embedding

Lemma 3.1 (Parseval normalization)**.** *If a ten-vector family $A = (a_j)_{j=1}^{10} \subset \mathbb{C}^4$ performs phase retrieval, then there is a phase-retrieving family $B = (b_j)_{j=1}^{10}$ satisfying*

$$\sum_{j=1}^{10} b_j b_j^* = I.$$

Proof. A phase-retrieving family spans $\mathbb{C}^4$. Otherwise one could choose

$$0 \neq z \in \operatorname{span}\{a_1, \ldots, a_{10}\}^{\perp}.$$

Then $\mathcal{M}_A(z) = 0 = \mathcal{M}_A(0)$, contrary to phase retrieval. Hence its frame operator

$$S = \sum_{j=1}^{10} a_j a_j^*$$

is positive definite, since for every $z \neq 0$,

$$z^* S z = \sum_{j=1}^{10} |a_j^* z|^2 > 0.$$

Put $T = S^{-1/2}$ and $b_j = T a_j$. The operator T is invertible, complex-linear, and Hermitian. Therefore

$$\sum_j b_j b_j^* = \sum_j T a_j a_j^* T^* = TST = I,$$

and

$$b_j^* x = (T a_j)^* x = a_j^* T^* x = a_j^* T x, \qquad \mathcal{M}_B(x) = \mathcal{M}_A(Tx).$$

If $\mathcal{M}_B(x) = \mathcal{M}_B(y)$, phase retrieval for A gives $Ty = \lambda T x$ for some $|\lambda| = 1$. Since T is invertible and complex-linear, $y = \lambda x$. Thus B performs phase retrieval. The same calculation with T^{-1} shows that this change of variables preserves the phase-retrieval property in both directions; unitarity is not required. □

Assume toward a contradiction that $B = (b_j)_{j=1}^{10}$ is a phase-retrieving Parseval family in $\mathbb{C}^4$. Define the affine hyperplane and its translation space by

$$H = \left\{ y \in \mathbb{R}^{10} : \sum_{j=1}^{10} y_j = 1 \right\}, \qquad H_0 = \left\{ z \in \mathbb{R}^{10} : \sum_{j=1}^{10} z_j = 0 \right\}.$$

Thus H is an affine Euclidean space isomorphic to $\mathbb{R}^9$, and $T_y H = H_0$ at every $y \in H$.

There is no global smooth choice of unit representative on projective space. We therefore define the measurement coordinates by the homogeneous formula

$$f_j([x]) = \frac{|b_j^* x|^2}{\|x\|^2}, \qquad F = (f_1, \ldots, f_{10}) : \mathbb{CP}^3 \longrightarrow H. \tag{3.1}$$

For every $\alpha \in \mathbb{C}^{\times}$, the numerator and denominator in (3.1) are both multiplied by $|\alpha|^2$, so the formula is well-defined on projective classes. The Parseval identity gives $\sum_j f_j = 1$, and phase retrieval makes F injective. Indeed, if $F([x]) = F([y])$, choose unit representatives x and y. Their unnormalized intensity vectors then agree, so phase retrieval implies $y = \lambda x$ for some $|\lambda| = 1$ and hence $[x] = [y]$.

Lemma 3.2. *The map $F : \mathbb{CP}^3 \to H$ is a smooth embedding.*

Proof. On the standard affine chart $U_k = \{[x] : x_k \neq 0\}$, choose the representative $s_k(z)$ whose kth coordinate is one. There (3.1) has the form

$$f_j([s_k(z)]) = \frac{|b_j^* s_k(z)|^2}{\|s_k(z)\|^2}.$$

The numerator and denominator are smooth real functions of the real and imaginary parts of z, and the denominator is strictly positive. Hence each f_j, and therefore F, is smooth.

Fix a unit representative x. The tangent space of the unit sphere is

$$T_x S^7 = \{w \in \mathbb{C}^4 : \mathrm{Re}(x^* w) = 0\},$$

the vertical tangent to the phase orbit is $\mathbb{R}(ix)$, and the horizontal space $x^\perp = \{v : x^* v = 0\}$ is a real six-dimensional complement. Thus the quotient differential identifies the full projective tangent space with

$$T_{[x]}\mathbb{CP}^3 \cong x^\perp$$

as a real vector space.

For clarity, differentiate the homogeneous coordinate $\widetilde{f}_j(x) = |b_j^* x|^2/\|x\|^2$ on $\mathbb{C}^4 \setminus \{0\}$. At a unit x and in a real direction w the quotient rule gives

$$d\widetilde{f}_{j,x}(w) = 2\,\mathrm{Re}\Big((b_j^* x)^*(b_j^* w)\Big) - 2|b_j^* x|^2\,\mathrm{Re}(x^* w).$$

For $v \in x^\perp$ the second term vanishes, and a local unit lift therefore gives

$$(df_j)_{[x]}(v) = 2\,\mathrm{Re}\Big((b_j^* x)^*(b_j^* v)\Big) = b_j^*(xv^* + vx^*)b_j. \tag{3.2}$$

This expression is independent of the chosen unit representative: replacing (x, v) by $(e^{i\theta}x, e^{i\theta}v)$ leaves $xv^* + vx^*$ unchanged. If $v \neq 0$, the matrix $Q = xv^* + vx^*$ is nonzero and Hermitian. On the complex span of x and $v/\|v\|$, its matrix is

$$\begin{pmatrix} 0 & \|v\| \\ \|v\| & 0 \end{pmatrix},$$

and it vanishes on the orthogonal complement. Hence Q has rank exactly two and inertia $(1, 1)$. If $dF_{[x]}(v) = 0$, equation (3.2) puts Q in $\ker \mathcal{A}_B$, contradicting Lemma 2.1. Thus F is an immersion.

The domain $\mathbb{CP}^3$ is compact and H is Hausdorff, so the continuous injective map $F : \mathbb{CP}^3 \to F(\mathbb{CP}^3)$ is a homeomorphism. Together with the immersion property, this makes F a smooth embedding. □

4 The proposed normal-bundle obstruction

Equip $H_0 \subset \mathbb{R}^{10}$ with the induced Euclidean metric. Let ν be the metric normal bundle of the embedding *inside* H:

$$\nu_{[x]} = \big(dF_{[x]}(T_{[x]}\mathbb{CP}^3)\big)^\perp \subset H_0. \tag{4.1}$$

Because F is an immersion, $dF(T\mathbb{CP}^3)$ is a smooth rank-six subbundle of the trivial Euclidean bundle $\mathbb{CP}^3 \times H_0$. Its orthogonal complement is smooth: locally, a smooth frame of $dF(T\mathbb{CP}^3)$ and its invertible Gram matrix give a smooth orthogonal projection, whose kernel is the complement in (4.1). Since $\dim H_0 = 9$, this is an actual smooth real rank-three bundle. The normal bundle obtained by viewing the image in $\mathbb{R}^{10}$ would have one additional trivial direction and is not the bundle used below. The complex orientation of $\mathbb{CP}^3$ and a choice of orientation of H_0 orient ν, and

$$T_{\mathbb{R}}\mathbb{CP}^3 \oplus \nu \cong \varepsilon^9. \tag{4.2}$$

Here ε^r denotes the trivial real rank-r bundle.
Let $u = c_1(\mathcal{O}(1)) \in H^2(\mathbb{CP}^3;\mathbb{Z})$. The Euler sequence

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(1)^{\oplus 4} \longrightarrow T\mathbb{CP}^3 \longrightarrow 0$$

for the holomorphic tangent bundle $T\mathbb{CP}^3$ gives

$$c_1(T\mathbb{CP}^3) = 4u, \qquad c_2(T\mathbb{CP}^3) = 6u^2.$$

For the underlying real tangent bundle,

$$p_1(T_{\mathbb{R}}\mathbb{CP}^3) = c_1^2 - 2c_2 = 4u^2.$$

This identity is the $k = 1$ case of [6, Corollary 15.5]. With integral coefficients, the general Whitney product formula for total Pontryagin classes is multiplicative modulo elements of order two [6, Theorem 15.3]. Its degree-four part gives

$$2\Big(p_1(T_{\mathbb{R}}\mathbb{CP}^3 \oplus \nu) - p_1(T_{\mathbb{R}}\mathbb{CP}^3) - p_1(\nu)\Big) = 0.$$

The class in parentheses belongs to $H^4(\mathbb{CP}^3;\mathbb{Z}) \cong \mathbb{Z}u^2$, which has no 2-torsion. Therefore the degree-four equality is exact. Using (4.2) and the triviality of ε^9 gives

$$0 = p_1(\varepsilon^9) = p_1(T_{\mathbb{R}}\mathbb{CP}^3) + p_1(\nu).$$

Consequently,

$$p_1(\nu) = -4u^2. \tag{4.3}$$

We use standard characteristic-class conventions as in [6, Sections 9, 14, and 15].

Lemma 4.1 (Coordinate-divisor normal section)**.** *For every index j with $b_j \neq 0$, the restriction of ν to the projective hyperplane $D_j = \{[x] \in \mathbb{CP}^3 : b_j^* x = 0\} \cong \mathbb{CP}^2$ has a smooth nowhere-zero section.*

Proof. Fix an index j with $b_j \neq 0$. Then D_j is a smooth projective hyperplane. Let e_j be the jth standard basis vector of $\mathbb{R}^{10}$, let $\mathbf{1} = (1, \dots, 1)$, and put

$$c_j = e_j - \frac{1}{10}\mathbf{1} \in H_0. \tag{4.4}$$

It is nonzero; in fact, $\|c_j\|^2 = 9/10$. To check that $df_j = 0$ along D_j without choosing a projective representative, write on $\mathbb{C}^4 \setminus \{0\}$

$$N(x) = |b_j^* x|^2, \qquad R(x) = \|x\|^2, \qquad f_j = N/R.$$

For every real tangent direction v in $\mathbb{C}^4 \setminus \{0\}$,

$$dN_x(v) = 2\,\mathrm{Re}\Big((b_j^* x)^*(b_j^* v)\Big).$$

Thus, at a point lying over D_j, both N and dN vanish in every real direction, including directions transverse to the hyperplane. Hence the quotient rule gives

$$d(N/R) = \frac{R\,dN - N\,dR}{R^2} = 0. \tag{4.5}$$

If $\pi : \mathbb{C}^4 \setminus \{0\} \to \mathbb{CP}^3$ is the quotient map, then $N/R = f_j \circ \pi$. Since π is a submersion, equation (4.5) says exactly that $df_j = 0$ on D_j. Differentiating $\sum_k f_k = 1$ also gives $\sum_k df_k = 0$. Therefore, for every $[x] \in D_j$ and $\xi \in T_{[x]}\mathbb{CP}^3$,

$$\left\langle c_j, dF_{[x]}(\xi) \right\rangle = df_j(\xi) - \frac{1}{10}\sum_{k=1}^{10} df_k(\xi) = 0. \tag{4.6}$$

Equations (4.1), (4.4), and (4.6) show that

$$s_j : D_j \longrightarrow \nu|_{D_j}, \qquad s_j([x]) = ([x], c_j),$$

is a smooth section of the actual metric normal bundle. It is nowhere zero because $\|c_j\|^2 = 9/10$. $\square$

Candidate argument for Claim 1.1. Because B spans $\mathbb{C}^4$, choose an index j with $b_j \neq 0$, and let $i : D_j \hookrightarrow \mathbb{CP}^3$ be the inclusion. Normalize the section in Lemma 4.1. It spans an oriented trivial line subbundle, and its metric orthogonal complement gives an actual splitting

$$\nu|_{D_j} \cong \varepsilon^1 \oplus \eta, \tag{4.7}$$

where η is a real rank-two bundle. Orient η using the orientations of ν and the chosen trivial section.

An oriented metric real two-plane bundle is equivalently a complex line bundle L. Under this identification $e(\eta) = c_1(L)$ and consequently

$$p_1(\eta) = e(\eta)^2. \tag{4.8}$$

The last identity is the rank-two case of [6, Corollary 15.8]. Put $u_D = i^*u$. Because D_j is a linear projective hyperplane, $\mathcal{O}(1)|_{D_j}$ is the hyperplane line bundle on $D_j \cong \mathbb{CP}^2$. Thus u_D generates $H^2(D_j;\mathbb{Z}) \cong \mathbb{Z}$, so there is an integer k such that

$$e(\eta) = k u_D. \tag{4.9}$$

By (4.3), naturality of Pontryagin classes, the splitting (4.7), and (4.8),

$$k^2 u_D^2 = p_1(\eta) = p_1(\nu|_{D_j}) = i^* p_1(\nu) = -4u_D^2.$$

But $H^4(D_j;\mathbb{Z}) \cong \mathbb{Z}$ is generated by u_D^2 and is torsion-free. Thus $k^2 = -4$, which is impossible for an integer k. This contradiction completes the proposed argument for the claim. $\square$

Conditional implication for Corollary 1.2. Assume that Candidate Claim 1.1 is valid. If a phase-retrieving family with fewer than ten vectors existed, adjoining zero measurement vectors would produce a phase-retrieving family with exactly ten vectors, contrary to Candidate Claim 1.1. Hence $m_{\mathbb{C}}(4) \geq 11$. Vinzant's explicit eleven-vector construction [7, Theorem 1] gives $m_{\mathbb{C}}(4) \leq 11$, so the equality would follow. $\square$

5 Application to six-dimensional Hermitian subspaces

Write $\mathrm{Herm}_0(4)$ for the traceless Hermitian matrices, with the real Hilbert–Schmidt inner product. Causin's K-theoretic bound gives dimension at most six for the related problem of real subspaces whose nonzero Hermitian matrices have rank at least three [2, Theorem 3.5]. The following candidate consequence would exclude equality in that upper bound under an additional rank-one positive-semidefinite spanning hypothesis.

Candidate Corollary 5.1. *Let $K \subset \mathrm{Herm}_0(4)$ be a real six-dimensional subspace and put $W = K^\perp \subset \mathrm{Herm}(4)$, where the orthogonal complement is taken in all of $\mathrm{Herm}(4)$. If W is spanned over $\mathbb{R}$ by ten rank-one positive semidefinite operators $a_j a_j^*$, then K contains a nonzero Hermitian matrix of rank at most two.*

Proof. Since K is traceless, $I \in W$. This forces the vectors a_j to span $\mathbb{C}^4$: otherwise a common orthogonal nonzero vector would be annihilated by every element of W, including I. The kernel of the associated Hermitian measurement map is

$$\left(\mathrm{span}_{\mathbb{R}}\{a_j a_j^* : 1 \leq j \leq 10\}\right)^\perp = W^\perp = K.$$

If K contained no nonzero matrix of rank at most two, the kernel criterion would make $(a_j)_{j=1}^{10}$ phase retrieving, contradicting Candidate Claim 1.1. Thus this consequence is conditional on the candidate claim. □

Research-process disclosure

This mathematical candidate and much of its exposition arose in an AI-assisted research workflow. Generative AI systems made substantial contributions to literature exploration, proof development, drafting, auxiliary code generation, and adversarial checks; no computation is presented as evidence for the candidate's proof-critical normal-bundle step. Jihun Kim directed the workflow and curated this proof-candidate release for audit but does not claim to have independently derived or personally verified every mathematical step. No qualified human specialist has validated the complete argument. Reviewer comments would not imply authorship, endorsement, certification, or peer review. This work was conducted in a personal capacity and is not affiliated with, sponsored by, or endorsed by the curator's employer.

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