NINETEEN POINTS IN THREE-SPACE

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Abstract. We show that every finite subscheme of length at most nineteen in complex affine three-space is a limit of reduced subschemes. In particular, the Hilbert schemes of at most nineteen points in $\mathbb{A}^3$ are irreducible. The cubic equations of the tangent cone determine directions; along those directions we either split off a point, or measure the excess needed to link the scheme to a shorter one. Compared with the case of fourteen points [23], the new ingredient is linkage: we use inductively the known fact that linkage preserves smoothability, together with bounds for the excess intersection that are governed by the cubic base scheme and the separation degrees of its points.

1. Introduction

A finite subscheme of $\mathbb{A}^n$ is *smoothable* if it is a limit of reduced subschemes, that is, of collections of distinct points. The finite subschemes of length d are parametrized by the Hilbert scheme $\mathrm{Hilb}_d(\mathbb{A}^n)$, and the smoothable ones form an irreducible component of dimension nd. Thus $\mathrm{Hilb}_d(\mathbb{A}^n)$ is irreducible exactly when every finite subscheme of $\mathbb{A}^n$ of length d is smoothable.

On a smooth surface this is always the case, by a theorem of Fogarty [8]:

> *Every finite subscheme of a smooth surface is a limit of reduced subschemes.*

In higher dimensions it fails. Iarrobino showed [16, 17]:

> *If $n \geq 3$ and $d \geq 78$, then the Hilbert scheme of d points in $\mathbb{A}^n$ is reducible.*

When $n \geq 4$, reducibility already starts at $d = 8$ [18, 3]. When $n = 3$, the Hilbert scheme of d points is irreducible for $d \leq 7$ by Mazzola [22], for $d = 8$ by Cartwright, Erman, Velasco and Viray [3], for $d = 9$ and 10 by Henni and Jardim [14], building on work of Šivic [27], for $d = 11$ by Douvropoulos, Jelisiejew, Nødland and Teitler [7], and for $d = 12$, 13 and 14 by the first part of this work [23]. Thus the smallest d for which $\mathrm{Hilb}_d(\mathbb{A}^3)$ is reducible lies between 15 and 78. This is [19, Problem V], where the bounds $12 \leq d \leq 78$ are recorded. In this paper we continue the count.

Theorem 1.1. *Every finite subscheme of $\mathbb{A}^3$ of length at most nineteen is smoothable. In particular, the Hilbert schemes $\mathrm{Hilb}_d(\mathbb{A}^3)$ for $d = 15, \ldots, 19$ are irreducible, of dimension $3d$.*

Consequently, the smallest d for which $\mathrm{Hilb}_d(\mathbb{A}^3)$ is reducible lies between 20 and 78; the sequels [25, 26] treat lengths twenty and twenty-one, which

raises the lower bound to 22. The question is local, so Theorem 1.1 holds for any smooth irreducible quasi-projective threefold in place of $\mathbb{A}^3$, see [7, Section 1]. For lengths at most eighteen, Theorem 1.1 has been formalized in the proof assistant Lean 4 [24].

1.2. The proof is by induction on the length. Let $S = \mathbb{C}[x, y, z]$ and $\mathfrak{m} = (x, y, z)$. It suffices to consider a subscheme $Z = V(I)$ supported at the origin whose ideal I lies in $\mathfrak{m}^2$; otherwise Z is isomorphic to a subscheme of a smooth surface, or it is a disjoint union of shorter schemes. The local algebra $A = S/I$ is then organized by its Hilbert function $H_A = (1, 3, H_A(2), \dots)$, the Hilbert function of the tangent cone of Z. There are 52, 70, 91, 119 and 155 possibilities in lengths fifteen to nineteen. We smooth Z in three ways: along a surface, by linkage, and by splitting off a point. We glue them by limits: an irreducible family of finite schemes in which the smoothable ones are dense consists of smoothable schemes.

1.3. The first way is to find a surface through Z with a rational double point, in suitable formal coordinates. The Hilbert schemes of points on such a surface are irreducible by a theorem of Zheng [28], see also [6], so Z is smoothable. The difficult case is a tangent cone that lies on a single quadric, the square x^2 of a linear form. Then the ideal of Z contains an equation $x^2 + g(y, z)$ in suitable formal coordinates. Its *transverse cubic* is the cubic part of g. If the transverse cubic is not a cube, an equation in the ideal can be chosen to define a rational double point of type D; a term of higher order may be needed to isolate the singularity. For instance, $x^2 + y^2 z$ is singular along a line, but $x^2 + y^2 z + z^4$ is of type D_5, and both lie in the ideal $(x^2 + y^2 z) + \mathfrak{m}^4$. In Section 4 we *turn* the transverse cubic by a flat deformation of Z, until it is not a cube or until an equation of type E_6, E_7 or E_8 appears.

1.4. The second way is linkage. Let $Z' = V(f, g, h)$ be a finite complete intersection containing Z. The *residual* scheme $Z'' = V\big((f, g, h) : I\big)$ has length $\ell(Z') - \ell(Z)$, and Z is smoothable if and only if Z'' is. This is known; we deduce it from the results of Jelisiejew, Ramkumar and Sammartano on nested Hilbert schemes [20]. Hence the following principle.

Theorem 1.5. *Let $Z \subset \mathbb{A}^3$ be a finite subscheme of length d. Suppose that $Z \subseteq Z' \subseteq X$, where $X \subseteq \mathbb{A}^3$ is open and Z' is a finite local complete intersection in X (for instance, defined by three equations), of length less than $2d$. If every finite subscheme of $\mathbb{A}^3$ of length less than d is smoothable, then Z is smoothable.*

For example, the scheme of length fifteen of Example 3.9, with Hilbert function $(1, 3, 6, 4, 1)$, lies on three surfaces whose tangent cones are cubic cones without a common line. They cut out a complete intersection of length 27, and the residual scheme has length twelve.

Compared with [23], linkage is the new ingredient. That linkage preserves smoothability is known; what we add to [23] is its inductive use through Theorem 1.5, together with bounds for the length of the complete intersection that are governed by the cubic base scheme and the separation degrees of its points, which we now define.

1.6. Suppose that the tangent cone lies on no quadric. Its cubic equations define a subscheme $\mathfrak{B}$ of the plane $\mathbb{P}^2$ of directions, the *cubic base scheme*; it is the preimage of a linear section of the cubic Veronese surface (Section 3.5). If $\mathfrak{B}$ is empty and the length is at least 14, three surfaces through Z with cubic tangent cones cut out, near the origin, a complete intersection of length at most 27, and Z is linked to a shorter scheme (Theorem 3.4). If $\mathfrak{B}$ is finite, linkage can still work. If three surfaces through Z with cubic tangent cones meet at the origin in an isolated point, their intersection multiplicity there is 27 plus a correction, which comes from the lines through the origin in the directions of the points of $\mathfrak{B}$. Over a point p of $\mathfrak{B}$, suitable surfaces are separated by their terms of degree $k(p)$, the *separation degree*. It is the first degree in which the tangent cone has an equation not vanishing along the line of p, or the socle degree s of A, the largest k with $H_A(k) \neq 0$, if this is smaller. After a deformation of Z that keeps its Hilbert function, there are three such surfaces through the deformed scheme whose correction at each point p is at most $k(p) - 3$ times the multiplicity e_p of $\mathfrak{B}$ at p, the intersection number at p of two general cubics of the tangent cone. We prove this by rescaling the coordinates: two of the surfaces become a cone over nine points of the plane, counted with multiplicity, and the third meets the line of each of these points at the origin with multiplicity three, or $k(p)$ at a point p of $\mathfrak{B}$. This gives a numerical test.

Theorem 1.7 (Excess linkage)**.** *Let $A = S/I$ be a local algebra of length d with $H_A(2) = 6$ and socle degree at least four, and suppose that every finite subscheme of $\mathbb{A}^3$ of length less than d is smoothable. Suppose that the cubics on which the tangent cone of $V(I)$ lies span a space of dimension at least three and define a finite subscheme $\mathfrak{B}$ of $\mathbb{P}^2$. If*

$$27 + \sum_{p \in \mathfrak{B}} \big(k(p) - 3\big)\, e_p < 2d,$$

then A is smoothable.

This is Theorem 3.12. For example, let A be as in the theorem, and suppose that $\mathfrak{B}$ consists of three reduced points, at each of which some quartic of the tangent cone does not vanish. Then $k(p) = 4$ and $e_p = 1$ at each point, the left side is 30, and A is smoothable if $d \geq 16$. The theorem settles at once most algebras whose cubics and quartics are governed by two or three points of the plane, and all of them for ten Hilbert functions (Corollaries 3.14, 3.15 and D.4). For schemes of length nineteen there is often a cheaper way: three equations of degrees 3, 3 and 4 that form a global

complete intersection of length 36, by Bézout's theorem, so that no local multiplicity has to be computed (Proposition 3.18).

1.8. The third way is to split off a single point [23, Section 5]. A point leaves the origin along a curve $V(b, c)$, and a scheme of smaller length stays behind; the ideal (b, c) is then called a *ray*. In Section 5 the curves are the lines of the points of $\mathfrak{B}$, or curves tangent to them. A parabola or a curve of higher order is needed when lower terms of the equations bend the path, and we find these curves as jets of curves contained in Z. Often the directions come from a finite subscheme Γ of the plane. A sufficiently late plateau of the Hilbert function, $H_A(k) = H_A(k+1) = n$ with $n \leq k$, makes the equations of the tangent cone in degrees k and $k+1$ those of a scheme Γ of length n, by Gotzmann's persistence theorem (Lemma 2.16). For example, for the Hilbert function $(1, 3, 6, 3, 3)$ the scheme Γ has length three. When it consists of three points, a point of Z can leave along each of the three lines (Figure 2), and every Γ is a limit of three distinct points, as $\mathrm{Hilb}_3(\mathbb{P}^2)$ is irreducible [8].

1.9. This last step hides the main difficulty, the *tail-lifting problem*. The algebra A is encoded by its *inverse system* (Macaulay), a space of polynomials in dual variables that is closed under differentiation. The parts of highest degree of its elements, the *top forms*, describe the tangent cone, and their parts of lower degree are the *tails*. Deforming the scheme of directions deforms the top forms, but not automatically the algebra. Over colliding directions the tails may acquire extra freedom, so an algebra over a nonreduced Γ need not be a limit of algebras over reduced ones. We control the tails by parameter spaces of constant rank, by dimension counts, and, in specified cases, by exact saturation computations over $\mathbb{Q}$ (Section 5.15). Some families are handled instead by a Gorenstein member, or by a member linked to a shorter scheme. The most economical control is a count. Suppose that all inverse systems with a given Hilbert function, tails included, are cut out in an irreducible parameter space by fewer equations than the codimension of every degenerate stratum, and that the nondegenerate ones form an irreducible set. Then they form a single irreducible family (Lemma 2.18), and one member that is Gorenstein, or linked to a short scheme, suffices.

1.10. The algebras with $\mathfrak{m}^4 \subseteq I \subseteq \mathfrak{m}^3$ are the *very compressed* algebras of lengths ten to twenty. Douvropoulos, Jelisiejew, Nødland and Teitler proved that all very compressed algebras of length at most 95 are smoothable [7, Proposition 1.2]. They let a general configuration of points shrink to the origin, and they check by a computer calculation that the limits are dense [7, Section 5]. In Appendix G we give an explicit form of their argument for $\mathfrak{m}^4 \subseteq I \subseteq \mathfrak{m}^3$: one configuration of twenty integer points, and a formula for the differential that reduces the density to the rank of one small matrix for each length.

1.11. The methods divide the Hilbert functions as follows. Of the 155 Hilbert functions of length nineteen, 91 are settled by surfaces as in [23]; 27 more by the transverse cubics of Section 4; 22 by general results on linkage, rays and very compressed algebras; and 15 need an individual argument, which combines the same tools. Table 1 gives the counts for all lengths fifteen to nineteen, and Table 2 lists the individual cases.

Section 2 collects what we use from [23], from the theory of inverse systems and on finite subschemes of the plane, together with the open conditions that certify smoothability in families: being Gorenstein, and lying in a short complete intersection. Section 3 organizes the cubic directions, measures the excess of linkage, and constructs global complete intersections. Section 4 treats tangent cones on a single quadric. Section 5 constructs rays along lines and curves and controls the tails in limits, in collisions and by counting dimensions, and Section 6 proves Theorem 1.1. The appendices contain auxiliary lemmas on plane schemes and pencils, the proofs of several general results of Table 1, the remaining cases grouped by method, the explicit proof for very compressed algebras, and a list of the computer calculations.

2. Foundations

2.1. Throughout, we work over the complex numbers. We write $S = \mathbb{C}[x, y, z]$, $\mathfrak{m} = (x, y, z)$, S_k for the space of forms of degree k, and $S_{\leq k} = S_0 \oplus \cdots \oplus S_k$. A local algebra $A = S/I$ is always supported at the origin, that is, $V(I) = \{0\}$, and has finite length $\ell(A) = \dim_{\mathbb{C}} A$. We write $T = \mathbb{C}[X, Y, Z]$, with T_k, $T_{\leq k}$ and $T_{<N} = T_{\leq N-1}$ defined as for S; the ring S acts on T by differentiation (Section 2.4). We use the notation and results of the first part [23]. In particular, smoothability of a finite scheme depends only on its algebra [23, Proposition 2.4(3)], so it makes sense to call a finite $\mathbb{C}$-algebra smoothable. *Polynomial coordinates centered at the origin* are polynomials a, b, c with $S = \mathbb{C}[a, b, c]$ that vanish at the origin [23, Section 5.1].

Proposition 2.2 ([23, Propositions 2.4 and 2.7])**.** (1) *Let U be a variety with a morphism to $\mathrm{Hilb}_d(\mathbb{A}^3)$. If the subschemes parametrized by a dense subset of U are smoothable, then all subschemes parametrized by U are smoothable.*
(2) *A disjoint union of smoothable finite subschemes is smoothable.*
(3) *Suppose that every finite subscheme of $\mathbb{A}^3$ of length less than d is smoothable. If every local algebra S/I of length d with $I \subseteq \mathfrak{m}^2$ is smoothable, then every finite subscheme of $\mathbb{A}^3$ of length d is smoothable.*

We construct morphisms to the Hilbert scheme in two ways.

Lemma 2.3. (1) *Let $\mathcal{I} \subseteq S[t]$ be an ideal with $\mathfrak{m}^N S[t] \subseteq \mathcal{I}$ for some N, such that the ideal $\mathcal{I}|_{t=t_0}$ obtained by the substitution $t = t_0$ has colength d for every $t_0 \in \mathbb{C}$. Then $S[t]/\mathcal{I}$ is a free $\mathbb{C}[t]$-module of rank d, and $t_0 \mapsto \mathcal{I}|_{t=t_0}$ is a morphism $\mathbb{A}^1 \to \mathrm{Hilb}_d(\mathbb{A}^3)$.*

(2) *Let U be a variety, and let $m_1, \dots, m_d \colon U \to T_{<N}$ be morphisms such that, for every $u \in U$, the polynomials $m_1(u), \dots, m_d(u)$ are linearly independent and span a space M_u that is closed under differentiation. Then the ideals I_u with inverse system M_u (Section 2.4) form a morphism $U \to \mathrm{Hilb}_d(\mathbb{A}^3)$.*

Proof. (1) The module $S[t]/\mathcal{I}$ is generated over $\mathbb{C}[t]$ by the monomials of degree less than N, so it is the direct sum of a free module $\mathbb{C}[t]^r$ and of finitely many modules $\mathbb{C}[t]/(t-a)^e$. Its fiber $S/\mathcal{I}|_{t=t_0}$ at t_0 has dimension r plus the number of summands with $a = t_0$. As this number is constant, there are no torsion summands. (2) By Lemma 2.5(1) below, $I_u/\mathfrak{m}^N$ is the orthogonal complement of M_u in $S/\mathfrak{m}^N$. These complements form a subbundle of corank d of the trivial bundle with fiber $S/\mathfrak{m}^N$ over U, and the quotient bundle is a finite flat family of algebras over U, as in [23, Section 2.2]. □

2.4. The *leading form* of a nonzero polynomial is its homogeneous component of lowest degree, and the leading forms of the elements of I span a homogeneous ideal I^*. The *tangent cone* of $Z = V(I)$ is the subscheme of $\mathbb{A}^3$ defined by I^*, and it *lies on* a form q if $q \in I^*$. The Hilbert function of $A = S/I$ is $H_A(k) = \dim S_k/I^*_k$; its values add up to $\ell(A)$, and $H_A(k) = 0$ implies $\mathfrak{m}^k \subseteq I$ [23, Lemma 3.2]. By Macaulay's theorem [1, Theorem 4.2.10], $H_A(k+1) \le H_A(k)^{\langle k \rangle}$ for $k \ge 1$. Here, if $h = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \dots + \binom{a_j}{j}$ with $a_k > a_{k-1} > \dots > a_j \ge j \ge 1$, then $h^{\langle k \rangle} = \binom{a_k+1}{k+1} + \dots + \binom{a_j+1}{j+1}$ [1, Section 4.2]. We use that $h^{\langle k \rangle} = h$ for $h \le k$, and that $4^{\langle 3 \rangle} = 5$, $5^{\langle 3 \rangle} = 6$ and $5^{\langle 4 \rangle} = 6$. We write s for the largest k with $H_A(k) \neq 0$, the *socle degree* of A.

Let $T = \mathbb{C}[X, Y, Z]$, on which S acts by differentiation: x, y and z act as $\partial/\partial X$, $\partial/\partial Y$ and $\partial/\partial Z$, and we write $g \circ F$ for the action. The pairing $\langle g, F \rangle = (g \circ F)(0)$ is perfect between $S/\mathfrak{m}^N$ and $T_{<N}$. For a nonzero $P \in T_1$ with coefficient vector $p \in \mathbb{C}^3$ we have $g \circ P^k = k!\, g(p)$ for $g \in S_k$. The *inverse system* of I is $M = \{F \in T : g \circ F = 0 \text{ for all } g \in I\}$.

Lemma 2.5 (Macaulay; see [23, Lemma 3.5])**.** *Let $\mathfrak{m}^N \subseteq I$, $A = S/I$, and let M be the inverse system of I.*

(1) *$M \subseteq T_{<N}$ is closed under differentiation, $\dim M = \ell(A)$, and I is the orthogonal complement of M. Conversely, every subspace of $T_{<N}$ closed under differentiation is the inverse system of an ideal containing $\mathfrak{m}^N$.*
(2) *Let $D_k \subseteq T_k$ be the space of top forms of degree k of elements of M, together with 0; here the* top form *of a nonzero polynomial is its homogeneous component of highest degree. Then $\dim D_k = H_A(k)$, and D_k is the orthogonal complement of I^*_k in T_k.*

The second assertion of (2) is not stated in [23]; it follows from the first: if $g \in I^*_k$ is the leading form of an element of I and F is the top form of an element of M of degree k, then $\langle g, F \rangle = 0$, and the dimensions are complementary. We call D_k the space of *top forms of degree k*; the spaces

D_k form the inverse system of I^*, and we call them the *graded data* of A. In particular, for $P \neq 0$ in T_1,

$$P^k \in D_k \quad \Longleftrightarrow \quad \text{all forms of degree } k \text{ in } I^* \text{ vanish at } p. \tag{1}$$

A form $F \in T_k$ has r *essential variables* if its first derivatives span a space of dimension r; then F is a polynomial in r linear forms. Forms with one essential variable are powers L^k, and forms with three essential variables are called *concise*. For subspaces $U \subseteq S$ and $V \subseteq T$ we write $U \circ V$ for the span of the elements $g \circ F$ with $g \in U$ and $F \in V$, and for $v \in S$ and $D \subseteq T$ we write $v^{-1}(D) = \{F \in T : v \circ F \in D\}$.

A nonzero $P \in T_1$ determines the point $p = [P]$ of the plane $\mathbb{P}^2 = \mathbb{P}(T_1)$ of directions. For $g \in S$ we write $g(p)$ for the value of g at the coefficient vector of P in the basis X, Y, Z; it depends on the choice of P, but for homogeneous g its vanishing does not. When no confusion is possible we call P itself a point and write $P \in \Gamma$ for a subscheme $\Gamma \subset \mathbb{P}^2$. The *line of* P is the line through the origin of $\mathbb{A}^3$ with this direction. We use ordinary powers P^k throughout, so that the factorials appear in formulas such as $g \circ P^k = k!\, g(p)$; the form $P^k \in T_k$ is a point of the Veronese image of $\mathbb{P}^2$ in $\mathbb{P}(T_k)$. The linear space $\nu_k(\Gamma) \subseteq T_k$ of Section 2.13 is the scheme-theoretic linear span of the Veronese image of Γ; when Γ is reduced, it is spanned by the powers P^k of its points.

2.6. When $H_A(2) = 6$, the tangent cone lies on no quadric, so $I \subseteq \mathfrak{m}^3$, and M contains $T_{\leq 2}$ [23, Section 3.4]. Then every element of M is congruent modulo $T_{\leq 2}$ to an element of M without terms of degree at most two, and we will usually drop these terms. For instance, $M \cap T_{\leq 3} = T_{\leq 2} \oplus D_3$.

Lemma 2.7. *Let $A = S/I$ be a local algebra with $H_A(2) = 6$ and $\mathfrak{m}^5 \subseteq I$. Then there is a linear map $\tau\colon D_4 \to T_3$ such that*

$$M = T_{\leq 2} \oplus D_3 \oplus \{F + \tau(F) : F \in D_4\}.$$

Conversely, for subspaces $D_3 \subseteq T_3$ and $D_4 \subseteq T_4$ with $S_1 \circ D_4 \subseteq D_3$, and for every linear map $\tau\colon D_4 \to T_3$, the displayed space is the inverse system of a local algebra with Hilbert function $(1, 3, 6, \dim D_3, \dim D_4)$.

Proof. Since $M \subseteq T_{\leq 4}$ and $M \cap T_{\leq 3} = T_{\leq 2} \oplus D_3$, every $F \in D_4$ is the top form of an element $F + G + Q$ of M with $G \in T_3$ and $Q \in T_{\leq 2}$, unique up to D_3 and $T_{\leq 2}$. Subtracting Q and choosing G linearly in F gives τ. Conversely, $v \circ (F + \tau(F)) = v \circ F + v \circ \tau(F)$ lies in $D_3 + T_2$ for $v \in S_1$, so the space is closed under differentiation, and Lemma 2.5 applies. □

When $H_A(2) = 6$, an element of M with top form F and without terms of degree at most two is a *lift* of F, and its homogeneous parts below F are its *tails*. In the lemma, $F + \tau(F)$ is a lift of $F \in D_4$, and $\tau(F)$ is its tail. The local algebras with $H_A(2) = 6$ and $\mathfrak{m}^5 \subseteq I$ over fixed spaces D_3, D_4 are parametrized by the vector space $\operatorname{Hom}(D_4, T_3)$ (two maps give the same algebra if they agree modulo D_3), and as D_3 and D_4 vary in a family,

these parameter spaces form a vector bundle. We will use this together with Lemma 2.3(2) without further mention.

2.8. We recall the results of [23] on surfaces. Let I be an ideal of finite colength with $V(I) = \{0\}$. If $f \in I$ has leading form x^2, the *transverse cubic* $\gamma_f \in \mathbb{C}[y,z]_3$ of f is the cubic part of $f(0,y,z)$, the restriction of the cubic part of f to the plane $x = 0$. We regard $0 = 0^3$ as a cube.

Theorem 2.9 ([23])**.** (1) *If the tangent cone of $Z = V(I)$ lies on a quadric that is not the square of a linear form, or on a nonzero quadric and on eight linearly independent cubics, then Z is smoothable. In particular, S/I is smoothable if $H_A(2) \leq 4$, or if $H_A(2) = 5$ and $H_A(3) \leq 2$.*

(2) *If $f \in I$ has leading form x^2 and γ_f is not a cube, then I contains an equation of type D, and S/I is smoothable.*

(3) *If $f \in I$ has leading form x^2, γ_f is a nonzero cube, and $\mathfrak{m}^5 \subseteq I$, then I contains an equation of type E, and S/I is smoothable.*

Part (1) is [23, Theorem 1.2 and Remark 4.10], and parts (2) and (3) are [23, Proposition 4.5 and Corollary 4.3]. Here *I contains an equation of type D* or E if, in suitable formal coordinates, $I\widehat{S}$ contains uf for a unit $u \in \widehat{S} = \mathbb{C}[[x,y,z]]$ and an ADE polynomial f of that type [23, Section 4.1]. Then $V(I)$ lies on a surface with a rational double point, and smoothability follows from the irreducibility of the Hilbert schemes of points on such surfaces, a theorem of Zheng [28, Theorem 3.3]. For the quotient surfaces $\mathbb{C}^2/G$, which are the surfaces $V(f)$ with f an ADE polynomial used here, a second proof of the irreducibility was given by Craw and Yamagishi [6, Proposition 5.1, Theorem 5.5 and Remark 5.6]; the Lean formalization [24] also proves the statement used here, for all types.

2.10. We recall the results of [23] on rays. Let a, b, c be polynomial coordinates centered at the origin. A *ray* of I is an ideal $L = (b, c)$ with $(I : L) \not\subseteq I + L$ [23, Definition 5.2].

Theorem 2.11 ([23])**.** (1) *If I has colength d and a ray L, then $V(I)$ is a flat limit of disjoint unions of a reduced point, which moves along the curve $V(L)$, and a scheme of length $d-1$. In particular, if every finite subscheme of $\mathbb{A}^3$ of length $d-1$ is smoothable, then so is $V(I)$.*

(2) *For a nonzero $P \in T_1$ with coefficient vector p, let $N_P \subseteq S_1$ be the plane of linear forms vanishing at p, and L_P the ideal it generates. If $P^k \in M$ but $P^k \notin N_P \circ M$ for some k, then L_P is a ray of I.*

Part (1) is [23, Proposition 5.3 and Corollary 5.4], and part (2) is [23, Lemma 5.6]. Here $N_P \circ M$ is the span of the elements $v \circ F$ with $v \in N_P$ and $F \in M$; it equals $L_P \circ M$ because M is closed under differentiation. We quote two known cases.

Proposition 2.12. (1) ([23, Lemma 8.2]) *If $I \subseteq \mathfrak{m}^3$ and $H_A(4) = \cdots = H_A(s) = 1$ with $s \geq 5$, then I has a ray.*

(2) ([7, Proposition 4.2]) *Suppose $H_A(1) \geq 2$, let k_0 be the largest k with $H_A(k) > 1$, and suppose $s \geq 2k_0$, that is, the Hilbert function ends with at least k_0 ones. Then $V(I)$ is a flat limit of schemes with disconnected support; in particular, $V(I)$ is smoothable if every finite subscheme of $\mathbb{A}^3$ of smaller length is.*

In [7], a scheme that is a flat limit of schemes with disconnected support is called *cleavable*; by [7, Corollary 3.8], $V(I)$ is cleavable if $a^k b, a^k c \in I$ and $a^k \notin I + (b, c)$ for some k, that is, if (b, c) is a ray because a^k lies in $(I : (b, c))$ but not in $I + (b, c)$.

2.13. Finite subschemes of the plane of directions govern many spaces of top forms. For a closed subscheme $\Gamma \subset \mathbb{P}^2 = \mathbb{P}(T_1)$ with saturated homogeneous ideal $I_\Gamma \subseteq S$, let

$$\nu_k(\Gamma) = \{F \in T_k : \ g \circ F = 0 \text{ for all } g \in (I_\Gamma)_k\},$$

the scheme-theoretic linear span of the k-th Veronese image of Γ. If Γ is reduced, consisting of the points $P_1, \dots, P_n$, then $\nu_k(\Gamma)$ is spanned by $P_1^k, \dots, P_n^k$. For the double point $\Gamma = V(z, y^2)$ at $[X]$ we have $\nu_k(\Gamma) = \langle X^k, X^{k-1}Y \rangle$, and for $k \geq 2$ its only k-th powers are the multiples of X^k. For the line $\Lambda = V(x)$ we have $\nu_k(\Lambda) = \mathbb{C}[Y, Z]_k$, the forms annihilated by x. Mostly Γ will be finite. We also need the description of pairs of binary forms from [23, Lemma 8.6], which we restate in terms of finite subschemes of the plane (Lemma 2.15).

Lemma 2.14. *Let $n \geq 1$ and $k \geq n - 1$. For every subscheme $\Gamma \subset \mathbb{P}^2$ of length n we have $\dim \nu_k(\Gamma) = n$, and $\Gamma \mapsto \nu_k(\Gamma)$ is a morphism $\mathrm{Hilb}_n(\mathbb{P}^2) \to \mathrm{Gr}(n, T_k)$.*

Proof. The dimension of $\nu_k(\Gamma)$ is the value $h_\Gamma(k)$ of the Hilbert function of S/I_Γ. A linear form v that vanishes at no point of Γ is a nonzerodivisor on S/I_Γ, so $h_\Gamma(k) - h_\Gamma(k-1)$ is the Hilbert function of the standard graded algebra $S/(I_\Gamma + (v))$, which is positive up to some degree and zero afterwards. Since $h_\Gamma(0) = 1$ and $h_\Gamma(k) = n$ for large k, we get $h_\Gamma(k) \geq \min(k+1, n)$, hence $h_\Gamma(k) = n$ for $k \geq n - 1$. For the second claim, let $\pi \colon \mathcal{Z} \to \mathrm{Hilb}_n(\mathbb{P}^2)$ be the universal family. The sheaf $\pi_* \mathcal{O}_\mathcal{Z}(k)$ is locally free of rank n, and by the first claim the restriction map from the trivial bundle with fiber S_k to it is surjective on fibers. Its kernel is a subbundle with fibers $(I_\Gamma)_k$, and $\nu_k(\Gamma)$ is their orthogonal complement. □

Lemma 2.15 ([23, Lemma 8.6])**.** *Let $D_4 \subseteq T_4$ and $D_3 \subseteq T_3$ be two-dimensional subspaces with $S_1 \circ D_4 \subseteq D_3$. Then $D_4 = \nu_4(\Gamma)$ and $D_3 = \nu_3(\Gamma)$ for a subscheme $\Gamma \subset \mathbb{P}^2$ of length two. In suitable coordinates, either Γ consists of two points and $D_k = \langle X^k, Y^k \rangle$, or Γ is a double point and $D_k = \langle X^k, X^{k-1}Y \rangle$.*

Indeed, [23, Lemma 8.6] provides a binary quadric $q \in \mathbb{C}[x, y]_2$ with $D_k = \{F \in \mathbb{C}[X, Y]_k : q \circ F = 0\}$, and this is $\nu_k(\Gamma)$ for the scheme $\Gamma = V(z, q)$.

More generally, equal consecutive values of the Hilbert function produce a finite subscheme of the plane. This follows from Gotzmann's persistence theorem.

For a subspace $D \subseteq T_k$ let $D' = \{F \in T_{k+1} : S_1 \circ F \subseteq D\}$ be the space of forms all of whose first derivatives lie in D.

Lemma 2.16. *Let $1 \leq n \leq k$, and let $D \subseteq T_k$ be a subspace of dimension n with $\dim D' \geq n$. Then there is a subscheme $\Gamma \subset \mathbb{P}^2$ of length n such that $D = \nu_k(\Gamma)$, $D' = \nu_{k+1}(\Gamma)$, and every $F \in T_j$, $j \geq k$, with $S_{j-k} \circ F \subseteq D$ lies in $\nu_j(\Gamma)$. In particular, if A is a local algebra with $H_A(k) = H_A(k+1) = n$, then $D_k = \nu_k(\Gamma)$, $D_{k+1} = \nu_{k+1}(\Gamma)$, and $D_j \subseteq \nu_j(\Gamma)$ for all $j \geq k$.*

Proof. Let K be the ideal generated by the orthogonal complement $D^\perp \subseteq S_k$. For $j \geq k$, the orthogonal complement of $K_j = S_{j-k}D^\perp$ in T_j is $\{F : S_{j-k} \circ F \subseteq D\}$; in particular $H_{S/K}(k) = n$ and $H_{S/K}(k+1) = \dim D' \geq n$. Macaulay's bound gives $H_{S/K}(k+1) \leq n^{\langle k \rangle} = n$, so equality holds, and by Gotzmann's persistence theorem [11, Satz on p. 61], [1, Theorem 4.3.3], $H_{S/K}(j) = n$ for all $j \geq k$. So K defines a subscheme $\Gamma \subset \mathbb{P}^2$ of length n, and K is contained in its saturation I_Γ. By Lemma 2.14, $(I_\Gamma)_j$ has codimension n in S_j for $j \geq n-1$, hence $K_j = (I_\Gamma)_j$ for all $j \geq k$, which gives the first assertions. For the last one, $D_{k+1} \subseteq D'_k$, and both D_k and D_{k+1} have dimension n; moreover $S_{j-k} \circ D_j \subseteq D_k$ for $j \geq k$. □

Lemma 2.15 is the case $n = 2$, $k = 3$, up to the explicit normal forms. When Γ consists of n distinct points, the forms P^k for $P \in \Gamma$ lie in D_k, so all forms of degree k and $k+1$ of the tangent cone vanish on the n lines through the origin and these points. Along these lines a point can leave Z (Section 5).

Families of graded data are often cut out by linear conditions, and the following elementary bound, an *expected-dimension count*, will show that small strata cannot be components.

Lemma 2.17. *Let $\mathcal{Q}$ be an irreducible variety, $\mathcal{E}$ a vector bundle of rank N on $\mathcal{Q}$, and ψ a section of $\mathcal{E}$. Every irreducible component of the zero set of ψ has dimension at least $\dim \mathcal{Q} - N$.*

Proof. Locally the zero set is defined by N functions, so by Krull's height theorem each of its components has codimension at most N in $\mathcal{Q}$. □

The bound is most useful when it is applied to all the data of an inverse system at once: the spaces D_k *and* all tails. The conditions for closedness under differentiation are then the equations, and a stratum of smaller dimension than the bound cannot be a component.

Lemma 2.18 (One family)**.** *Let $\mathcal{Q}$, $\mathcal{E}$ and ψ be as in Lemma* 2.17*, with zero set $\mathcal{Y}$. Suppose that $\mathcal{Y}$ is the union of an irreducible open subset $\mathcal{U}$ and of finitely many locally closed subsets of dimension less than $\dim \mathcal{Q} - N$. Then $\mathcal{Y}$ is irreducible, the closure of $\mathcal{U}$.*

Proof. An irreducible component $\mathcal{C}$ of $\mathcal{Y}$ has dimension at least $\dim \mathcal{Q} - N$, so it is not contained in the union of the closures of the small subsets. Hence $\mathcal{C}$ meets the open set $\mathcal{U}$, and $\mathcal{C} \cap \mathcal{U}$ is dense in $\mathcal{C}$. So $\mathcal{C}$ lies in the closure of $\mathcal{U}$, and as $\mathcal{U}$ is irreducible, $\mathcal{Y} = \overline{\mathcal{U}}$. □

When $\mathcal{Y}$ maps onto the set of all algebras with a given Hilbert function (Lemma 2.3(2)), these algebras then form a single irreducible family, and one smoothable member of $\mathcal{U}$ that is Gorenstein, or linked to a short scheme, shows that all of them are smoothable. Theorem C.10 is an example; there this count replaces a computation of limits over colliding points.

Only strata of dimension *less* than the bound are excluded. A stratum whose dimension equals the bound may be a component, and its algebras must then be treated directly; this happens in step (iv) of the proof of Theorem E.11.

Conventions for counts. The space $\mathcal{Q}$ parametrizes data, not isomorphism classes of algebras, and the bound and the dimensions of all strata are computed in the same space $\mathcal{Q}$. Spaces of top forms are points of Grassmannians. A single form, such as a top form F spanning D_5, is counted projectively, as a point $[F]$ of a projective space, unless we say that it is *framed*, that is, fixed as a nonzero vector. A tail is counted either as a *raw* polynomial, in T_k, or as a class modulo D_k, in T_k/D_k, that is, as an element of a complement of D_k chosen locally on $\mathcal{Q}$; each count says which, and every algebra is reached with either choice, after changing the tails of lower degree. The conditions for closedness are compatible with both choices, since $S_1 \circ D_k \subseteq D_{k-1}$; the raw choice adds $\dim D_k$ for each tail polynomial ($\dim D \cdot \dim D_k$ for a tail map on a space D) to the dimension of $\mathcal{Q}$ and to that of every stratum, and framing adds one, so the comparison does not depend on the choice. Free cubic tails are counted as classes in T_3/D_3. Table 4 in Appendix H lists the counts of this paper with their conventions.

2.19. Two open conditions certify smoothability in families: being Gorenstein, and lying in a short complete intersection. The first concerns one more class of smoothable schemes. A local algebra is *Gorenstein* if its socle $(0 : \mathfrak{m}_A)$ is one-dimensional, equivalently, if its inverse system is generated by a single polynomial. Every finite Gorenstein subscheme of $\mathbb{A}^3$ is smoothable: this follows from the structure theorem of Buchsbaum and Eisenbud for Gorenstein ideals of codimension three [2] and the deformation theory of Pfaffian schemes [21, Corollary 4.3]; see [5, Proposition 2.3 and Corollary 2.4] and [4, Section 2.4]. Being Gorenstein is an open condition in families: for a finite flat family of algebras A_u, the fiber A_u is Gorenstein exactly when the A_u-module $\operatorname{Hom}_{\mathbb{C}}(A_u, \mathbb{C})$ is generated by one element, and the minimal number of generators is upper semicontinuous in u by Nakayama's lemma. Hence:

Proposition 2.20. *Let U be an irreducible variety with a morphism $U \to \mathrm{Hilb}_d(\mathbb{A}^3)$. If one of the parametrized schemes is Gorenstein, then all of them are smoothable.*

Proof. The parametrized schemes that are Gorenstein form an open subset of U, which is nonempty, hence dense. They are smoothable, and Proposition 2.2(1) applies. □

2.21. The second open condition is linkage. A finite subscheme Z' of a smooth threefold X is a *local complete intersection* if at each of its points its ideal is generated by three elements. For a subscheme $Z \subseteq Z'$, the *residual* scheme is $Z'' = V(\mathcal{I}_{Z'} : \mathcal{I}_Z)$, defined by the ideal quotient of the ideal sheaves. For example, $Z = V(x, y, z)$ lies in $Z' = V(x, y, z^3)$, and its residual scheme is $V(x, y, z^2)$; Example 3.9 below links a scheme of length fifteen to one of length twelve. The following theorem is folklore; Jelisiejew, Ramkumar and Sammartano call it known in the proof of [20, Corollary 5.7], and we deduce it from their results on nested Hilbert schemes [20, Propositions 5.2 and 5.3].

Theorem 2.22 ([20, Section 5.1])**.** *Let $X \subseteq \mathbb{A}^3$ be open, and let $Z \subseteq Z'$ be finite subschemes of X of lengths $d \leq d'$, with Z' a local complete intersection. Then the residual scheme Z'' has length $d' - d$, and Z is smoothable if and only if Z'' is smoothable.*

Proof. We transport smoothability through the linkage isomorphism of nested Hilbert schemes of Jelisiejew, Ramkumar and Sammartano. Let $\mathrm{Hilb}^{(d,d')}_{\mathrm{lci}}(X)$, denoted $\mathrm{Hilb}^{(d,d')}_{\star,\mathrm{lci}}(X)$ in [20], be the open subscheme of the nested Hilbert scheme parametrizing pairs $[Z_1 \subseteq Z_1']$ of lengths d and d' with Z_1' a local complete intersection. By [20, Proposition 5.3] there is an isomorphism

$$L_{d,d'} \colon \mathrm{Hilb}^{(d,d')}_{\mathrm{lci}}(X) \to \mathrm{Hilb}^{(d'-d,d')}_{\mathrm{lci}}(X)$$

with $L_{d'-d,d'} \circ L_{d,d'} = \mathrm{id}$, which sends $[Z_1 \subseteq Z_1']$ to $[Z_1'' \subseteq Z_1']$, where Z_1'' is defined by the annihilator of the dual of the $\mathcal{O}_{Z_1'}$-module $N = \mathcal{I}_{Z_1}/\mathcal{I}_{Z_1'}$, and $\mathcal{O}_{Z_1''}$ is isomorphic to this dual locally. A finite-dimensional module and its dual have the same annihilator, and the annihilator of N is $(\mathcal{I}_{Z_1'} : \mathcal{I}_{Z_1})/\mathcal{I}_{Z_1'}$. So Z_1'' is the residual scheme, of length $\dim N = d' - d$. If Z_1' is reduced, then so is Z_1'', which consists of the points of Z_1' not in Z_1. Hence $L_{d,d'}$ maps the locus of pairs with Z_1' reduced onto the corresponding locus, and their closures onto each other. By [20, Proposition 5.2], the pairs $[Z_1 \subseteq Z_1']$ with Z_1 smoothable are exactly those in the closure of the pairs with Z_1' reduced. Applying this to $[Z \subseteq Z']$ and to $[Z'' \subseteq Z']$ proves the claim. □

Smoothability in an open subset X of $\mathbb{A}^3$ is the same as in $\mathbb{A}^3$: the Hilbert scheme $\mathrm{Hilb}_d(X)$ is open in $\mathrm{Hilb}_d(\mathbb{A}^3)$, and the reduced subschemes of X are dense in its intersection with the irreducible smoothable component.

Linkage preserves more than smoothability: by [20, Theorem 5.6], if Z is linked to Z'', then the Hilbert schemes have smoothly equivalent singularities at $[Z]$ and $[Z'']$. So the passage from Z to Z'' is not numerical bookkeeping:

it transports the whole local deformation problem through a common smooth correspondence, the nested Hilbert scheme of lci pairs, whose projection to Hilb_d is smooth [20, Proposition 5.2]; this is why one example of short linkage controls an irreducible family (Proposition 2.24 below).

Corollary 2.23 (Theorem 1.5)**.** *Let $Z \subseteq Z'$ be as in Theorem* 2.22*, with $d' < 2d$. If every finite subscheme of $\mathbb{A}^3$ of length less than d is smoothable, then Z is smoothable.*

Proof. The residual scheme has length $d' - d < d$, so it is smoothable, and so is Z. □

Linkage to a short scheme is an open condition, so in an irreducible family one example suffices.

Proposition 2.24. *Suppose that every finite subscheme of $\mathbb{A}^3$ of length less than d is smoothable. Let U be an irreducible variety with a morphism $U \to \mathrm{Hilb}_d(\mathbb{A}^3)$. If one of the parametrized schemes is contained, in some open subset X of $\mathbb{A}^3$, in a finite local complete intersection of length less than $2d$, then all of them are smoothable.*

Proof. Let $d' < 2d$ be the length of the complete intersection. By [20, Proposition 5.2], the projection $p\colon \mathrm{Hilb}^{(d,d')}_{\mathrm{lci}}(X) \to \mathrm{Hilb}_d(X)$ is smooth, hence open. Its image is an open subset of $\mathrm{Hilb}_d(X)$, and so of $\mathrm{Hilb}_d(\mathbb{A}^3)$, and its preimage in U is open and nonempty, hence dense. The schemes it parametrizes are smoothable by Corollary 2.23, and Proposition 2.2(1) applies. □

We call a member of an irreducible family that lies, in some open subset of $\mathbb{A}^3$, in a finite local complete intersection of length less than $2d$ a *linkage anchor* of the family.

Local lengths are certified as follows. Let $J \subseteq S$ be an ideal. If $\dim_{\mathbb{C}} S/(J + \mathfrak{m}^N) = \dim_{\mathbb{C}} S/(J + \mathfrak{m}^{N+1}) = e$ for some N, then $\mathfrak{m}^N \subseteq J + \mathfrak{m}^{N+1}$, so $\mathfrak{m}^N \subseteq JS_{\mathfrak{m}}$ by Nakayama's lemma, and the local ring of $V(J)$ at the origin has length e. Conversely, if this length is at most e, then $\mathfrak{m}^e \subseteq JS_{\mathfrak{m}}$ and $\dim_{\mathbb{C}} S/(J + \mathfrak{m}^{e+1}) \le e$; and if $\dim_{\mathbb{C}} S/(J + \mathfrak{m}^{e+1}) \le e$, then $\mathfrak{m}^k \subseteq J + \mathfrak{m}^{k+1}$ for some $k \le e$, and the length is at most e. So for an ideal generated by polynomials depending polynomially on parameters, the condition that the local length at the origin is at most e is Zariski open in the parameters, being a condition on the rank of a matrix. In particular, if one triple of elements of $I \cap \mathfrak{m}^3$ defines a complete intersection of local length at most e at the origin, then so do general triples of elements of $I \cap \mathfrak{m}^3$; the condition depends only on the triple modulo $\mathfrak{m}^{e+1}$, so "general" refers to a nonempty Zariski-open subset of the finite-dimensional space of triples of elements of $(I \cap \mathfrak{m}^3)/(I \cap \mathfrak{m}^{e+1})$.

2.25. We collect the notation used throughout. Finite subschemes of the plane $\mathbb{P}^2 = \mathbb{P}(T_1)$ of directions are denoted Γ, with ideals $I_\Gamma \subseteq S$ and spans

$\nu_k(\Gamma) \subseteq T_k$; a point is a nonzero $P \in T_1$, and $N_P \subseteq S_1$ is the plane of linear forms vanishing at it. The inverse system of I is M, and $D_k \subseteq T_k$ are its spaces of top forms. A top form is usually called F and its lift Φ (other elements of M are called Ψ). The tails are written $\tau\colon D_4 \to T_3$ for the lifts of quartics, R (degree four) and E (degree three) for the lift $F + R + E$ of a quintic, and $R_{s-1}, \dots, R_3$, indexed by their degrees, for the lift $\Phi = F + R_{s-1} + \cdots + R_3$ of a top form of higher degree s. The cubic base scheme of A is $\mathfrak{B} \subset \mathbb{P}^2$ (Section 3.5), with multiplicities e_p and the separation degrees $k(p)$ of (4). Rays are ideals (b, c) for polynomial coordinates a, b, c. Powers are ordinary powers.

3. Linkage and cubic directions

3.1. In this section, except in Theorem 3.6, the tangent cone lies on no quadric. Take three equations of Z whose leading forms are cubics. When the three cubic cones have no common line, the equations cut out, near the origin, a complete intersection of length at most 27 that contains Z, and Z is linked to a scheme of length at most $27 - \ell(Z)$ (Theorem 3.4). In general the common lines form the cubic base scheme $\mathfrak{B}$. When $\mathfrak{B}$ is finite, the length exceeds 27 by a contribution of each point of $\mathfrak{B}$, which we bound after deforming Z (Theorem 3.12). This numerical test settles most algebras whose cubics and quartics are governed by two or three points of the plane (Corollaries 3.14 and 3.15).

3.2. Complete intersections containing a local scheme are found among the leading forms.

Lemma 3.3. *Let $Z = V(I)$ be local, and let $f, g, h \in I$ have leading forms of degrees k_1, k_2, k_3 without a common zero in $\mathbb{P}^2$. Then there is an open set $X \subseteq \mathbb{A}^3$ containing the origin such that $Z' = V(f, g, h) \cap X$ is a finite complete intersection in X containing Z, of length at most $k_1k_2k_3$.*

Proof. Let $J = (f, g, h)$ and let J^* be the ideal of leading forms of the elements of J. As in the proof of [23, Lemma 3.2], the associated graded ring of the local ring $S_{\mathfrak{m}}/JS_{\mathfrak{m}}$ is S/J^*. Since J^* contains the leading forms f^*, g^*, h^*, which form a regular sequence because they have no common zero, the length of $S_{\mathfrak{m}}/JS_{\mathfrak{m}}$ is at most $\dim S/(f^*, g^*, h^*) = k_1k_2k_3$. So the origin is an isolated point of $V(J)$, and we take X to be the complement of the other points and components of $V(J)$. □

Theorem 3.4. *Let $Z = V(I)$ be a local scheme of length d, and suppose that every finite subscheme of $\mathbb{A}^3$ of length at most $27 - d$ is smoothable. If the cubics on which the tangent cone of Z lies have no common zero other than the origin, then Z is smoothable. Equivalently, the conclusion holds if D_3 contains no cube P^3 of a nonzero linear form.*

Proof. The equivalence is (1). A general cubic $f \in I_3^*$ defines a plane curve with finitely many components. Since the cubics in I_3^* have no common zero,

a general $g \in I_3^*$ contains none of these components, and a general $h \in I_3^*$ avoids the finitely many common points of f and g. Lifting f, g, h to elements of I and applying Lemma 3.3, we find Z inside a complete intersection of length at most 27. The residual scheme has length at most $27 - d$, and Theorem 2.22 applies. If $d \geq 14$, then $27 - d < d$, so the hypothesis holds when all schemes shorter than Z are smoothable. □

3.5. The cubics of the tangent cone organize what follows. Let $\mathfrak{B} \subset \mathbb{P}^2$ be the subscheme defined by the cubics in I_3^*, the *cubic base scheme* of A. Only the cubic piece of I^* is used here: the whole ideal I^* contains a power of $\mathfrak{m}$ and defines the empty subscheme of $\mathbb{P}^2$. A cubic $g \in S_3$ is a linear form on T_3, with $\langle g, P^3 \rangle = 6g(p)$, and by Lemma 2.5(2) the linear forms vanishing on D_3 are the cubics of I_3^*. Hence $\mathfrak{B}$ is, as a scheme, the preimage of the linear space $\mathbb{P}(D_3) \subseteq \mathbb{P}(T_3)$ under the Veronese embedding $\mathbb{P}^2 \to \mathbb{P}(T_3)$, $[P] \mapsto [P^3]$; on points, this is (1) for $k = 3$. If $\mathbb{P}(D_3)$ misses the cubic Veronese surface, Z is linked to a shorter scheme by Theorem 3.4. If they meet in finitely many points, these are directions along which a point may leave Z (Section 5), and when there are few of them, they measure the excess of linkage (Theorem 3.12). Nonreduced points of $\mathfrak{B}$ record colliding directions (Section 5.15). In short: the cubic equations determine directions; along those directions one either separates a point or measures the excess needed to link to a shorter scheme.

When the tangent cone lies on a quadric x^2, three equations of the tangent cone are found differently: one of them is x^2, and the other two need only have no common zero on the line $x = 0$ of $\mathbb{P}^2$.

Theorem 3.6 (Quadric linkage)**.** *Let $A = S/I$ be a local algebra of length d with $I_2^* = \mathbb{C}x^2$, and suppose that every finite subscheme of $\mathbb{A}^3$ of length less than d is smoothable. Let $J \subseteq \mathbb{C}[y, z]$ be the image of I^* under restriction to the plane $x = 0$. If J contains forms of degrees k_1 and k_2 without common factor and $k_1 k_2 < d$, then A is smoothable.*

Proof. Let $g, h \in I^*$ restrict to such forms. A common zero of x^2, g and h in $\mathbb{P}^2$ lies on the line $x = 0$ and is a common zero of the restrictions; so there is none. Lifting x^2, g, h to elements of I, Lemma 3.3 gives a complete intersection containing $V(I)$ of length at most $2k_1k_2 < 2d$, and Corollary 2.23 applies. □

3.7. We now treat the Hilbert functions $(1, 3, 6, c, 1)$ with $4 \leq c \leq 8$. They show the interplay of the three ways; for $c = 4$, length fifteen, this is the simplest case in which linkage is needed. By Lemma 2.7, the inverse system of such an algebra is

$$M = T_{\leq 2} \oplus D_3 \oplus \mathbb{C}(F + \tau), \tag{2}$$

where F spans D_4, the space D_3 has dimension c and contains the first derivatives $S_1 \circ F$, and the tail $\tau \in T_3$ is arbitrary.

Theorem 3.8. *Let $4 \le c \le 8$, and suppose that every finite subscheme of $\mathbb{A}^3$ of length less than $11+c$ is smoothable. Then every local algebra with Hilbert function $(1,3,6,c,1)$ is smoothable. In particular, every local algebra with Hilbert function $(1,3,6,4,1)$ is smoothable.*

Proof. We use the three ways in turn: linkage when D_3 contains no cube, a ray when the top form is a fourth power, and limits otherwise. The last assertion follows from the first and [23]. Let $c \le 7$; the case $c = 8$, in which I_3^* is a pencil and three cubics of the tangent cone are not available, is proved in Appendix B by an expected-dimension count and one complete intersection of degrees $3, 3, 4$. Let r be the number of essential variables of F.

Linkage. If D_3 contains no cube, then A is linked to an algebra of length at most $27-(11+c) = 16-c$ by Theorem 3.4, and it is smoothable.

Splitting off a point. If $r = 1$, then $F = P^4$ and $P^3 \in S_1 \circ F \subseteq D_3 \subseteq M$. Every $v \in N_P$ annihilates P^4, so $N_P \circ M$ lies in $T_{\le 2}$ and does not contain P^3. By Theorem 2.11(2), L_P is a ray, and A is smoothable by Theorem 2.11(1).

Limits. Let $r = 3$, so that F is concise. Consider the tuples of a quartic F and cubics $G_1, \dots, G_{c-3}, \tau$ for which the cubics $F_X, F_Y, F_Z, G_1, \dots, G_{c-3}$ are linearly independent. They form a nonempty open subset of an affine space, hence an irreducible variety. By Lemma 2.3(2) they parametrize the algebras (2) with $D_3 = \langle F_X, F_Y, F_Z, G_1, \dots, G_{c-3}\rangle$; these are all algebras in question with $r = 3$. The tuples for which D_3 contains a cube form a closed subset, the image of a closed incidence $\{(P, F, G_1, \dots) : P^3 \in D_3\}$ under a proper projection.

We claim that it is a proper subset. Let $F_0 = (X^2+Y^2+Z^2)^2$. The space $S_1 \circ F_0 = (X^2+Y^2+Z^2)\,T_1$ contains no cube, since $X^2+Y^2+Z^2$ is irreducible. Hence $D_3 = S_1 \circ F_0 + K$, with $K = \langle G_1, \dots, G_{c-3}\rangle$, contains a cube only if the image of K in $T_3/(S_1 \circ F_0) \cong \mathbb{C}^7$, a subspace of dimension $c-3$, meets the image of the cone of cubes, which has dimension at most three; it is closed, because it is the image of the cone of cubes under a linear projection whose kernel $S_1 \circ F_0$ meets this cone only in 0. For $c \le 7$ a general subspace of dimension $c-3$ meets this image only in 0, since $(c-3)+3 < 7+1$; projectively, a general $\mathbb{P}^{c-4}$ misses a surface in $\mathbb{P}^6$. This proves the claim. So the algebras with D_3 free of cubes are dense in the family; they are smoothable, and so are all algebras with $r = 3$, by Proposition 2.2(1).

Finally let $r = 2$, so that $F \in \mathbb{C}[X, Y]$ in suitable coordinates and $D_3 = \langle F_X, F_Y, G_1, \dots, G_{c-2}\rangle$ for some cubics G_i. Choose a quartic F' with $\partial F'/\partial Z = G_1$, and for $t \in \mathbb{C}$ let $F_t = F + tF'$ and $D_{3,t} = \langle F_X + tF'_X,\ F_Y + tF'_Y,\ G_1, \dots, G_{c-2}\rangle$. These c cubics are independent for t near 0, and they contain the derivatives of F_t, since $\partial F_t/\partial Z = tG_1$. For small $t \ne 0$ the three derivatives of F_t are independent, so F_t is concise. The spaces $T_{\le 2} \oplus D_{3,t} \oplus \mathbb{C}(F_t + \tau)$, with these polynomial bases, form a family as in Lemma 2.3(2), whose member at $t = 0$ is M and whose other members near 0 are smoothable by the case $r = 3$. So A is smoothable. □

Example 3.9. Let $F_0 = (X^2+Y^2+Z^2)^2$ and $D_3 = S_1 \circ F_0 + \mathbb{C}XYZ$. The algebra with inverse system $T_{\leq 2} \oplus D_3 \oplus \mathbb{C}F_0$ is S/I with

$$I = (x^3 - 3xy^2,\ y^3 - 3x^2y,\ z^3 - 3x^2z,\ xz^2 - xy^2,\ yz^2 - x^2y,\ y^2z - x^2z).$$

The three cubics $x^3 - 3xy^2$, $y^3 - 3yz^2$ and $z^3 - 3x^2z$ lie in I and have no common zero in $\mathbb{P}^2$: a zero coordinate forces the others to vanish, and otherwise $x^2 = 3y^2$, $y^2 = 3z^2$ and $z^2 = 3x^2$ give $1 = 27$. They generate a complete intersection of length 27, and the residual scheme is defined by

$$J = (x^2+y^2+z^2,\ xyz,\ y^3 - 3yz^2,\ 4xy^2 + xz^2,\ 3y^2z + 4z^3),$$

an ideal with Hilbert function $(1, 3, 5, 3)$. Indeed, JI lies in the ideal K of the three cubics, so $J \subseteq K : I$, and both ideals have colength $12 = 27 - 15$ (Theorem 2.22). So this scheme of length fifteen is linked to a scheme of length twelve, which is smoothable by [23, Theorem 6.7]. (Script `linkage15.sage`.) Together with Proposition 2.24, this single example gives a second proof that the algebras with $c = 4$ and $r = 3$ are smoothable: the family of the proof of Theorem 3.8 is irreducible and contains this example.

Remark 3.10. Theorem 3.4 also gives a second proof of Theorem G.4 for the Hilbert functions $(1, 3, 6, c)$ with $4 \leq c \leq 7$, given smoothability in smaller lengths. Such an algebra is $S/(\mathcal{C} + \mathfrak{m}^4)$ for a subspace $\mathcal{C} \subseteq S_3$ of codimension c, and its space $D_3 = \mathcal{C}^\perp$ is a general c-dimensional subspace of T_3 when $\mathcal{C}$ is general. The cubes form a surface in $\mathbb{P}(T_3) \cong \mathbb{P}^9$, so a general linear subspace of dimension $c - 1 \leq 6$ misses it, and the general very compressed algebra is linked to an algebra of length $17 - c$. Since the very compressed algebras of fixed length form an irreducible family over a Grassmannian, they are all smoothable.

3.11. When the cubic base scheme $\mathfrak{B}$ is finite but not empty, three cubics of the tangent cone no longer cut out a finite scheme: the three cubic cones share the lines through the origin with directions in $\mathfrak{B}$. The surfaces themselves, however, need not share these lines, because their terms of higher degree pull them apart. We measure the excess on a cone: two cubic cones without common component meet in nine lines through the origin, counted with multiplicity, and the third surface meets each of these lines at the origin with some multiplicity.

Cone lemma. *Let $h_1, h_2 \in S$ be forms without common factor, and let $g = \sum_{j \geq 1} g_j \in S$ with $g_j \in S_j$. For a point p of the finite subscheme $V(h_1, h_2) \subset \mathbb{P}^2$, let n_p be the length of $V(h_1, h_2)$ at p, and let r_p be the smallest j such that g_j does not vanish at p. If every r_p is finite, then the local ring of $V(h_1, h_2, g) \subset \mathbb{A}^3$ at the origin has length $\sum_p n_p r_p$.*

In other words, $V(h_1, h_2) \subset \mathbb{A}^3$ is the union of the lines of the points p, with multiplicities n_p; the restriction of g to the line of p vanishes at the origin to order r_p, and the length is the sum of these orders, counted with the multiplicities.

Proof. Choose a linear form l vanishing at no point of $V(h_1, h_2) \subset \mathbb{P}^2$, and let $R = S/(h_1, h_2)$. As h_1, h_2, l have only the origin as common zero in $\mathbb{A}^3$, they form a regular sequence; so l is a nonzerodivisor on the graded ring R, the quotient R/lR is finite, and R is a free $\mathbb{C}[l]$-module with a homogeneous basis $b_1, \dots, b_N$, of degrees $d_1, \dots, d_N$. The ring $\bar{R} = R/(l-1)R$ is the coordinate ring of $V(h_1, h_2)$ in the affine chart $l \neq 0$ of $\mathbb{P}^2$, the product of the local rings $\bar{R}_p$ of $V(h_1, h_2)$ at its points, and the images of the b_i form a basis of $\bar{R}$. Let M_j be the matrix, in this basis, of multiplication by the image of g_j in $\bar{R}$. Comparing homogeneous parts shows that multiplication by g on R has the matrix $\Delta^{-1}(\sum_j l^j M_j)\Delta$ in the basis $b_1, \dots, b_N$, where Δ is the diagonal matrix with entries $l^{d_1}, \dots, l^{d_N}$. In a basis of $\bar{R}_p$ adapted to a composition series of ideals, multiplication by an element a of $\bar{R}_p$ is triangular with diagonal entries $a(p)$; so on $\bar{R}_p \otimes \mathbb{C}[l]$ the matrix $\sum_j l^j M_j$ is triangular with diagonal entries $\sum_j g_j(p)\, l^j$, where $g_j(p)$ is the value at the coefficient vector of p with $l(p) = 1$. Hence the determinant of multiplication by g is

$$\det\Big(\sum_j l^j M_j\Big) = \prod_p \Big(\sum_j g_j(p)\, l^j\Big)^{n_p},$$

and its order at $l = 0$ is $\sum_p n_p r_p$. By the elementary divisor theorem, this order is the length of the localization of the torsion $\mathbb{C}[l]$-module $R/gR = S/(h_1, h_2, g)$ at the ideal (l) of $\mathbb{C}[l]$. This localization is the product of the local rings of $V(h_1, h_2, g)$ at its points with $l = 0$, and the origin is the only such point. □

On the blow-up of $\mathbb{A}^3$ at the origin, the same count is given by the excess formula of Fulton and Lazarsfeld [10, Theorem 3(A), its proof, and the Remark on p. 104]: if effective Cartier divisors V_1, V_2, V_3 on a smooth threefold meet only at a point o, then their intersection multiplicity at o, which is the length of the local ring of $V_1 \cap V_2 \cap V_3$ at o [9, Proposition 7.1(b)], is

$$i_o(V_1 \cdot V_2 \cdot V_3) = e_o(V_1)\, e_o(V_2)\, e_o(V_3) + \deg(\widetilde{V}_1 \cdot \widetilde{V}_2 \cdot \widetilde{V}_3), \tag{3}$$

where e_o denotes the multiplicity at o, and $\widetilde{V}_i$ is the proper transform of V_i on the blow-up at o. For three surfaces with cubic tangent cones the first term is 27, and the proper transforms meet the exceptional plane along the three plane cubics, so the second term comes from their common zeros. We do not use (3); on a cone, the cone lemma computes the length directly.

The local contributions are multiplicities. Let p be a point of the finite subscheme $\mathfrak{B}$, and $\mathfrak{b}_p$ the ideal of $\mathfrak{B}$ in the local ring $\mathcal{O}_p$ of $\mathbb{P}^2$ at p. For cubics $w_1, w_2 \in I_3^*$ without common factor, $V(w_1, w_2)$ contains $\mathfrak{B}$. By the criterion following Proposition 2.24, applied in an affine chart around p, the pairs (w_1, w_2) for which the length of $V(w_1, w_2)$ at p is as small as possible form a nonempty open subset of $(I_3^*)^2$. This smallest length is the *multiplicity* e_p of $\mathfrak{B}$ at p, the intersection number at p of two general cubics of the tangent

cone.[1] The cubics of I_3^* generate $\mathfrak{b}_p$, so Lemma A.4 in Appendix A gives $e_p = \ell(\mathcal{O}_p/\mathfrak{b}_p)$ when $\mathfrak{b}_p$ is generated by two elements, for instance when $\mathfrak{B}$ is curvilinear at p, and $e_p = 4$ for $\mathfrak{b}_p = (u, v)^2$ and $e_p = 5$ for $\mathfrak{b}_p = (u^2, uv, v^3)$, in local coordinates u, v; by that lemma, these are all the ideals of colength at most four. We write $e(\mathfrak{B}) = \sum_p e_p$. So $e(\mathfrak{B})$ is the length of $\mathfrak{B}$ when $\mathfrak{B}$ is curvilinear, and $e(\mathfrak{B}) \le \ell(\mathfrak{B}) + 1$ when $\ell(\mathfrak{B}) \le 4$.

The local contribution at a point $p = [P]$ of $\mathfrak{B}$ depends on how late the tangent cone stops vanishing at p. Fix a coefficient vector for each point p of $\mathfrak{B}$, so that a form q has a value $q(p)$. Along the line of p, the cubic part of an element $f \in I$ with leading form $w \in I_3^*$ vanishes, and the restriction of f to this line starts with the values at p of its homogeneous parts of degrees four, five, ... By (1), all forms of degree k of the tangent cone vanish at p exactly when $P^k \in D_k$. Let s be the socle degree of A. The *separation degree* of p is

$$k(p) = \min\{k : \ 4 \le k \le s, \ P^k \notin D_k\}, \tag{4}$$

and $k(p) = s$ if $P^k \in D_k$ for all $4 \le k \le s$. So $k(p) = 4$ if some quartic of the tangent cone does not vanish at p, and $k(p) = 5$ if all quartics vanish at p but some quintic does not. When $k(p) = s$ because $P^k \in D_k$ for all $4 \le k \le s$, we say that $k(p)$ is *capped*; then the separation need not come from an equation of the tangent cone: it comes from a deformation of A (Step 1 of the proof below). For example, if $s = 5$, $D_4 = \langle X^4, Y^4, Z^4\rangle$ and $D_5 = \mathbb{C}X^5$, then $k(p) = 5$ at the three points $[X]$, $[Y]$, $[Z]$: at $[Y]$ and $[Z]$ because some quintic of the tangent cone does not vanish there, at $[X]$ because it is capped (Example E.6).

Theorem 3.12 (Excess linkage)**.** *Let $A = S/I$ be a local algebra of length d with $H_A(2) = 6$ and socle degree $s \ge 4$, and suppose that every finite subscheme of $\mathbb{A}^3$ of length less than d is smoothable. Suppose that* $\dim I_3^* \ge 3$ *and that the cubic base scheme $\mathfrak{B}$ is finite. If*

$$27 + \sum_{p \in \mathfrak{B}} \big(k(p) - 3\big)\, e_p < 2d,$$

then A is smoothable.

Proof. We deform A along a line of algebras with the same Hilbert function, so that one element of the ideal acquires, near each point p of $\mathfrak{B}$, large terms of degree $k(p)$ that do not vanish at p; then we rescale the coordinates and count on a cone by the cone lemma.

Step 1: the free cubic tails. For each $F \in D_s$ choose, linearly in F, an element $\Phi_F \in M$ with top form F, and let $M_{<s}$ be the space of elements of

[1] Two general elements of $\mathfrak{b}_p$ generate a reduction of $\mathfrak{b}_p$ [15, Theorem 8.6.6], so e_p is the Hilbert–Samuel multiplicity of $\mathfrak{b}_p$ [15, Corollary 1.2.5, Propositions 11.2.1 and 11.1.10(2)]. We do not use this.

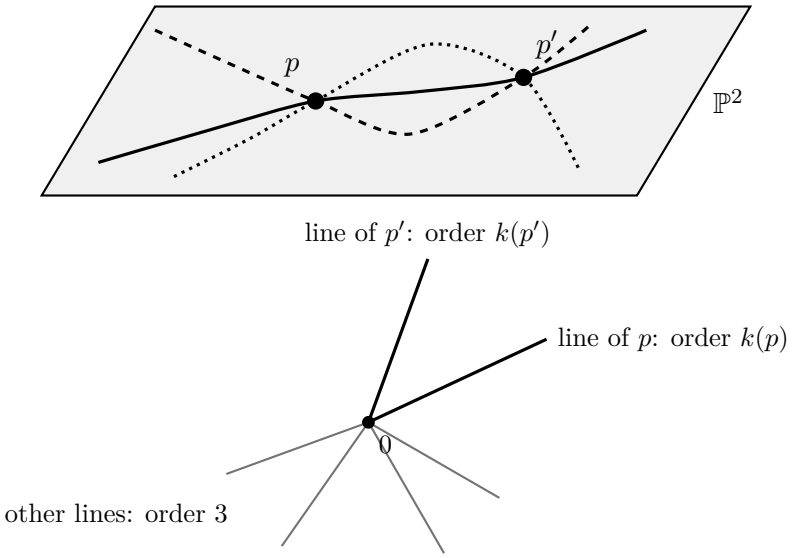


FIGURE 1. Excess linkage. Above: the cubics $w_1 = 0$ and $w_2 = 0$ (solid, dashed) of the tangent cone meet in nine points of the plane $\mathbb{P}^2$ of directions, counted with multiplicity, among them the points p, p' of the cubic base scheme $\mathfrak{B}$; the cubic $w_3 = 0$ (dotted) passes through the points of $\mathfrak{B}$ but through no other common point of w_1 and w_2. Below: after the deformation and the rescaling of Theorem 3.12, two of the three surfaces become the cone $V(w_1, w_2)$, nine lines through the origin counted with multiplicity; the equation of the third surface, restricted to one of these lines, vanishes at the origin to order three, or to order $k(p)$ on the line of a point p of $\mathfrak{B}$. The length at the origin is the sum of these orders times the multiplicities (the cone lemma), that is, $27 + \sum_p (k(p) - 3)e_p$.

M of degree less than s; then $M = M_{<s} \oplus \{\Phi_F : F \in D_s\}$. For a linear map $Q' \colon D_s \to T_3$ and $\lambda \in \mathbb{C}$ let

$$M_\lambda = M_{<s} \oplus \{\Phi_F + \lambda Q'(F) : F \in D_s\}.$$

It is closed under differentiation, because the derivatives of the cubics $Q'(F)$ lie in $T_2 \subseteq M_{<s}$, and it has the same spaces D_k. By Lemma 2.3(2), the ideals I_λ of the spaces M_λ form a family over the line, with $I_0 = I$. Since I_λ is the orthogonal complement of M_λ: (a) every element of I of order at least four lies in I_λ, because it is orthogonal to all cubics; (b) if $f \in I$ has leading form $w \in I_3^*$, and $q \in S_s$ satisfies $\langle q, F\rangle = -\langle w, Q'(F)\rangle$ for all $F \in D_s$, then $f + \lambda q \in I_\lambda$, because q is orthogonal to $M_{<s}$ and

$$\langle f + \lambda q,\ \Phi_F + \lambda Q'(F)\rangle = \langle f, \Phi_F\rangle + \lambda\big(\langle w, Q'(F)\rangle + \langle q, F\rangle\big) = 0;$$

here $\langle f, Q'(F)\rangle$ equals $\langle w, Q'(F)\rangle$ and $\langle q, \Phi_F\rangle$ equals $\langle q, F\rangle$ for reasons of degree, and $\langle q, Q'(F)\rangle = 0$ because $s > 3$. Now let $w_1, w_2, w_3 \in I_3^*$ be linearly independent, let $f_i \in I$ have leading form w_i, and let $L \in S_1$ vanish at no point of $\mathfrak{B}$. The w_i are linearly independent linear forms on T_3, so there is Q' with $\langle w_1, Q'(F)\rangle = \langle w_2, Q'(F)\rangle = 0$ and $\langle w_3, Q'(F)\rangle = -\langle L^s, F\rangle$ for all F; then f_1, f_2 and $f_3 + \lambda L^s$ lie in I_λ. For $4 \le k < s$ let q_k be a general element of I_k^*, and $g_k \in I$ an element with leading form q_k (or $g_k = 0$ if $I_k^* = 0$); put $q_s = L^s$. If $k(p) = k < s$, some form of I_k^* does not vanish at p, so $q_k(p) \ne 0$; and $q_k(p) = 0$ whenever $k < k(p)$. Thus, at every point p of $\mathfrak{B}$, $k(p)$ is the smallest k with $q_k(p) \ne 0$. By (a) and (b), for $\varepsilon \ne 0$ the polynomials f_1, f_2 and

$$f_3^\varepsilon = f_3 + \sum_{4 \le k < s} \varepsilon^{-(k-3)} g_k + \varepsilon^{-(s-3)} L^s \tag{5}$$

lie in the ideal I_λ with $\lambda = \varepsilon^{-(s-3)}$.

Step 2: the local bound. We prove the following claim, in which the elements f_i^ε need not come from Step 1. Let f_1, f_2, f_3 have leading forms $w_1, w_2, w_3 \in I_3^*$, and let $f_i^\varepsilon = f_i + \sum_{j=4}^N \varepsilon^{-(j-3)} g_{i,j}$ for $\varepsilon \ne 0$, where $g_{i,j}$ has order at least j and its part of degree j is $\theta_i q_j$, with $\theta \in \mathbb{C}^3$ and forms $q_j \in S_j$ that do not depend on i. Suppose that $\theta_3 \ne 0$; that the cubics $h_1 = w_1 - \frac{\theta_1}{\theta_3} w_3$ and $h_2 = w_2 - \frac{\theta_2}{\theta_3} w_3$ have no common factor, and $V(h_1, h_2) \subset \mathbb{P}^2$ has length e_p at every point p of $\mathfrak{B}$; that w_3 vanishes at no other point of $V(h_1, h_2)$; and that for every $p \in \mathfrak{B}$ there is $k_p \ge 4$ with $q_j(p) = 0$ for $j < k_p$ and $q_{k_p}(p) \ne 0$. Then for all but finitely many ε, the local ring of $V(f_1^\varepsilon, f_2^\varepsilon, f_3^\varepsilon)$ at the origin has length at most $27 + \sum_{p \in \mathfrak{B}} (k_p - 3) e_p$.

For $\varepsilon \ne 0$ the substitution σ_ε of $\varepsilon x, \varepsilon y, \varepsilon z$ for x, y, z is an automorphism of S that fixes the origin, so the ideals generated by the f_i^ε and by the $F_i = \varepsilon^{-3} \sigma_\varepsilon(f_i^\varepsilon)$ have local rings of the same length at the origin. An element g of order at least j satisfies $\sigma_\varepsilon(g) = \varepsilon^j (g_j + \varepsilon g')$, where g_j is its part of degree j and g' is a polynomial in ε, x, y, z. So the F_i are polynomials in ε, x, y, z, and at $\varepsilon = 0$ they are $F_i = w_i + \theta_i q$ with $q = \sum_{j=4}^N q_j$. By the criterion following Proposition 2.24, the condition that the local length of $V(F_1, F_2, F_3)$ at the origin is at most a given number is Zariski open in ε; so it suffices to bound this length at $\varepsilon = 0$. There the ideal is generated by h_1, h_2 and $w_3 + \theta_3 q$, and the cone lemma applies with $g_3 = w_3$ and $g_j = \theta_3 q_j$ for $j \ge 4$. At a point p of $V(h_1, h_2)$ outside $\mathfrak{B}$ we have $w_3(p) \ne 0$, so $r_p = 3$; at a point p of $\mathfrak{B}$ we have $w_3(p) = 0$, so $r_p = k_p$, and $n_p = e_p$. As the n_p add up to nine, the length is $27 + \sum_{p \in \mathfrak{B}} (k_p - 3) e_p$. This proves the claim.

Step 3: conclusion. Choose $w_1, w_2 \in I_3^*$ without common factor such that $V(w_1, w_2)$ has length e_p at every point p of $\mathfrak{B}$; by the definition of e_p and the argument of the proof of Theorem 3.4, general pairs have these properties. Choose $w_3 \in I_3^*$ outside $\langle w_1, w_2\rangle$ that vanishes at none of the finitely many points of $V(w_1, w_2)$ outside $\mathfrak{B}$; they are not base points of I_3^*. Let f_1, f_2 and f_3^ε be the elements of Step 1. They have the form of Step 2 with $\theta = (0, 0, 1)$,

$h_i = w_i$, and the forms $q_4, \dots, q_s$, so that $k_p = k(p)$. Hence, for all but finitely many ε, the origin is an isolated point of $V(f_1, f_2, f_3^\varepsilon)$, and in a neighborhood of it these polynomials define a complete intersection of length at most $27 + \sum_p (k(p) - 3)e_p < 2d$. They lie in the ideal I_λ, $\lambda = \varepsilon^{-(s-3)}$, of the line of Step 1. Small ε corresponds to large λ, far from A; but Proposition 2.24 applies to the whole λ-line, an irreducible family, and shows that all its algebras are smoothable, among them $A = S/I_0$. □

In other words: over a point p of $\mathfrak{B}$, three general surfaces through Z cannot be separated by their cubic terms, and the deformation separates them by terms of degree $k(p)$: the third equation, restricted to the line of p, vanishes at the origin to order $k(p)$ instead of three, and this line has the multiplicity e_p in the cone (Figure 1). A point at which all forms of the tangent cone of degree less than s vanish still has $k(p) = s$: the cubic tails of the lifts of the top forms are free, and they move the values of degree s (Step 1). When $\mathfrak{m}^5 \subseteq I$, every point has $k(p) = 4$, and the condition reads $27 + e(\mathfrak{B}) < 2d$.

The weights are large for the Hilbert functions governed by points of the plane: if $D_k = \nu_k(\Gamma)$ for a plateau of degrees k (Lemma 2.16), then $P^k \in D_k$ for every point P of Γ and every such k. Excess linkage still applies when the length is large compared with the weights (Corollary 3.14 below); otherwise we use rays (Section 5), or a single example together with Proposition 2.24.

Example 3.13. Let $D_3 = \langle X^3, X^2Y, X^2Z, G\rangle$ with a general cubic G, and $D_4 = \langle X^4, X^3Y, X^3Z\rangle$, so that $H = (1, 3, 6, 4, 3)$ and $d = 17$. Then $\mathfrak{B}$ is the point $p = [1 : 0 : 0]$ with local ideal $\mathfrak{m}_p^2$, so $e_p = 4$, and $k(p) = s = 4$. The example shows why Theorem 3.12 deforms A. Take first the graded algebra, with tail $\tau = 0$, so that $I = I^*$. Every cubic and every quartic of I vanishes to order at least two at p, as $D_3 \supseteq X^2T_1$ and $D_4 = X^3T_1$; so only the terms of degree five separate the surfaces along the line of p, and Step 2 of the proof of Theorem 3.12 gives only the bound $27 + 2 \cdot 4 = 35 > 2d$. The bound is attained: for $G = Y^3 + Z^3$, for instance, the elements $xy^2 - yz^2$, $xz^2 - y^2z$ and $xyz + y^3 - z^3 + x^5$ of I define a complete intersection of local length 35 (script `excess_graded.sage`). For general cubic tails, or for a suitable member of the tail deformation of Step 1, the quartic terms do not vanish at p, and three general elements of order three of the ideal define a complete intersection of local length at most $31 = 27 + 4$, with a residual scheme of length at most 14. Script `excess.sage` computes the local lengths for random tails in four examples, with $\mathfrak{B}$ the fat point above, a curvilinear point of length three, and points with local ideals (u^2, v^2) and (u^2, uv, v^3), and finds $27 + e(\mathfrak{B})$ in each.

Theorem 3.12 is a numerical test, and it settles at once most algebras whose cubics and quartics are governed by a subscheme of the plane of length two or three, for every socle degree.

Corollary 3.14 (Two points of the plane)**.** *Let $A = S/I$ be a local algebra of length d with $H_A(2) = 6$, $H_A(3) = c \leq 4$ and socle degree s, and suppose that every finite subscheme of $\mathbb{A}^3$ of length less than d is smoothable. Suppose that $D_4 = \nu_4(\Gamma)$ for a subscheme $\Gamma \subset \mathbb{P}^2$ of length two, and that the cubic base scheme $\mathfrak{B}$ is finite. Then A is smoothable in each of the following cases:*

1. *near the support of Γ, the scheme $\mathfrak{B}$ coincides with Γ, and $2s+c+19 < 2d$;*
2. *near the support of Γ, the scheme $\mathfrak{B}$ is curvilinear of length three, and $3s + c + 15 < 2d$.*

In particular, every local algebra with $H_A(2) = 6$ and $H_A(3) = H_A(4) = H_A(5) = 2$ is smoothable if all shorter schemes are.

Proof. The space D_3 contains $S_1 \circ \nu_4(\Gamma) = \nu_3(\Gamma)$, so the cubics of I_3^* vanish on Γ, and $\Gamma \subseteq \mathfrak{B}$. The length of $\mathfrak{B}$ is at most c: if $c \leq 3$, the ideal K generated by I_3^* has $H_{S/K}(3) = c$, so $H_{S/K}(j) \leq c$ for $j \geq 3$ by Macaulay's bound; if $c = 4$, this is Lemma A.11. At a point $q = [Q]$ of $\mathfrak{B}$ outside the support of Γ, we have $Q^4 \notin \nu_4(\Gamma) = D_4$, because Γ is cut out by a linear form and a quadric; so $k(q) = 4$. In case (1), the points of Γ have multiplicities adding up to two and weights at most $s-3$, and the rest of $\mathfrak{B}$ has length at most $c-2 \leq 2$, so it is curvilinear, with multiplicities adding up to at most $c-2$ and weight one. So the bound of Theorem 3.12 is at most $27 + 2(s-3) + c - 2 = 2s + c + 19$. In case (2) it is at most $27 + 3(s-3) + c - 3 = 3s + c + 15$. Theorem 3.12 applies, as $\dim I_3^* = 10 - c \geq 6$.

For the last assertion, Lemma 2.16 with $k = 3$ and $n = 2$ gives $D_3 = \nu_3(\Gamma)$ and $D_4 = \nu_4(\Gamma)$. Then $I_3^* = (I_\Gamma)_3$ defines $\mathfrak{B} = \Gamma$, and $c = 2$. Since $H_A(k) \geq 1$ for $k \leq s$, we have $d \geq 1+3+6+2+2+2+(s-5) = s+11$, so $2s+21 < 2d$, and (1) applies. □

The weights grow with the socle degree, but so does the length: each further value of the Hilbert function adds one to s and at least one to d, hence two to each side of the inequality. In lengths sixteen to nineteen, the last assertion covers the Hilbert functions $(1,3,6,2,2,2)$, $(1,3,6,2,2,2,1)$, $(1,3,6,2,2,2,2)$, $(1,3,6,2,2,2,1,1)$, $(1,3,6,2,2,2,2,1)$ and $(1,3,6,2,2,2,1,1,1)$, with no ray and no computation of limits over colliding points. It does not apply to $(1,3,6,2,2,1)$, where $27 + 2 \cdot 2 > 30$; there rays are needed (Theorem 5.10). Parts (1) and (2) settle most algebras of the Hilbert functions of Appendix C; for $(1,3,6,3,2,2)$, for instance, the bounds are 32 and 33, both less than 34. What remains are the algebras whose cubic base scheme has a fat point at a point of Γ, of weight $4(s-3)$, and a few others in which $\mathfrak{B}$ is long.

Corollary 3.15 (Three points of the plane)**.** *Let $A = S/I$ be a local algebra of length d with $H_A(2) = 6$, $H_A(3) = H_A(4) = 3$ and socle degree s, and suppose that every finite subscheme of $\mathbb{A}^3$ of length less than d is smoothable. Let $\Gamma \subset \mathbb{P}^2$ be the subscheme of length three with $D_3 = \nu_3(\Gamma)$ and $D_4 = \nu_4(\Gamma)$ (Lemma 2.16). Then A is smoothable if $27 + 4(s-3) < 2d$, or if Γ is curvilinear and $27 + 3(s-3) < 2d$.*

Proof. The ideal of Γ is generated in degrees at most three, so $I_3^* = (I_\Gamma)_3$ defines $\mathfrak{B} = \Gamma$. If Γ is curvilinear, its multiplicities add up to three; otherwise Γ is the fat point $V(\mathfrak{m}_p^2)$, with $e_p = 4$. Every weight is at most $s-3$, and Theorem 3.12 applies. □

The corollary covers $(1,3,6,3,3)$, $(1,3,6,3,3,2)$ and $(1,3,6,3,3,3)$, with bounds $31 < 32$, $35 < 36$ and $35 < 38$, and the algebras with Γ curvilinear for $(1,3,6,3,3,1)$ and $(1,3,6,3,3,2,1)$, with bounds $33 < 34$ and $36 < 38$. A fat point in socle degree six, of weight $3 \cdot 4 = 12$, is too heavy. There a refinement helps: when the top form is a power X^s, the closedness of M often leaves some quartic tails free, and adding such a tail to the lift of X^s lowers by one the weight $s-3$ of every point with $k(p) = s$ (Lemma D.3 and Corollary D.4 in Appendix D).

3.16. For schemes of length nineteen there is a cheaper way around the base points: a complete intersection whose length is computed globally, by Bézout's theorem, so that no local multiplicity is needed.

Lemma 3.17. *Let $w_1, w_2, w_3 \in S_3$ and $L \in S_1$, where w_1 and w_2 have no common factor and L vanishes at no point of the finite subscheme $V(w_1, w_2)$ of $\mathbb{P}^2$. Then $V(w_1, w_2, w_3 + L^4) \subset \mathbb{A}^3$ is a finite complete intersection of length* 36*.*

Proof. Homogenize with a new variable h. In $\mathbb{P}^3$ the cubics w_1, w_2 define the cone over the finite scheme $V(w_1, w_2) \subset \mathbb{P}^2$ with vertex $[0:0:0:1]$, a complete intersection curve, hence without embedded points, whose components are the lines joining the vertex to the points p of $V(w_1, w_2)$. On such a line, parametrized by $[\sigma p : h]$, the quartic $hw_3 + L^4$ restricts to $\sigma^3(h\, w_3(p) + \sigma L(p)^4)$, which is not identically zero. So $hw_3 + L^4$ is a nonzerodivisor modulo (w_1, w_2), and the three forms define a zero-dimensional complete intersection in $\mathbb{P}^3$ of length $3 \cdot 3 \cdot 4 = 36$. It has no point on the plane $h = 0$, where $w_1 = w_2 = L = 0$ would be needed. So all of it lies in $\mathbb{A}^3$, where it is $V(w_1, w_2, w_3 + L^4)$. □

Proposition 3.18 (Global linkage)**.** *Let $A = S/I$ be a local algebra of length $d \geq 19$ with $H_A(2) = 6$ and $\mathfrak{m}^5 \subseteq I$, and suppose that every finite subscheme of $\mathbb{A}^3$ of length less than d is smoothable. If $\dim I_3^* \geq 3$ and the cubics of I_3^* have no common factor, then A is smoothable.*

Proof. By Lemma 2.7, the algebras with the spaces D_3, D_4 of A form an affine space of maps $\tau\colon D_4 \to T_3$, which contains A. Let I_τ be the ideal of the algebra with tail τ. Since I_τ is the orthogonal complement of its inverse system, a polynomial $h_3 + h_4$ with $h_3 \in S_3$ and $h_4 \in S_4$ lies in I_τ if and only if

$$h_3 \in I_3^* \qquad \text{and} \qquad \langle h_4, F\rangle = -\langle h_3, \tau(F)\rangle \quad \text{for all } F \in D_4. \tag{6}$$

Two general cubics $w_1, w_2 \in I_3^*$ have no common factor: a general w_1 has finitely many irreducible factors, and for each of them the cubics of I_3^*

divisible by it form a proper subspace. Complete them to linearly independent $w_1, w_2, w_3 \in I_3^*$, and choose $L \in S_1$ vanishing at no point of $V(w_1, w_2) \subset \mathbb{P}^2$. As w_1, w_2, w_3 are linearly independent linear forms on T_3, there is τ with $\langle w_1, \tau(F)\rangle = \langle w_2, \tau(F)\rangle = 0$ and $\langle w_3, \tau(F)\rangle = -\langle L^4, F\rangle$ for all $F \in D_4$. By (6), w_1, w_2 and $w_3 + L^4$ lie in I_τ, and by Lemma 3.17 they define a finite complete intersection of length $36 < 2d$ containing $V(I_\tau)$. Proposition 2.24 concludes. □

No condition at the points of the cubic base scheme is needed, in contrast to Theorem 3.12; the residual scheme, of length $36 - d$, need not be local. The same construction works in socle degree five when the tails can still prescribe the values $\langle w_i, \tau(F)\rangle$ (Lemma E.8).

When the cubics of I_3^* do have a common factor, the algebra can often be deformed to one where they do not, by an expected-dimension count; Theorem B.5 in Appendix B shows this on the Hilbert function $(1, 3, 6, 6, 3)$.

4. Surfaces: turning the transverse cubic

4.1. In this section the tangent cone lies on exactly one quadric, up to scalar, and this quadric is the square x^2 of a linear form; by Theorem 2.9(1) this is the only case with $H_A(2) \leq 5$ that needs attention. We extend x to linear coordinates x, y, z. By Theorem 2.9(2), A is smoothable as soon as some element of I with leading form x^2 has a transverse cubic that is not a cube. The transverse cubics of such elements form an affine space [23, Lemma 4.8], and when all of them are cubes we would like to change it. This is possible after deforming I.

Lemma 4.2 (Turning family)**.** *Let I be an ideal of finite colength with $V(I) = \{0\}$ and $I_2^* = \mathbb{C}x^2$, let $f \in I$ have leading form x^2, and let $u \in \mathfrak{m}^3$ satisfy $\mathfrak{m}^2 u \subseteq I$. Then there is an ideal $\mathcal{I} \subseteq S[t]$ such that $S[t]/\mathcal{I}$ is a free $\mathbb{C}[t]$-module of rank $\ell(S/I)$, $\mathcal{I}|_{t=0} = I$, and for every $t_0 \in \mathbb{C}$ the ideal $\mathcal{I}|_{t=t_0}$ contains $f + t_0 u$ and $I \cap \mathfrak{m}^4$.*

Proof. Since $x^2 \in I^*$, the space I_3^* contains $x^2 S_1$; choose a complement W of $x^2 S_1$ in I_3^*, and let

$$K = \{g \in I \cap \mathfrak{m}^3 : \text{ the cubic part of } g \text{ lies in } W\}.$$

This is an ideal: if $g \in K$ and $h \in S$, the cubic part of hg is $h(0)$ times that of g. It contains $I \cap \mathfrak{m}^4$, and $I = K + (f)$: an element of I of order two becomes, after subtracting a multiple of f, an element of order at least three, whose cubic part lies in $I_3^* = x^2 S_1 \oplus W$; subtracting vf for a suitable $v \in S_1$ moves it into K.

Let $\mathcal{I} = KS[t] + (f + tu)S[t]$, so that $\mathcal{I}|_{t=t_0} = K + (f + t_0 u)$ and $\mathcal{I}|_{t=0} = I$. Fix t_0 and let $S' = S/K$. Since $\mathfrak{m}^2 f$ and $\mathfrak{m}^2 u$ lie in $I \cap \mathfrak{m}^4 \subseteq K$, the ideal $(f + t_0 u)S'$ is spanned by the classes of $f + t_0 u$ and $x(f + t_0 u)$, $y(f + t_0 u)$, $z(f + t_0 u)$. These four classes are linearly independent. Indeed, if $\alpha(f + t_0 u) + v(f + t_0 u) \in K$ with $\alpha \in \mathbb{C}$ and $v \in S_1$, then $\alpha = 0$ because

$K \subseteq \mathfrak{m}^3$, and then the cubic part vx^2 lies in W, so $v = 0$. Hence $K+(f+t_0u)$ has colength $\ell(S') - 4$ for every t_0, and this is $\ell(S/I)$ by the case $t_0 = 0$. Since K contains a power of $\mathfrak{m}$, Lemma 2.3(1) applies. □

The condition $\mathfrak{m}^2 u \subseteq I$ says that u lies in the ideal quotient $(I : \mathfrak{m}^2)$. Every element of $I \cap \mathfrak{m}^3$ satisfies it, and so does every element of $\mathfrak{m}^{s-1}$. In terms of the inverse system, $u \in (I : \mathfrak{m}^2)$ if and only if $\langle u, \mathfrak{m}^2 \circ M\rangle = 0$, since $\mathfrak{m}^2 \circ M$ is the inverse system of $(I : \mathfrak{m}^2)$.

4.3. The deformed ideal contains $f+tu$, and the type of the surface $f+tu = 0$ is read off from its transverse equation. We first record the recognition of the singularities of type E by weights; it replaces the hypothesis $\mathfrak{m}^5 \subseteq I$ of Theorem 2.9(3). Let g_0 be one of $y^3 + z^4$, $y^3 + yz^3$, $y^3 + z^5$, and give y and z the weights $\operatorname{wt}(y) = \frac{1}{3}$ and $\operatorname{wt}(z) = \frac{1}{4}$, $\frac{2}{9}$ or $\frac{1}{5}$, respectively; the weight of a monomial $y^i z^j$ is $i \operatorname{wt}(y) + j \operatorname{wt}(z)$. Then g_0 is homogeneous of weight one, and its Jacobian ideal $(\partial_y g_0, \partial_z g_0)$ is homogeneous for the weights. The Milnor algebra $\mathbb{C}[y,z]/(\partial_y g_0, \partial_z g_0)$ has the monomial basis

$$y^i z^j \ (i \le 1,\ j \le 2), \qquad 1, y, z, yz, z^2, z^3, z^4, \qquad y^i z^j \ (i \le 1,\ j \le 3),$$

for the Jacobian ideals (y^2, z^3), $(3y^2 + z^3, yz^2)$ and (y^2, z^4), of the largest weights $\frac{5}{6}$, $\frac{8}{9}$ and $\frac{14}{15}$, all less than one (script `ade_weights.sage`). Consequently, every monomial of weight greater than one lies in the Jacobian ideal, and it is a combination $a\,\partial_y g_0 + b\,\partial_z g_0$ with a and b homogeneous for the weights. There are no normal-form parameters of weight greater than one; this is the classical argument behind the normal forms of semiquasihomogeneous functions, see [12, Lemma 1.2(i),(ii)].

Lemma 4.4. *Let $\mathfrak{m}^N \subseteq I$, and suppose that, in suitable formal coordinates, $I\widehat{S}$ contains a unit multiple of $x^2 + g(y,z)$, where all monomials of g have weight at least one and the part of weight one is $\alpha y^3 + \beta g_1$ with $\alpha\beta \neq 0$, for one of the following: $g_1 = z^4$ with weights $(\frac{1}{3}, \frac{1}{4})$, $g_1 = yz^3$ with weights $(\frac{1}{3}, \frac{2}{9})$, $g_1 = z^5$ with weights $(\frac{1}{3}, \frac{1}{5})$. Then I contains an equation of type E_6, E_7 or E_8, respectively.*

Proof. After rescaling y and z we may assume $\alpha = \beta = 1$; let $g_0 = y^3 + g_1$ and $g = g_0 + r$. We change the coordinates y, z formally, keeping x. Let $\delta > 1$ be the smallest weight of a monomial of r, and write the part of r of weight δ as $a\,\partial_y g_0 + b\,\partial_z g_0$ as above, with a and b of weights $\delta - 1 + \operatorname{wt}(y)$ and $\delta - 1 + \operatorname{wt}(z)$. Substituting $y - a$ for y and $z - b$ for z, which is a formal change of coordinates, removes this part: the quadratic terms of the Taylor expansion of g_0 have weight at least $2\delta - 1 > \delta$, and r changes only by terms of weight greater than δ. Repeating, we reach $g \equiv g_0$ modulo $\widehat{\mathfrak{m}}^N$, since only finitely many weights occur below any given degree. Since $\widehat{\mathfrak{m}}^N$ lies in the transformed ideal $I\widehat{S}$, so does $x^2 + g_0$, which is E_6, E_7 or E_8. □

Theorem 4.5 (Transverse jets)**.** *Let $A = S/I$ be a local algebra with $I_2^* = \mathbb{C}x^2$, let $f \in I$ have leading form x^2, and choose formal coordinates, fixing*

y and z, in which $f = x^2 + g(y,z)$. For $u \in (I : \mathfrak{m}^2) \cap \mathfrak{m}^3$, written in these coordinates, and $t \in \mathbb{C}$, let

$$G_t = g + t\,u(0,y,z) - \tfrac{t^2}{4}\Big(\frac{\partial u}{\partial x}(0,y,z)\Big)^2.$$

Suppose that for all but finitely many t, after a linear change of y and z, the terms of G_t of degree at most four, and in case (4) the indicated term of degree five, satisfy one of the following, with $\alpha\beta \neq 0$:

1. *the cubic part is not a cube;*
2. *the cubic part is αy^3 and the coefficient of z^4 is β;*
3. *the cubic part is αy^3, the coefficient of z^4 is zero, and the coefficient of yz^3 is β;*
4. *the cubic part is αy^3, the coefficients of z^4 and yz^3 are zero, and the coefficient of z^5 in $G_t + \frac{t^3}{8}\big(\frac{\partial u}{\partial x}\big)^2\frac{\partial^2 u}{\partial x^2}(0,y,z)$ is β.*

Then A is smoothable.

Proof. Consider the family of Lemma 4.2; its member $\mathcal{I}|_{t=t_0}$ contains $f + t_0 u$. Write $h = x^2 + g + t_0 u$, and let $a = \partial_x u(0,y,z)$ and $b = \partial_x^2 u(0,y,z)$, of orders at least two and one. Since $\partial_x u$ has order at least two, the equation $\partial_x h = 0$ has a solution $x = \xi(y,z)$ with $\xi = -\frac{t_0}{2}a + \xi'$, where ξ' has order at least three, and $h = h(\xi,y,z) + (x-\xi)^2\upsilon$ with a unit υ, as in the proof of [23, Proposition 4.5]. Expanding u in powers of x and using $\xi^2 + t_0 a\xi = \xi'^2 - \frac{t_0^2}{4}a^2$, we find

$$h(\xi,y,z) \equiv g + t_0 u(0,y,z) - \tfrac{t_0^2}{4}a^2 + \tfrac{t_0^3}{8}a^2 b = G_{t_0} + \tfrac{t_0^3}{8}a^2 b \pmod{(y,z)^6},$$

since ξ'^2, $ab\xi'$ and ξ^3 have order at least six. As $a^2 b$ has order at least five, $h(\xi,y,z) \equiv G_{t_0}$ modulo $(y,z)^5$. In case (1), $\mathcal{I}|_{t=t_0}$ contains an equation of type D by Theorem 2.9(2). In cases (2) and (3), all monomials of degree at least five have weight greater than one for the weights $(\frac{1}{3},\frac{1}{4})$, respectively $(\frac{1}{3},\frac{2}{9})$, and the hypotheses say that the monomials of weight at most one in $h(\xi,y,z)$ are αy^3 and βz^4, respectively βyz^3. So Lemma 4.4 gives an equation of type E_6 or E_7 in $\mathcal{I}|_{t=t_0}$. In case (4) we use the weights $(\frac{1}{3},\frac{1}{5})$; the monomials of degree at least six then have weight greater than one, and so do the quintic monomials other than z^5 and the quartic monomials other than z^4 and yz^3. So the congruence modulo $(y,z)^6$ and Lemma 4.4 give an equation of type E_8. In all cases $\mathcal{I}|_{t=t_0}$ is smoothable for all but finitely many t_0, and so is I by Proposition 2.2(1). □

The theorem turns the problem into linear algebra. By the remark after Lemma 4.2, the elements u form the orthogonal complement of $\mathfrak{m}^2 \circ M$, and only their parts of degree three and four enter, together with the part of degree five in case (4). We now derive the cases used in this paper.

Corollary 4.6. *Let $A = S/I$ be a local algebra with $I_2^* = \mathbb{C}x^2$. Suppose that $(I : \mathfrak{m}^2)$ contains an element of $\mathfrak{m}^3$ whose cubic part, restricted to the plane $x = 0$, is not a cube. Then A is smoothable.*

Proof. Let δ be that restriction. The cubic part of G_t is $\gamma_f + t\delta$. The cubes form a closed cone in $\mathbb{C}[y,z]_3$. If $\gamma_f + t\delta$ were a cube for all t, then so would be δ, the limit of $(\gamma_f + t\delta)/t$ as $t \to \infty$. So case (1) of Theorem 4.5 holds. □

Corollary 4.7. *Every local algebra $A = S/I$ with $H_A(2) \leq 5$ and $\mathfrak{m}^5 \subseteq I$ is smoothable.*

Proof. By Theorem 2.9(1) we may assume that $I_2^* = \mathbb{C}x^2$. Since $\mathfrak{m}^5 \subseteq I$, the element $u = y^3 + z^3$ satisfies $\mathfrak{m}^2 u \subseteq I$, and $y^3 + z^3$ is not a cube. □

No induction on the length is needed here. Corollary 4.7 contains [23, Lemma 7.2], which treated the Hilbert functions $(1,3,5,k)$ by a different family.

Corollary 4.8. *Let $A = S/I$ be a local algebra with $I_2^* = \mathbb{C}x^2$ and $\mathfrak{m}^7 \subseteq I$. Suppose that $(I : \mathfrak{m}^2)$ contains an element of $\mathfrak{m}^3$ whose cubic part does not vanish identically on the plane $x = 0$. Then A is smoothable.*

Proof. Let u be such an element and $\delta \neq 0$ the restriction of its cubic part to the plane $x = 0$. If δ is not a cube, Corollary 4.6 applies, and if γ_f is not a cube, Theorem 2.9(2) does. So let $\delta = v^3$ and $\gamma_f = v'^3$ with linear forms v, v'. If v' is not a multiple of v, then $v'^3 + tv^3$ has three distinct roots for $t \neq 0$, and case (1) of Theorem 4.5 holds. So we may assume $v = y$ and $\gamma_f = ay^3$; then the cubic part of G_t is $(a+t)y^3$. Its coefficients of z^4 and yz^3 are polynomials in t. If one of them is not identically zero, case (2) or (3) of Theorem 4.5 holds. Otherwise, for $b \in \mathbb{C}$ the element $u + bz^5$ also lies in $(I : \mathfrak{m}^2)$, because $\mathfrak{m}^2 z^5 \subseteq \mathfrak{m}^7 \subseteq I$. The monomial z^5 is not affected by the coordinate change of Theorem 4.5, which fixes y and z, and $\partial_x(z^5) = 0$. So the replacement adds $tb\,z^5$ to the polynomial of case (4) and changes nothing else; for all but one value of b case (4) holds. □

If $\mathfrak{m}^6 \subseteq I$, the types E_7 and E_8 are not needed: then $z^4 \in (I : \mathfrak{m}^2)$, and replacing u by $u + bz^4$ adds $tb\,z^4$ to G_t, so that case (2) holds for all but one value of b. This applies in Corollaries 4.9 and 4.10 below.

In particular, if $\mathfrak{m}^7 \subseteq I$, then Corollary 4.8 applies to any element of I whose leading form is a cubic not divisible by x. So a local algebra with $I_2^* = \mathbb{C}x^2$ and $\mathfrak{m}^7 \subseteq I$ is smoothable unless $I_3^* \subseteq xS_2$, that is, unless D_3 contains all cubic forms in Y and Z. We call this remaining case the *binary case*. If moreover $H_A(3) = 4$, then $D_3 = \mathbb{C}[Y,Z]_3$, and all spaces D_k, $k \geq 3$, consist of forms in Y and Z, because their derivatives do.

When the socle degree is five, $(I : \mathfrak{m}^2)$ can be read off from the top forms. Indeed, if $\mathfrak{m}^6 \subseteq I$ and $h \in S_3$ satisfies $h \circ D_5 = 0$, then $hq \in S_5$ is orthogonal to D_5 for every $q \in S_2$, so $hq \in I_5^*$ by Lemma 2.5(2), and $hq \in I$ because $\mathfrak{m}^6 \subseteq I$. Since also $h\,\mathfrak{m}^3 \subseteq \mathfrak{m}^6 \subseteq I$, we get $h \in (I : \mathfrak{m}^2)$.

Corollary 4.9. *Every local algebra $A = S/I$ with $H_A(2) = 5$, $H_A(5) = 1$ and $\mathfrak{m}^6 \subseteq I$ is smoothable.*

Proof. By Theorem 2.9(1) we may assume $I_2^* = \mathbb{C}x^2$. Let F span D_5. Since $x^2 \circ F = 0$, the form F has degree at most one in X, so $F = F_0 + XF_1$ with $F_0 \in \mathbb{C}[Y,Z]_5$ and $F_1 \in \mathbb{C}[Y,Z]_4$. For $h = h' + xq' + x^2r$ with $h' \in \mathbb{C}[y,z]_3$, $q' \in \mathbb{C}[y,z]_2$ and $r \in S_1$ we have

$$h \circ F = (h' \circ F_0 + q' \circ F_1) + X\,(h' \circ F_1).$$

So there is such h with $h \circ F = 0$ if and only if $h' \circ F_1 = 0$ and $h' \circ F_0$ lies in the space $\{q' \circ F_1 : q' \in \mathbb{C}[y,z]_2\}$. Let a and b be the ranks of the maps $\mathbb{C}[y,z]_3 \to \mathbb{C}[Y,Z]_1$ and $\mathbb{C}[y,z]_2 \to \mathbb{C}[Y,Z]_2$ given by $h' \mapsto h' \circ F_1$ and $q' \mapsto q' \circ F_1$. These are a and $3-b$ linear conditions on h' in a space of dimension four, and $a \le b$: if $F_1 = 0$ then $a = b = 0$; if F_1 is a fourth power then $a = b = 1$; and otherwise F_1 has two essential variables, so $a = 2$, and $b \ge 2$: if the second derivatives of F_1 spanned a line $\mathbb{C}Q$, both first derivatives of F_1 would be cubes of linear forms whose squares are multiples of Q, hence they would be proportional. Hence there is a solution with $h' \neq 0$. The corresponding h lies in $(I : \mathfrak{m}^2)$, and it restricts to $h' \neq 0$ on the plane $x = 0$. Corollary 4.8 applies. □

In length fifteen, Corollary 4.9 covers $(1,3,5,4,1,1)$; Table 1 counts the Hilbert functions it covers in each length.

Corollary 4.10. *Every local algebra $A = S/I$ with $H_A(2) = 5$, $\mathfrak{m}^6 \subseteq I$ and $1 \le H_A(4) = H_A(5) \le 3$ is smoothable.*

Proof. By Theorem 2.9(1) we may assume $I_2^* = \mathbb{C}x^2$. Let $n = H_A(4)$. By Lemma 2.16 with $k = 4$, $D_5 = \nu_5(\Gamma)$ for a subscheme $\Gamma \subset \mathbb{P}^2$ of length n. Every cubic $h \in (I_\Gamma)_3$ satisfies $hS_2 \subseteq (I_\Gamma)_5$, hence $h \circ D_5 = 0$ and $h \in (I : \mathfrak{m}^2)$. As $\dim (I_\Gamma)_3 = 10 - n > 6 = \dim xS_2$, some such h does not vanish identically on the plane $x = 0$, and Corollary 4.8 applies. □

When the socle degree is six or seven, the cubic part of $(I : \mathfrak{m}^2)$ is no longer determined by the top forms alone; it depends on the tails. It can still be found when the Hilbert function ends with a short plateau.

Proposition 4.11 (Tails on one quadric)**.** *Let $A = S/I$ be a local algebra with $H_A(2) = 5$. Then A is smoothable in each of the following cases:*

1. *$\mathfrak{m}^7 \subseteq I$ and $H_A(5) = H_A(6) = 1$;*
2. *$\mathfrak{m}^7 \subseteq I$, $H_A(4) = H_A(5) = 2$ and $H_A(6) \neq 0$;*
3. *A has socle degree seven and $H_A(5) = H_A(6) = H_A(7) = 1$;*
4. *A has socle degree seven, $H_A(4) = H_A(5) = 2$ and $H_A(6) = H_A(7) = 1$.*

No induction on the length is needed. In each case we find an element u of $(I : \mathfrak{m}^2) \cap \mathfrak{m}^3$ whose cubic part does not vanish on the plane $x = 0$, and apply Theorem 4.5, through Corollary 4.6 or 4.8 when possible. The proofs are in Appendix F. The idea is most visible in (2) when $H_A(6) = 2$: the plateau makes the top forms those of a scheme Γ of length two supported on the line $x = 0$, a cubic u_3 of the ideal of Γ that is not a cube on this line

annihilates all top forms of degree at least four, and a quartic correction u_4 kills the remaining tails, so that $u_3 + u_4 \in (I : \mathfrak{m}^2)$.

In length sixteen, Proposition 4.11 covers $(1,3,5,4,1,1,1)$; in length eighteen, for example, $(1,3,5,4,3,1,1)$, $(1,3,5,4,2,2,1)$ and $(1,3,5,4,2,1,1,1)$; and in length nineteen, for example, $(1,3,5,4,2,2,2)$ and $(1,3,5,4,2,2,1,1)$.

Theorem 4.12. *Every local algebra A with $H_A(2) = 5$ and $H_A(3) = 3$ is smoothable.*

Proof. By Theorem 2.9(1) we may assume $I_2^* = \mathbb{C}x^2$. Let $\pi\colon S_3 \to \mathbb{C}[y,z]_3$ be the restriction to the plane $x = 0$. The transverse cubics of the elements of I with leading form x^2 form the affine space $\gamma_f + \pi(I_3^*)$, and $\dim \pi(I_3^*) \geq \dim I_3^* - 6 = 1$ [23, Lemma 4.8]. If this affine space contains a non-cube, Theorem 2.9(2) applies. Otherwise, as in the proof of [23, Lemma 8.3], the space $\pi(I_3^*)$ is a line $\mathbb{C}v^3$ spanned by a cube and $\gamma_f \in \mathbb{C}v^3$. In coordinates with $y = v$ we may then choose f with $\gamma_f = y^3$. Moreover, I_3^* meets the kernel xS_2 of π in a space of dimension $7-1 = 6$, so $xS_2 \subseteq I_3^*$, and I contains an element r with leading form xz^2.

Apply Theorem 4.5 with $u = r$, which lies in I and hence in $(I : \mathfrak{m}^2)$. In the formal coordinates of the theorem, r still has leading form xz^2, because the change of coordinates has linear part the identity. So $r(0,y,z)$ has order at least four, and $\partial_x r(0,y,z) = z^2 + (\text{higher terms})$. Hence G_t has cubic part y^3 and z^4-coefficient

$$a_0 + a_1 t - \tfrac{1}{4}t^2,$$

where a_0 and a_1 are the coefficients of z^4 in g and in $r(0,y,z)$. This polynomial in t is not identically zero, so case (2) of Theorem 4.5 holds. □

Here the ideal is not deformed: if the complement W in the proof of Lemma 4.2 is chosen to contain xz^2, then $r \in K$, and the family is constant. The pencil $f + tr$ consists of elements of one and the same ideal, among which one defines a surface with an E_6 singularity. In length fifteen, Theorem 4.12 covers $(1,3,5,3,3)$, $(1,3,5,3,2,1)$ and $(1,3,5,3,1,1,1)$.

5. Rays and tail families

5.1. When the tangent cone lies on no quadric, the cubes P^3 in D_3 correspond to the lines through the origin on which all cubics of the tangent cone vanish, by (1). They are natural paths along which a point can leave Z. This section finds rays along such lines and along curves tangent to them, and then controls the tails in the families through which we pass to limits.

Lemma 5.2. *Let $A = S/I$ be a local algebra with $H_A(2) = 6$ and $\mathfrak{m}^5 \subseteq I$, and let $P \in T_1$ be nonzero with $P^3 \in D_3$. If $P^3 \notin N_P \circ D_4$, then L_P is a ray of I.*

Proof. By Lemma 2.7, $P^3 \in M$, and $N_P \circ M$ is spanned by $N_P \circ T_{\leq 2}$, $N_P \circ D_3$ and the elements $v \circ F + v \circ \tau(F)$ with $v \in N_P$ and $F \in D_4$. The cubic part

of each element of $N_P \circ M$ therefore lies in $N_P \circ D_4$. So $P^3 \notin N_P \circ M$, and Theorem 2.11(2) applies. □

The condition depends only on the tangent cone, not on the map τ. For example, if D_4 is spanned by fourth powers of linear forms $P_1, \dots, P_n$ whose cubes are linearly independent, and $P = P_1$, then $N_P \circ D_4$ is spanned by the cubes $P_2^3, \dots, P_n^3$, and the lemma applies.

5.3. Sometimes the point has to leave along a curve. The following variant of Theorem 2.11(2) detects such rays; rays along curves of this kind already appear in [23, Lemma 8.2].

Lemma 5.4. *Let a, b, c be polynomial coordinates centered at the origin, $k \geq 1$, and suppose that $a^k b$ and $a^k c$ lie in I. If the inverse system of I contains an element Ψ with $b \circ \Psi = c \circ \Psi = 0$ and $\langle a^k, \Psi \rangle \neq 0$, then (b, c) is a ray of I.*

Proof. The linear form $\langle \cdot, \Psi \rangle$ vanishes on I, and on (b, c) because $\langle bh, \Psi \rangle = \langle h, b \circ \Psi \rangle = 0$ and similarly for c. It does not vanish at a^k. So $a^k \in (I : (b, c))$ but $a^k \notin I + (b, c)$. □

The element Ψ can be read off from the curve $\Lambda = V(b, c)$. Take $a = x$, $b = y - \beta(x)$ and $c = z - \gamma(x)$ for polynomials $\beta, \gamma \in \mathbb{C}[t]$ divisible by t^2, so that Λ is parametrized by $t \mapsto (t, \beta(t), \gamma(t))$. For $k \geq 0$ let $E_k(\Lambda) \in T$ be the polynomial with

$$\langle g, E_k(\Lambda) \rangle = \frac{d^k}{dt^k}\, g(t, \beta(t), \gamma(t))\Big|_{t=0} \qquad \text{for all } g \in S.$$

Since b and c vanish on Λ, we have $b \circ E_k(\Lambda) = c \circ E_k(\Lambda) = 0$, and $\langle a^k, E_k(\Lambda) \rangle = k!$. Writing $\beta = \sum \beta_i t^i$ and $\gamma = \sum \gamma_i t^i$, and comparing coefficients, one finds

$$\begin{aligned} E_3(\Lambda) &= X^3 + 6X(\beta_2 Y + \gamma_2 Z) + 6(\beta_3 Y + \gamma_3 Z), \\ E_4(\Lambda) &= X^4 + 12X^2(\beta_2 Y + \gamma_2 Z) + (\text{terms of degree at most two}). \end{aligned} \tag{7}$$

In other words, $E_k(\Lambda)$ is the k-th jet of the curve Λ at the origin, seen as a functional. More precisely, let $\Lambda_k = V(b, c, a^{k+1})$ be the curvilinear subscheme of length $k+1$ of Λ at the origin; its ideal consists of the g for which $g(t, \beta(t), \gamma(t))$ is divisible by t^{k+1}. This ideal is the annihilator of $E_k(\Lambda)$, and the derivatives of $E_k(\Lambda)$ span the inverse system of Λ_k, with basis $E_0(\Lambda), \dots, E_k(\Lambda)$. So the inverse system of the jet Λ_k is generated by $E_k(\Lambda)$, and $E_k(\Lambda) \in M$ says that $\Lambda_k \subseteq Z$. Containing a curvilinear jet does not by itself split off a point; the conditions $a^k b, a^k c \in I$ of Lemma 5.4 supply the deformation, along which the end of the jet moves away along Λ. Lemma 5.4 gives:

Corollary 5.5. *If $x^k b$ and $x^k c$ lie in I and $E_k(\Lambda)$ lies in the inverse system of I, then (b, c) is a ray of I, and a point leaves $V(I)$ along the curve Λ.*

When $H_A(2) = 6$, the inverse system contains $T_{\leq 2}$, so by (7) only the parts of degree at least three of $E_3(\Lambda)$ and $E_4(\Lambda)$ matter.

Rays found by operators of order at least four survive when a cubic is added to the inverse system, and conversely many algebras arise from shorter ones by adding a cubic; Lemmas A.14 and A.15 in Appendix A make this precise.

5.6. We now show the simplest way in which points of the plane govern a Hilbert function: the cubics and quartics of the tangent cone are those of a plane scheme Γ, and a point leaves Z along the line of any point of Γ when Γ is reduced.

Proposition 5.7 (Plane directions)**.** *Let $A = S/I$ be a local algebra with $H_A(2) = 6$ and $\mathfrak{m}^5 \subseteq I$. Suppose that $D_3 = \nu_3(\Gamma)$ and $D_4 \subseteq \nu_4(\Gamma)$ for a subscheme $\Gamma \subset \mathbb{P}^2$ of length n whose points impose independent conditions on cubics, that is, $\dim \nu_3(\Gamma) = n$. Then A is a limit of algebras with the same Hilbert function that have a ray along the line of a point. In particular, A is smoothable if all finite subschemes of length $\ell(A) - 1$ are.*

Proof. We deform Γ to reduced schemes, carrying the tails along, and find a ray when Γ is reduced. Let $k = \dim D_4$. The subschemes Γ of length n with $\dim \nu_3(\Gamma) = n$ form an open subset $\mathcal{U}$ of $\mathrm{Hilb}_n(\mathbb{P}^2)$, by semicontinuity, which is irreducible [8, Proposition 2.3 and Theorem 2.4] and contains Γ; the reduced subschemes are dense in it. Since the Hilbert function of Γ is nondecreasing and bounded by n, also $\dim \nu_4(\Gamma) = n$ on $\mathcal{U}$, and ν_3, ν_4 are vector bundles of rank n over $\mathcal{U}$, as in the proof of Lemma 2.14. The pairs (Γ, D_4) with D_4 a k-dimensional subspace of $\nu_4(\Gamma)$ form a Grassmann bundle over $\mathcal{U}$, and $S_1 \circ D_4 \subseteq S_1 \circ \nu_4(\Gamma) \subseteq \nu_3(\Gamma)$. By Lemma 2.7, the algebras with these $D_3 = \nu_3(\Gamma)$ and D_4, and arbitrary τ, form an irreducible family containing A, in which the algebras with Γ reduced are dense. Let Γ consist of n distinct points $P_1, \dots, P_n$. Then $P_1^3 \in D_3$, and $N_{P_1} \circ D_4 \subseteq N_{P_1} \circ \nu_4(\Gamma)$ is spanned by the cubes $P_2^3, \dots, P_n^3$, because $N_{P_1} \circ P_1^4 = 0$ and $v \circ P_i^4 = 4v(p_i)P_i^3$, where p_i is the coefficient vector of P_i. As the cubes of the points of Γ are linearly independent, $P_1^3 \notin N_{P_1} \circ D_4$, and Lemma 5.2 shows that the line of P_1 is a ray. Theorem 2.11(1) and Proposition 2.2(1) conclude. □

Corollary 5.8. *Every local algebra with Hilbert function $(1, 3, 6, 3, 3)$ is smoothable.*

Proof. By Lemma 2.16 with $n = k = 3$, there is a subscheme $\Gamma \subset \mathbb{P}^2$ of length three with $D_3 = \nu_3(\Gamma)$ and $D_4 = \nu_4(\Gamma)$, and $\dim \nu_3(\Gamma) = 3$ by Lemma 2.14. Proposition 5.7 applies, together with the case of fifteen points. □

Thus when Γ consists of three points, the forms of degree three and four of the tangent cone of Z vanish on the three lines through these points, and a point of Z can leave along each of them (Figure 2). When the points collide, the lines come together, and the rays may disappear; but these schemes are limits of schemes with three such lines. (Corollary 3.15 gives a second proof

of Corollary 5.8, by excess linkage with the bound $31 < 32$; we keep this one because it shows the rays.) The same pattern, a finite subscheme of the plane provided by Lemma 2.16, rays when it is reduced, and limits otherwise, recurs throughout the appendices. The Hilbert functions $(1, 3, 6, c, k)$ with $c \in \{4, 5\}$ and $k \in \{3, 4\}$ are treated in the same way (Theorem B.3), by Proposition 5.7 or its analogue for points of a line, or by excess linkage when the cubic base scheme is too short to carry the plane directions.

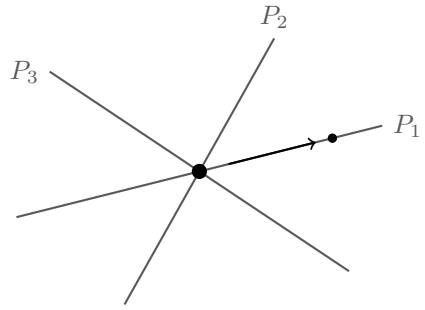


FIGURE 2. For $(1, 3, 6, 3, 3)$ with $\Gamma = \{P_1, P_2, P_3\}$ reduced, the cubics and quartics of the tangent cone of Z vanish on three lines through the origin. A point leaves Z along any of them; here along the line of P_1.

5.9. We apply these lemmas to the Hilbert function $(1, 3, 6, 2, 2, 1)$. By Lemma 2.15, $D_3 = \nu_3(\Gamma)$ and $D_4 = \nu_4(\Gamma)$ for a subscheme $\Gamma \subset \mathbb{P}^2$ of length two. The quintic F spanning D_5 has its derivatives in D_4, so it lies in $\nu_5(\Gamma)$: in the coordinates of Lemma 2.15, $q \circ (S_1 \circ F) = 0$ and $z \circ (S_1 \circ F) = 0$ imply $q \circ F = 0$ and $z \circ F = 0$, because a form all of whose derivatives vanish is zero. As in Lemma 2.7, the inverse system of A is

$$M = T_{\le 2} \oplus D_3 \oplus \{Q + \tau(Q) : Q \in D_4\} \oplus \mathbb{C}(F + R + E), \tag{8}$$

where $\tau\colon D_4 \to T_3$ is linear, $R \in T_4$ and $E \in T_3$. A space of the form (8) is closed under differentiation exactly when

$$v \circ R - \tau(v \circ F) \in D_3 \qquad \text{for all } v \in S_1, \tag{9}$$

since $v \circ F \in D_4$ and $v \circ E \in T_2$. Here the part $M \cap T_{\le 4}$ is the inverse system of $S/(I + \mathfrak{m}^5)$, to which Lemma 2.7 applies, and $F + R + E$ is an element of M with top form F, from which the terms of degree at most two have been removed. Conversely, any such data give an algebra with Hilbert function $(1, 3, 6, 2, 2, 1)$. Here τ, R and E are the tails (Section 2.6).

Theorem 5.10. *Every local algebra with Hilbert function* $(1, 3, 6, 2, 2, 1)$ *is smoothable.*

Proof. Two lines. Suppose first that Γ consists of two points, so that $D_k = \langle X^k, Y^k \rangle$ for $k = 3, 4$, and $D_5 = \mathbb{C}F$ with $F = \alpha X^5 + \beta Y^5$. By symmetry

we may assume $\alpha \neq 0$. Let $\Phi = F + R + E$ and let x, y, z be the dual coordinates. The operators x^3y and x^5 annihilate $T_{\leq 3}$, X^4 and Y^4; moreover $x^3y \circ F = 0$, so $x^3y \circ \Phi$ is the number $\rho = x^3y \circ R$, while $x^5 \circ \Phi = 120\alpha$. Hence $x^3(y - \beta_2 x^2)$ annihilates M for $\beta_2 = \rho/(120\alpha)$, and similarly $x^3(z - \gamma_2 x^2) \in I$ for a suitable γ_2. For the parabola Λ: $y = \beta_2 x^2$, $z = \gamma_2 x^2$, the jet

$$E_3(\Lambda) = X^3 + 6X(\beta_2 Y + \gamma_2 Z)$$

of (7) lies in M, as $X^3 \in D_3$ and $T_2 \subseteq M$. By Corollary 5.5, the ideal $(y - \beta_2 x^2, z - \gamma_2 x^2)$ is a ray: a point leaves the origin along Λ, tangent to the line of X (Figure 3), and A is smoothable by Theorem 2.11(1) and the case of fourteen points [23].

Limits. By Lemma 2.14, the pairs (Γ, F) with $F \in \nu_5(\Gamma)$ nonzero form the complement of the zero section in a vector bundle of rank two over $\mathrm{Hilb}_2(\mathbb{P}^2)$, which is irreducible [8, Proposition 2.3 and Theorem 2.4]; call it $\mathcal{G}$. Let $\mathcal{G}_\mathrm{p} \subset \mathcal{G}$ be the closed subset where F is a fifth power, that is, a multiple of P^5 for a point P of Γ, and $\mathcal{G}_\mathrm{b}$ its complement. The group GL_3 acts on $\mathcal{G}$, and up to scaling F there are four orbits: Γ consists of two points or is a double point, and F is a fifth power or not. For $u = (\Gamma, F)$, the solutions (τ, R) of (9), with $\tau \in \mathrm{Hom}(D_4, T_3)$ and $R \in T_4$, form a linear space $\mathcal{T}_u$, cut out by linear equations whose coefficients depend regularly on u in local frames of the bundles ν_k: 24 equations in the 35 coefficients of (τ, R). On the representatives $F = X^5 + Y^5$ and $F = X^5$ for $\Gamma = \{[X], [Y]\}$, and $F = X^4Y$ and $F = X^5$ for the double point with $\nu_5(\Gamma) = \langle X^5, X^4Y \rangle$, the equations have ranks 24, 19, 24 and 19 (script `tails136221.sage`). So the dimension of $\mathcal{T}_u$ is 11 on the two orbits in $\mathcal{G}_\mathrm{b}$ and 16 on the two orbits in $\mathcal{G}_\mathrm{p}$. Since the rank is constant on each of these two reduced varieties, the spaces $\mathcal{T}_u$ form a vector bundle over $\mathcal{G}_\mathrm{b}$ and one over $\mathcal{G}_\mathrm{p}$. Both bases are irreducible: $\mathcal{G}_\mathrm{b}$ is open in $\mathcal{G}$, and $\mathcal{G}_\mathrm{p}$ is a $\mathbb{C}^*$-bundle over the marked pairs (Γ, P) of length two, an irreducible variety by Lemma A.2. Local frames of these bundles, together with $E \in T_3$, give families as in Lemma 2.3(2). Hence the algebras with Hilbert function $(1, 3, 6, 2, 2, 1)$ form two irreducible families, and in each of them the algebras over two points are dense. These are smoothable, so all are, by Proposition 2.2(1). □

Why a parabola? The criterion of Theorem 2.11(2) for the line of X asks that $X^3 \notin N_X \circ M$. But $N_X \circ M$ contains $z \circ \Phi$, whose cubic part $z \circ R$ lies in $D_3 = \langle X^3, Y^3 \rangle$ by (9) and may involve X^3, and similarly $y \circ \Phi$ minus a multiple of the element of M with top form Y^4; then the criterion can fail. The corrections $\beta_2 x^2$ and $\gamma_2 x^2$ in the generators of the ray absorb exactly these terms.

The argument for two lines proves the following more general statement, which we use repeatedly in the appendices.

Lemma 5.11 (Parabolas)**.** *Let $A = S/I$ be a local algebra with $H_A(2) = 6$ and $\mathfrak{m}^6 \subseteq I$. Every $F \in D_5$ is the top form of an element $F + R(F) + E(F)$ of M with $R(F) \in T_4$ and $E(F) \in T_3$, chosen linearly in F. Let a, b, c be*

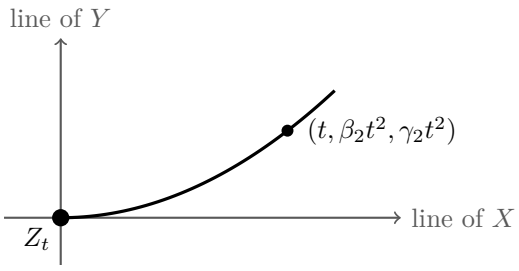


FIGURE 3. For $(1,3,6,2,2,1)$ with Γ two points, the cubics and quartics of the tangent cone of Z vanish on the lines of X and Y. A point leaves Z along a parabola tangent to one of them; a scheme Z_t of length fourteen stays at the origin.

linear coordinates and $P \in T_1$ the point with $a \circ P = 1$ and $b \circ P = c \circ P = 0$. Suppose that $P^3 \in D_3$, that a^3b and a^3c annihilate D_4 and D_5, and that $a^5 \circ F \neq 0$ for some $F \in D_5$.

1. *If $H_A(5) = 1$, then I has a ray along a parabola tangent to the line of P.*
2. *In general, the same holds if the numbers $a^3b \circ R(F)$ and $a^3c \circ R(F)$ vanish for all $F \in D_5$ with $a^5 \circ F = 0$.*

In particular, then A is smoothable if all finite subschemes of length $\ell(A) - 1$ are.

Proof. The space M is spanned by $T_{\leq 2}$, D_3, the elements $Q + \tau(Q)$ with $Q \in D_4$, and the lifts $F + R(F) + E(F)$. The operator a^3b has degree four, so it annihilates $T_{\leq 3}$ and the elements $Q + \tau(Q)$, and $a^3b \circ (F + R(F) + E(F)) = a^3b \circ R(F)$ is a number $\rho(F)$. The operator a^5 annihilates all elements of M except the lifts, and $a^5 \circ (F + R(F) + E(F)) = a^5 \circ F = \alpha(F)$. The linear form α on D_5 is nonzero, and ρ vanishes on its kernel, in case (1) because the kernel is zero. So $\rho = \beta_2 \alpha$ for a number β_2, and $a^3(b - \beta_2 a^2) \in I$; similarly $a^3(c - \gamma_2 a^2) \in I$ for a number γ_2. Let Λ be the parabola $b = \beta_2 a^2$, $c = \gamma_2 a^2$. By (7), in the coordinates a, b, c and their dual coordinates, $E_3(\Lambda)$ is P^3 plus a form of degree two, so it lies in M. Corollary 5.5 gives the ray, and Theorem 2.11(1) the last assertion. □

Part (2) says that one parabola serves all lifts at once, provided that their tails do not distinguish them in the directions b and c. This is often forced by the closedness of M under differentiation: as in (9), the derivative of a lift by a must lie in M, so that $a \circ R(F) \equiv \tau(a \circ F)$ modulo D_3. Hence, if a^2b and a^2c annihilate D_3, then $a^3b \circ R(F) = a^2b \circ \tau(a \circ F)$, and if moreover $a \circ D_5 \subseteq \mathbb{C}P^4$, these numbers are multiples of $a^5 \circ F$. This happens when $D_3 = \nu_3(\Gamma)$ for a reduced subscheme Γ of the plane, D_5 is spanned by fifth powers of points of Γ, and a vanishes at the points of Γ other than P. In case (a) of the proof of Theorem C.5 the numbers vanish for another reason: the tails of the second lift have no terms X^3Y, X^3Z.

5.12. When the Hilbert function ends with several ones, the operators $x^k b$ and $x^k c$ of Corollary 5.5 can be found without computing the tails, by linear algebra on the functionals $\langle x^j, \cdot \rangle$.

Lemma 5.13 (Rays from a functional)**.** *Let $A = S/I$ be a local algebra with $H_A(2) = 6$ and socle degree s, let x, y, z be linear coordinates with dual basis X, Y, Z, let $k \in \{3, 4\}$ with $X^k \in D_k$, and let M^0 be the space of elements of M in which no monomial X^j with $j \geq k+2$ occurs.*

1. *If no element of M^0 contains the monomial $X^k Y$, then $x^k(y - \beta(x)) \in I$ for a polynomial $\beta \in x^2\mathbb{C}[x]$.*
2. *If moreover no element of M^0 contains $X^k Z$, so that also $x^k(z - \gamma(x)) \in I$ with $\gamma \in x^2\mathbb{C}[x]$, and if $E_k(\Lambda) \in M$ for the curve Λ: $y = \beta(x)$, $z = \gamma(x)$, then $(y - \beta, z - \gamma)$ is a ray of I. For $k = 3$ the condition $E_3(\Lambda) \in M$ always holds.*

Proof. (1) An element F contains $X^k Y$ exactly when $\langle x^k y, F \rangle \neq 0$, and M^0 is the common kernel on M of the linear forms $\langle x^j, \cdot \rangle$, $k+2 \leq j \leq s$. So the linear form $\langle x^k y, \cdot \rangle$ on M vanishes on this common kernel, and it is a linear combination of these forms: $\langle x^k y - \sum_j \beta_j x^j, F \rangle = 0$ for all $F \in M$. A polynomial orthogonal to M lies in I (Lemma 2.5), and we put $\beta = \sum_j \beta_j x^{j-k}$. (2) This is Corollary 5.5 with $b = y - \beta$ and $c = z - \gamma$. For $k = 3$, (7) gives $E_3(\Lambda) \in X^3 + T_{\leq 2}$, and $X^3 \in D_3 \subseteq M$. □

The hypothesis can be tested one element at a time: if $M = N \oplus K'$, where $N \subseteq M^0$ and the $s - k - 1$ forms $\langle x^j, \cdot \rangle$ are linearly independent on the space K' of dimension $s - k - 1$, then $M^0 = N$, and it suffices that no element of N contains $X^k Y$. The operators $x^k b$ and $x^k c$ have order $k+1 \geq 4$, so these rays survive the addition of cubics (Lemma A.14).

When the Hilbert function ends with three ones, Lemma 5.13 with $k = 4$ gives a ray without any case distinction.

Lemma 5.14 (Septic tails)**.** *Let $A = S/I$ be a local algebra with $H_A(2) = 6$, socle degree seven and $H_A(5) = H_A(6) = H_A(7) = 1$. Then I has a ray along a curve tangent to the line of the point P with $D_7 = \mathbb{C}P^7$. In particular, A is smoothable if all finite subschemes of length $\ell(A) - 1$ are.*

Proof. The form spanning D_7 has all its derivatives in the line D_6, so it is a seventh power; choose coordinates with $D_7 = \mathbb{C}X^7$. Then $D_6 = \mathbb{C}X^6$, $D_5 = \mathbb{C}X^5$, $X^4 \in D_4$ and $X^3 \in D_3$. Let $\Phi = X^7 + R_6 + R_5 + R_4 + R_3$ be the lift of X^7. We apply Lemma 5.13 with $k = 4$. Let N be the space of elements of M of degree at most five; then $M = N \oplus \langle x \circ \Phi, \Phi \rangle$, and the forms $\langle x^6, \cdot \rangle$, $\langle x^7, \cdot \rangle$ are independent on $\langle x \circ \Phi, \Phi \rangle$, as $\langle x^7, \Phi \rangle = \langle x^6, x \circ \Phi \rangle = 7!$ and $\langle x^7, x \circ \Phi \rangle = 0$. So, by the remark after Lemma 5.13, it suffices that no element of N contains $X^4 Y$ or $X^4 Z$; and indeed their parts of degree five lie in $D_5 = \mathbb{C}X^5$. Hence $x^4(y - \beta_2 x^2 - \beta_3 x^3)$ and $x^4(z - \gamma_2 x^2 - \gamma_3 x^3)$ lie in I. Evaluating $\langle x^4 y, \cdot \rangle = \beta_2 \langle x^6, \cdot \rangle + \beta_3 \langle x^7, \cdot \rangle$ at $x \circ \Phi$ gives $5!\, r_Y = 7!\, \beta_2$, where

r_Y is the coefficient of X^5Y in R_6; likewise $5!\,r_Z = 7!\,\gamma_2$ for the coefficient r_Z of X^5Z.

The elements $y \circ \Phi$ and $z \circ \Phi$ of M have degree at most five, so $y \circ R_6$ and $z \circ R_6$ lie in $D_5 = \mathbb{C}X^5$, and monomial by monomial $R_6 \in \langle X^6, X^5Y, X^5Z\rangle$. So the element $\frac{4!}{7!}\,x^3 \circ \Phi$ of M, with top form X^4, has the cubic part $aX^3 + \frac{4!}{7!} \cdot \frac{5!}{2} X^2(r_Y\,Y + r_Z\,Z) = aX^3 + 12X^2(\beta_2 Y + \gamma_2 Z)$ for some $a \in \mathbb{C}$. By (7), the jet $E_4(\Lambda)$ of the curve Λ: $y = \beta_2 x^2 + \beta_3 x^3$, $z = \gamma_2 x^2 + \gamma_3 x^3$ differs from it by $-aX^3$ modulo $T_{\le 2}$, so $E_4(\Lambda) \in M$, and Lemma 5.13(2) gives the ray. Theorem 2.11(1) gives the last assertion. □

The curve is a parabola corrected by a cubic term, tangent to the line of X; its quadratic coefficients are read off from the tail of degree six of the lift of X^7. For socle degree six the element with top form X^4 is $x^2 \circ \Phi/30$, whose cubic part involves the quintic tail R_5, and a quartic ray exists only under a condition on R_5; Theorem 5.21 shows how such cases are handled. Lemma 5.14 covers $(1,3,6,2,2,1,1,1)$ in length seventeen, $(1,3,6,3,2,1,1,1)$ in length eighteen, and $(1,3,6,4,2,1,1,1)$ and $(1,3,6,3,3,1,1,1)$ in length nineteen; Proposition 2.12(2) does not apply to them, as $s = 7 < 2k_0$.

Over a double point the method breaks down for a structural reason. If Γ is the double point at $[X]$ tangent to the line through $[X]$ and $[Y]$, and $D_{k+1} = \nu_{k+1}(\Gamma)$, then $X^kY \in D_{k+1}$, and an element of M with this top form pairs with $x^k y$ but with no x^j, $j \ge k+2$; so no operator $x^k(y - \beta(x))$ lies in I. Such algebras are treated as limits of algebras over two points (Section 5.15), or by excess linkage. With $k = 4$, the argument of Lemma 5.14 also gives rays over two points (Proposition C.6).

5.15. The rest of this section controls the tails in limits. In most cases the graded data $D_3, D_4, \ldots$ of an algebra are governed by a finite subscheme Γ of the plane, and the algebras over reduced Γ have rays. Since $\mathrm{Hilb}_n(\mathbb{P}^2)$ is irreducible, every Γ is a limit of reduced ones. But deforming the graded data does not automatically deform the algebra: the algebra is determined by its graded data together with its *tails*, the parts of the elements of M below their top forms, and over a nonreduced Γ the tails may have more freedom than over reduced ones. Then some algebras over the collision might not be limits of algebras over distinct points. This is the *tail-lifting problem*. A toy model is the equation $t\xi = 0$ for a tail coefficient ξ over the line of a collision parameter t: for $t \ne 0$ it forces $\xi = 0$, but over $t = 0$ every ξ is a solution, and the solutions with $\xi \ne 0$ are not limits of solutions with $t \ne 0$. We control the problem in two ways: by showing that the tails form vector bundles over the relevant families (constant rank), or by an explicit family over a collision of two points whose ideal of equations has no component over the collision (saturation).

Lemma 5.16 (Constant rank)**.** *Let U be a reduced variety, and consider a system of linear equations $\sum_j a_{ij}(u)\xi_j = b_i(u)$ in unknowns $\xi \in \mathbb{C}^N$ whose coefficients are regular functions on U. If the matrix $(a_{ij}(u))$ and the*

augmented matrix $(a_{ij}(u)\,|\,b_i(u))$ *have the same rank* r *for all* $u \in U$*, then the solutions* (u,ξ) *form an affine bundle of rank* $N-r$ *over* U*, locally trivial in the Zariski topology; its total space is irreducible if* U *is. This holds in particular if Gaussian elimination, with pivots that are nonzero constants, reduces the system to* r *equations solved for* r *of the unknowns.*

Proof. Near a point u_0 some $r \times r$ minor of (a_{ij}) does not vanish. On the open set where it does not vanish, the r corresponding equations imply the others, because the rank of the augmented matrix is r, and Cramer's rule solves them for r of the unknowns as regular functions of u and the remaining $N-r$ unknowns. In the last case these minors are constants, and the elimination itself gives the solution. □

We use the lemma together with Lemma 2.3(2): local frames of the bundles of graded data and of the tails give families of inverse systems. Only constant rank is needed, not a particular rank; the ranks are computed exactly on each orbit, or by eliminations with constant pivots, by the scripts named in the proofs.

5.17. Two devices simplify the tails. The first is a change of coordinates. Let φ be the automorphism of S with $\varphi(x) = x$, $\varphi(y) = y + a_2x^m$ and $\varphi(z) = z+a_3x^m$, where $m \geq 2$. The ideal $\varphi^{-1}(I)$ defines a scheme isomorphic to $V(I)$, and its inverse system is $\varphi^\vee(M)$, where $\langle g, \varphi^\vee G\rangle = \langle \varphi(g), G\rangle$. For homogeneous g, the difference $\varphi(g)-g$ has larger order, so $\varphi^\vee G$ has the same top form as G, and the spaces D_k do not change. If $\Phi \in M$ has top form X^s, where s is the socle degree, then $\varphi^\vee\Phi$ agrees with Φ in degrees above $s-m+1$, and in degree $s-m+1$ it differs from Φ by $\frac{s!}{(s-m)!}X^{s-m}(a_2Y+a_3Z)$: the only monomials g of degree $s-m+1$ for which $\varphi(g)-g$ involves x^s are $x^{s-m}y$ and $x^{s-m}z$. So we may remove the terms $X^{s-m}Y$ and $X^{s-m}Z$ from the part of degree $s-m+1$ of the lift of X^s.

The second device is a frame for two colliding points. Let $\Gamma_t = \{X, X+tY\}$. For $t \neq 0$ this is a pair of points, and $\nu_k(\Gamma_t)$ has the basis X^k and

$$G_k(t) = \frac{(X+tY)^k - X^k}{kt} = X^{k-1}Y + \tfrac{k-1}{2}\,tX^{k-2}Y^2 + \dots,$$

the divided difference of the two Veronese points, which is polynomial in t. At $t=0$ it is the tangent vector $X^{k-1}Y$, and $X^k, X^{k-1}Y$ is a basis of $\nu_k(\Gamma_0)$ for the double point Γ_0 at $[X]$ with ideal (z, y^2) (Figure 4). We call these bases the *collision frame*. We also use $\Pi_t = \big((X+tY)^5 - X^5 - 5tX^4Y\big)/(10t^2) = X^3Y^2 + tX^2Y^3 + \dots$, which satisfies $y \circ \Pi_t = 2G_4(t)$ and $z \circ \Pi_t = 0$. Every subscheme of length two of $\mathbb{P}^2$ containing $[X]$ is, after a linear change of coordinates fixing X, one of the Γ_t.

5.18. Over the collision frame, the algebras with given graded data are found by a uniform computation. Given bases $\mathcal{L}_k$ of the spaces D_k, $k \geq 3$, depending polynomially on parameters, every algebra with these spaces has

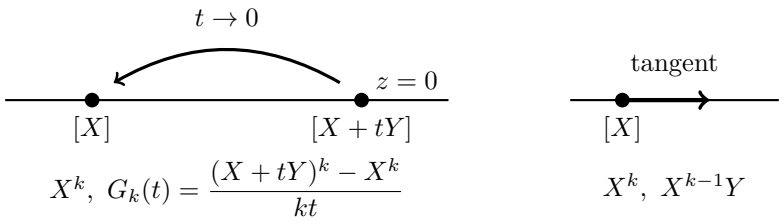


FIGURE 4. The collision frame. Two points of the plane of directions approach each other; the divided difference $G_k(t)$ of their Veronese images stays a basis vector of $\nu_k(\Gamma_t)$ and tends to the tangent vector $X^{k-1}Y$ of the double point. In $\mathbb{A}^3$, the two lines through the origin with these directions, and the curves along which points leave, merge.

a unique inverse system of the form

$$M = T_{\leq 2} \oplus D_3 \oplus \bigoplus_{k \geq 4} \{\, L(F) : F \in \mathcal{L}_k \,\}, \qquad L(F) = F + \sum_{3 \leq j < k} \tau_j(F),$$

where each tail $\tau_j(F)$ is a combination of a fixed set of monomials of degree j complementary to D_j; indeed, subtracting lifts of lower degree removes the other monomials. The space M is closed under differentiation if and only if each $v \circ L(F)$, $v \in \{x, y, z\}$, reduces to an element of $T_{\leq 2}$ when lifts are subtracted degree by degree. This gives polynomial equations in the parameters and the coefficients of the tails, which define a closed subscheme $\mathcal{X}$ of the product of the parameter space with an affine space, together with a morphism $\mathcal{X} \to \mathrm{Hilb}_d(\mathbb{A}^3)$ (Lemma 2.3(2)).

Lemma 5.19 (Collisions)**.** *Let $\mathcal{X} \subseteq \mathbb{A}^1_t \times \mathbb{A}^N$ be the closed subscheme defined by an ideal $\mathfrak{J} \subseteq \mathbb{C}[t, \xi_1, \ldots, \xi_N]$, with a morphism $\mathcal{X} \to \mathrm{Hilb}_d(\mathbb{A}^3)$. If $\mathfrak{J}$ is saturated with respect to t, that is, $(\mathfrak{J} : t^\infty) = \mathfrak{J}$, then no irreducible component of $\mathcal{X}$ lies in the hyperplane $t = 0$. Consequently, if the schemes parametrized by the points of $\mathcal{X}$ with $t \neq 0$ are smoothable, then so are those with $t = 0$.*

Proof. If $(\mathfrak{J} : t^\infty) = \mathfrak{J}$, then t is a nonzerodivisor modulo $\mathfrak{J}$, so it lies in no associated prime of $\mathfrak{J}$, and in particular in no minimal prime. So every irreducible component $\mathcal{C}$ of $\mathcal{X}$ meets the open set $t \neq 0$, in a dense open subset of $\mathcal{C}$, and Proposition 2.2(1) applies to $\mathcal{C}$. □

In the applications, $\mathfrak{J}$ is the ideal of the equations above, after the unknowns that occur linearly with a nonzero constant coefficient have been eliminated; such eliminations do not change the question: solving an equation $cu + r = 0$, with c a nonzero constant and u not occurring in r, identifies the subscheme with a graph over the subscheme defined by the remaining equations, compatibly with t, so t is a nonzerodivisor modulo the one ideal if and only if it is modulo the other. The parameters other than t are taken

in finitely many strata, and components over $t = 0$ that remain are treated directly; Appendix H explains why this suffices. When no equations remain, the algebras over the collision frame form an affine bundle over the t-line, and Lemma 5.16 applies directly.

5.20. We show the method on a family in which a critical point, a vanishing tail and a collision each play a role.

Theorem 5.21. *Let $c \geq 2$, and suppose that every finite subscheme of $\mathbb{A}^3$ of length $13 + c$ is smoothable. Then every local algebra with Hilbert function $(1, 3, 6, c, 2, 1, 1)$ is smoothable.*

Proof. The form spanning D_6 has all its derivatives in the line D_5, so it is a sixth power; choose coordinates with $D_6 = \mathbb{C}X^6$. Then $D_5 = \mathbb{C}X^5$ and $D_4 = \langle X^4, Q\rangle$ for a quartic Q without the term X^4, and D_3 is a space of dimension c containing $S_1 \circ D_4$. Let $\Phi = X^6 + R_5 + R_4 + R_3$ be the lift of X^6, with its terms of degree at most two dropped; then $x \circ \Phi$ and $x^2 \circ \Phi$ have top forms $6X^5$ and $30X^4$, and

$$M = T_{\leq 2} \oplus D_3 \oplus \mathbb{C}\,(Q + \tau_Q) \oplus \langle \Phi,\ x \circ \Phi,\ x^2 \circ \Phi\rangle$$

for some $\tau_Q \in T_3$. This space is closed under differentiation if and only if the first derivatives of Φ and of $Q + \tau_Q$ lie in it; the derivatives of $Q + \tau_Q$ lie in $D_3 + T_2$, and those of $x \circ \Phi$ and $x^2 \circ \Phi$ are derivatives of these, up to elements of $D_3 \oplus T_{\leq 2}$. So the condition concerns $y \circ \Phi$ and $z \circ \Phi$: their quartic parts must lie in D_4, and after subtracting the corresponding multiples of $x^2 \circ \Phi/30$ and $Q + \tau_Q$, their cubic parts must lie in D_3. Subtracting a multiple of $x \circ \Phi$ and changing coordinates as in Section 5.17 with $m = 2$, we may assume that R_5 has no terms X^5, X^4Y, X^4Z. Then the conditions read $y \circ R_5 = \beta Q$, $z \circ R_5 = \beta' Q$ and

$$y \circ R_4 - \beta\tau_Q \in D_3, \qquad z \circ R_4 - \beta'\tau_Q \in D_3. \tag{10}$$

A vanishing tail. If $R_5 = 0$, then $x^4 y$ and $x^4 z$ lie in I: they have order five, so they kill $T_{\leq 4}$; they kill X^6, R_4 and R_3; and $x^5 y$, $x^6 y$ kill X^6. Since $x^2 \circ \Phi \in 30X^4 + T_{\leq 2}$, we have $X^4 \in M$, and Lemma 5.4 with $k = 4$ and $\Psi = X^4$ shows that (y, z) is a ray, for every Q, D_3 and all tails.

A critical point. Let $R_5 \neq 0$. Then $v = \beta' y - \beta z \neq 0$ kills R_5, and, by $zy \circ R_5 = yz \circ R_5$, also Q. Choose coordinates with $v = z$: then $Q \in \mathbb{C}[X, Y]_4$, $\beta' = 0$, $R_5 = \beta R_Q$ with $\beta \neq 0$ and $R_Q \in \mathbb{C}[X, Y]_5$ the polynomial without terms X^5, X^4Y with $y \circ R_Q = Q$, and (10) says that τ_Q is determined by R_4 modulo D_3 and that $z \circ R_4 \in D_3$. If Q is not a multiple of X^3Y, the derivative of $q(\eta) = Q(1, \eta)$ has a root η_0, because q is a nonzero polynomial without constant term that is not a scalar multiple of η, so that q' is not constant; the substitution $Y \mapsto Y + \eta_0 X$, which fixes X, removes the term X^3Y from Q. It may create a term X^4 in Q, which we drop, as $X^4 \in D_4$, and terms X^5, X^4Y in R_5, which the normalization above removes again; after it, all terms of Q and R_Q have degree at most two in X. Put $\beta_3 = x^3 y \circ R_4/720$ and $\gamma_3 = x^3 z \circ R_4/720$. The operator $x^3(y - \beta_3 x^3)$ kills $T_{\leq 3}$ and $Q + \tau_Q$; it

kills Φ, as $x^3(y-\beta_3x^3)\circ\Phi = x^3y\circ R_4 - 720\beta_3 = 0$, because $x^3y\circ X^6 = 0$ and $x^3y\circ R_Q = 0$; it kills $x\circ\Phi$, as $x^4(y-\beta_3x^3)\circ\Phi = \beta\, x^4y\circ R_Q = 0$; and it kills $x^2\circ\Phi$. Likewise $x^3(z-\gamma_3x^3)\in I$. For the curve Λ given by $y=\beta_3x^3$, $z=\gamma_3x^3$, (7) gives $E_3(\Lambda) = X^3+6(\beta_3Y+\gamma_3Z)$, which lies in M, as $X^3\in D_3$. Corollary 5.5 gives a ray along a cubic curve. (Script `uniform_136c211.sage` checks the identities with all tails as indeterminates.)

A collision. If $Q=X^3Y$, then $D_4=\nu_4(\Gamma_0)$ for the double point Γ_0 at $[X]$, and $R_5 = \beta X^3Y^2/2 = \beta\Pi_0/2$. We use the collision frame. Let K be the space of elements of D_3 without the monomials X^3 and X^2Y; as $\nu_3(\Gamma_0)=\langle X^3, X^2Y\rangle\subseteq D_3$, it has dimension $c-2$, and $D_3=\nu_3(\Gamma_0)\oplus K$. For $t\in\mathbb{C}$ and $\beta\in\mathbb{C}^*$ let $Q_t = G_4(t)$, $R_{5,t}=\beta\Pi_t/2$, $D_{3,t}=\nu_3(\Gamma_t)\oplus K$, which has dimension c for all t, because the coefficients of X^3 and X^2Y in $\alpha X^3+\gamma G_3(t)$ are α and γ; let R_4 lie in the space $\{R_4 : z\circ R_4\in D_{3,t}\}$, which has dimension $5+c$ for all t, as $z\colon T_4\to T_3$ is onto with kernel $\mathbb{C}[X,Y]_4$, and a polynomial basis in t; let $\tau_{Q_t} = y\circ R_4/\beta$, and let R_3 be free. As $y\circ\Pi_t = 2G_4(t)$ and $z\circ\Pi_t=0$, we have $y\circ R_{5,t}=\beta Q_t$ and $z\circ R_{5,t}=0$, and (10) holds with Q_t, $D_{3,t}$ and $\beta'=0$, because $y\circ R_4-\beta\tau_{Q_t}=0$. So the spaces M_t so defined are closed under differentiation for all values of the parameters, and they form a family as in Lemma 2.3(2) over an irreducible base, which contains, at $t=0$, every algebra with $Q=X^3Y$ and the given D_3, and in which $t\neq 0$ is dense. For $t\neq 0$, $Q_t = X^3Y+\frac{3}{2}tX^2Y^2+\dots$ is not a multiple of X^3Y modulo X^4, and the previous case applies.

In all cases, Theorem 2.11(1) and Proposition 2.2(1) conclude. □

For $c=2,3,4,5$ the theorem covers $(1,3,6,2,2,1,1)$, $(1,3,6,3,2,1,1)$, $(1,3,6,4,2,1,1)$ and $(1,3,6,5,2,1,1)$, in lengths sixteen to nineteen. The three cases of the proof are the three ways in which tails behave throughout this paper: a tail that vanishes leaves a straight ray; a nonvanishing tail bends the ray, and a critical point of the quartic Q on the line of $[X]$ tells us how; and where no critical point exists, the algebra is a limit of algebras where one does.

5.22. Collisions can often be avoided altogether. When all algebras with a given Hilbert function, tails included, form a single irreducible family, one member in the open Gorenstein locus (Proposition 2.20), or in an open locus of short linkage (Proposition 2.24), suffices, and Lemma 2.18 proves irreducibility by counting. The member must lie in an open locus of smoothable schemes: smoothability itself is a closed condition, and a smoothable member alone says nothing about the general one. Theorem C.10 in Appendix C shows this on the Hilbert function $(1,3,6,5,2,2)$: the algebras are cut out by 20 equations in a smooth variety of dimension 55, so every component of the family has dimension at least 35, while every degenerate stratum has dimension at most 34; and one member lies in a complete intersection of local length $32<38$. The count replaces both a collision computation and a case analysis of the ranks of the tails.

6. Proof of the main theorem

6.1. We now prove Theorem 1.1. We proceed by induction on the length d, starting from [23]. By Proposition 2.2(3), it suffices to show that every local algebra $A = S/I$ of length d with $I \subseteq \mathfrak{m}^2$ is smoothable, assuming that all finite subschemes of smaller length are. The Hilbert function of A starts with $(1, 3)$, sums to d and satisfies Macaulay's bound; there are 52, 70, 91, 119 and 155 such sequences for $d = 15, \ldots, 19$ (script `enumerate.sage`). Table 1 lists the general results and how many Hilbert functions each of them covers, and Table 2 lists the remaining ones individually, with the method and the place of the proof (script `table_counts.sage`, which also prints the complete lists). Together they form a map of all Hilbert functions by method. This proves Theorem 1.1.

Four rows of Table 1 rest on the appendices: Proposition B.2 and the case $c = 8$ of Theorem 3.8 (Appendix B), Corollary D.4 (Appendix D), and the proofs of Proposition 4.11 (Appendix F). One row rests on a citation: the very compressed algebras, with Hilbert function $(1, 3, 6, c)$, are smoothable by [7, Proposition 1.2]. Appendix G gives an explicit form of that proof (Theorem G.4), and for $4 \leq c \leq 7$ Remark 3.10 gives a second proof by linkage.

TABLE 1. General results and the number of Hilbert functions of lengths fifteen to nineteen that they cover. Here $H = H_A$, s is the socle degree, and each Hilbert function is counted in the first row that applies.

Hilbert functions	result	15	16	17	18	19
$H(2) \le 4$, or $H(2) = 5$ and $H(3) \le 2$	Theorem 2.9(1)	38	48	60	73	91
$H(2) = 5$, $H(3) = 3$	Theorem 4.12	3	4	5	7	8
$H(2) = 5$, $s \le 4$	Corollary 4.7	3	4	4	4	3
$H(2) = 5$, $s = 5$, $H(5) = 1$	Corollary 4.9	1	2	3	4	4
$H(2) = 5$, $s = 5$, $H(4) = H(5) \le 3$	Corollary 4.10	0	0	1	1	2
$H(2) = 5$, and either $s = 6$ and $(H(5) = H(6) = 1$ or $H(4) = H(5) = 2)$, or $s = 7$, $H(6) = H(7) = 1$ and $(H(5) = 1$ or $H(4) = H(5) = 2)$	Proposition 4.11	0	1	3	6	10
$H(2) = 6$, $H(4) = \cdots = H(s) = 1$, $s \ge 5$	Proposition 2.12(1)	3	4	5	6	7
$s \ge 2k_0$, where k_0 is the largest k with $H(k) > 1$	Proposition 2.12(2)	0	0	0	2	5
$H(2) = 6$, $s = 7$, $H(5) = H(6) = H(7) = 1$	Lemma 5.14	0	0	1	1	2
$H(2) = 6$, $H(3) = H(4) = H(5) = 2$	Corollary 3.14	0	1	1	2	2
$H(2) = 6$, $H(3) = H(4) = 3$, and $s = 4$, or $s = 5$, $d \ge 18$, or $s = 6$, $H(6) = 1$, $d \ge 19$	Corollaries 3.15, D.4	0	1	0	1	2
$(1, 3, 6, c)$	Theorem G.4	1	1	1	1	1
$(1, 3, 6, c, 1)$, $4 \le c \le 8$	Theorem 3.8	1	1	1	1	1
$(1, 3, 6, c, 2)$, $3 \le c \le 7$	Proposition B.2	1	1	1	1	1
$(1, 3, 6, c, 2, 1, 1)$, $c \ge 2$	Theorem 5.21	0	1	1	1	1
individual	Table 2	1	1	4	8	15
total		52	70	91	119	155

TABLE 2. The Hilbert functions of lengths fifteen to nineteen not covered by Table 1, the result that treats them, its methods, and where it is proved. "Plane points" refers to a finite subscheme $\Gamma \subset \mathbb{P}^2$ governing D_3 and D_4 (Lemmas 2.16, A.11, A.12), and "cubic reduction" to Lemma A.15.

d	Hilbert function	result	methods	where
15	$(1,3,6,2,2,1)$	Theorem 5.10	parabolas, constant rank	§5
16	$(1,3,6,3,2,1)$	Theorem E.2	curves, Gorenstein, limits	App. E
17	$(1,3,6,3,2,2)$	Theorem C.3	excess linkage, linkage	App. C
	$(1,3,6,3,3,1)$	Theorem D.5	plane points, expected dim., parabolas	App. D
	$(1,3,6,4,2,1)$	Theorem E.3	Gorenstein, parabolas, limits	App. E
	$(1,3,6,4,3)$	Theorem B.3	excess linkage, plane points	App. B
18	$(1,3,5,4,3,2)$	Theorem F.2	turning, quadric linkage	App. F
	$(1,3,6,3,2,2,1)$	Theorem C.4	plane points, curves, collisions	App. C
	$(1,3,6,3,3,1,1)$	Theorem D.6	plane points, curves, free quartic tails	App. D
	$(1,3,6,4,2,2)$	Theorem C.5	plane points, rays, expected dim., linkage	App. C
	$(1,3,6,4,3,1)$	Theorem E.7	excess linkage, expected dim., Gorenstein	App. E
	$(1,3,6,4,4)$	Theorem B.3	excess linkage, plane points	App. B
	$(1,3,6,5,2,1)$	Theorem E.5	excess linkage, parabolas, cubic reduction	App. E
	$(1,3,6,5,3)$	Theorem B.3	excess linkage, plane points, rays	App. B
19	$(1,3,5,4,3,2,1)$	Theorem F.8	turning, quadric linkage, linkage	App. F
	$(1,3,5,4,4,2)$	Theorem F.6	turning, linkage	App. F
	$(1,3,5,5,3,2)$	Theorem F.7	turning, quadric linkage, constant rank	App. F
	$(1,3,6,3,2,2,1,1)$	Theorem C.8	quartic jets, excess linkage, collisions, rays	App. C
	$(1,3,6,3,2,2,2)$	Theorem C.7	excess linkage, collisions	App. C
	$(1,3,6,4,2,2,1)$	Theorem C.9	plane points, rays, excess linkage, collisions	App. C
	$(1,3,6,4,3,1,1)$	Theorem D.7	excess linkage, free quartic tails, one family	App. D
	$(1,3,6,4,3,2)$	Theorem E.10	excess linkage, parabolas, linkage	App. E
	$(1,3,6,4,4,1)$	Theorem E.9	excess linkage, plane points	App. E
	$(1,3,6,4,5)$	Theorem B.4	linkage	App. B
	$(1,3,6,5,2,2)$	Theorem C.10	one family (expected dim.), linkage	App. C
	$(1,3,6,5,3,1)$	Theorem E.11	excess and global linkage, expected dim., rays	App. E
	$(1,3,6,5,4)$	Theorem B.3	excess linkage, plane points, rays	App. B
	$(1,3,6,6,2,1)$	Theorem E.12	excess linkage, parabolas, cubic reduction	App. E
	$(1,3,6,6,3)$	Theorem B.5	global linkage, expected dim.	App. B

Appendix A. Pencils of quartics, spaces of cubics, and cubic reductions

A.1. Throughout the appendices, a statement about local algebras of length d assumes, without further mention, that every finite subscheme of $\mathbb{A}^3$ of length less than d is smoothable. This is the induction hypothesis of the proof of Theorem 1.1 (Section 6).

This appendix collects auxiliary facts: on finite subschemes of the plane (used in Theorem 5.10 and in Appendices C–E) and on ideals of small colength (used in Section 3.11); on pencils of quartics, and on four- and five-dimensional spaces of cubics (used in Appendix B); and, at the end, on adding a cubic to an inverse system or removing one (used in Appendices C–E).

Lemma A.2. *The* marked pairs (Γ, P) *of a subscheme* $\Gamma \subset \mathbb{P}^2$ *of length* n *and a point* P *of its support form an irreducible variety of dimension* $2n$.

Proof. The universal family $\mathcal{Z} \to \mathrm{Hilb}_n(\mathbb{P}^2)$ is finite and flat, so by going down for flat morphisms each irreducible component of $\mathcal{Z}$ dominates $\mathrm{Hilb}_n(\mathbb{P}^2)$, which is irreducible [8, Proposition 2.3 and Theorem 2.4]. Hence each component is the closure of its intersection with the preimage of the open set of reduced subschemes. This preimage, the set of pairs of a point and $n-1$ further distinct points, is irreducible of dimension $2n$, so $\mathcal{Z}$ is its closure. □

Spaces of forms whose derivatives in the directions vanishing at a point of Γ stay in $\nu_{m-1}(\Gamma)$ occur repeatedly; their dimension measures the socle of Γ at that point.

Lemma A.3 (Colon and socle)**.** *Let* $\Gamma \subset \mathbb{P}^2$ *be a subscheme of length* n, P *a point of* Γ, $N_P \subseteq S_1$ *the plane of linear forms vanishing at* P, *and* σ *the dimension of the socle of the local ring of* Γ *at* P. *For* $m \geq \max(n, 2)$ *the space*

$$\mathcal{K}_m(\Gamma, P) = \{F \in T_m : \ N_P \circ F \subseteq \nu_{m-1}(\Gamma)\}$$

has dimension $n + \sigma + 1$.

Proof. Let $J = I_\Gamma$ and $N_P = \langle b, c\rangle$. A form F lies in $\mathcal{K}_m(\Gamma, P)$ if and only if $\langle bg, F\rangle = \langle cg, F\rangle = 0$ for all $g \in J_{m-1}$, so $\mathcal{K}_m(\Gamma, P)$ is the orthogonal complement of $bJ_{m-1} + cJ_{m-1}$. The pairs $(g, h) \in J_{m-1}^2$ with $bg + ch = 0$ are the pairs $(ck, -bk)$ with $k \in S_{m-2}$ and $bk, ck \in J$, that is, $k \in (J : N_P)_{m-2}$. The ideal $(J : N_P)$ is saturated, and it defines the subscheme Γ' of Γ residual to the socle at P: at the other points of Γ it agrees with J, and at P its quotient by J is the socle. So Γ' has length $n - \sigma$ (it is empty if Γ is the reduced point P). By Lemma 2.14, $\dim J_{m-1} = \binom{m+1}{2} - n$ and $\dim(J : N_P)_{m-2} = \binom{m}{2} - (n - \sigma)$, as $m - 1 \geq n - 1$ and $m - 2 \geq n - \sigma - 1$. Hence $\dim \mathcal{K}_m(\Gamma, P) = \binom{m+2}{2} - 2\binom{m+1}{2} + 2n + \binom{m}{2} - n + \sigma = n + \sigma + 1$. □

For example, if Γ is reduced, then $\sigma = 1$, and $\mathcal{K}_m(\Gamma, P) = \nu_m(\Gamma) + P^{m-1}T_1$ has dimension $n + 2$; for the fat point $\Gamma = V(\mathfrak{m}_P^2)$, of length three, the dimension is 6. The jump at a fat point is not an accident of a rank

computation: it measures the larger socle. Script `colon_socle.sage` checks the lemma on representatives of many types of marked pairs of length at most five.

Lemma A.4. *Let $\mathfrak{b} \subset \mathbb{C}[[u,v]]$ be an ideal of finite colength, and V a finite-dimensional space of elements of $\mathfrak{b}$ that generate $\mathfrak{b}$. If $\mathfrak{b}$ is generated by two elements, then two general elements of V generate $\mathfrak{b}$. Every ideal of colength at most four is generated by two elements, or is $(u,v)^2$ or (u^2,uv,v^3) in suitable coordinates, and then two general elements of V generate an ideal of colength 4, respectively 5.*

Proof. If an ideal of colength at most four is not of the form (u,v^k), it contains $(u,v)^3$ and is generated modulo $(u,v)^3$ by $(u,v)^2$ or by a pencil of binary quadrics, which is $\langle u^2,v^2\rangle$ or $\langle u^2,uv\rangle$ in suitable coordinates; only $(u,v)^2$ and (u^2,uv,v^3) need three generators. By Nakayama's lemma, V maps onto $\mathfrak{b}/(u,v)\mathfrak{b}$, and two general elements of V map to two general elements of this space. If $\mathfrak{b}$ is generated by two elements, they map to a basis, so they generate $\mathfrak{b}$. If $\mathfrak{b}=(u,v)^2$, their quadratic leading forms have no common factor, so they generate an ideal of colength at most four, as in the proof of Lemma 3.3, and at least four, as it is a proper subideal of $\mathfrak{b}$. If $\mathfrak{b}=(u^2,uv,v^3)$, they generate a proper subideal, of colength at least $\ell(\mathbb{C}[[u,v]]/\mathfrak{b})+1=5$. Conversely, they can be replaced by combinations of the form $uv+\beta v^3+\dots$ and $\alpha u^2+\gamma v^3+\dots$ with $\alpha\gamma\neq 0$, where the dots are elements of $(u,v)\mathfrak{b}$. For the weights 3 and 2 of u and v, the dots have weighted order at least 7 and βv^3 has weighted order 6, so the weighted initial forms are uv and $\alpha u^2+\gamma v^3$. As $\mathbb{C}[u,v]/(uv,\alpha u^2+\gamma v^3)$ has the basis $1,u,v,v^2,v^3$, the two elements generate an ideal of colength at most five, as in the proof of Lemma 3.3. □

A.5. In what follows $W\subseteq T_4$ denotes a two-dimensional space of quartic forms, a *pencil*, and $r(W)=\dim S_1\circ W$. Since a pencil cannot consist of fourth powers of one linear form, $r(W)\geq 2$. We describe the pencils with $r(W)\leq 4$. We use three families of pencils.

- $\mathcal{V}$: the planes $W\subseteq\nu_4(\Gamma)$ for subschemes $\Gamma\subset\mathbb{P}^2$ of length three. By Lemma 2.14, the pairs (Γ,W) form a $\mathbb{P}^2$-bundle over $\mathrm{Hilb}_3(\mathbb{P}^2)$, which is irreducible [8, Proposition 2.3 and Theorem 2.4]. If $r(W)=3$, then $S_1\circ W=\nu_3(\Gamma)$, because $S_1\circ\nu_4(\Gamma)\subseteq\nu_3(\Gamma)$ and both have dimension three.
- $\mathcal{S}$: the pencils $W=\langle L^4,H\rangle$, where $L\in T_1$ and H is a binary quartic with two essential variables in a plane $\Pi\subset T_1$ not containing L. Then $S_1\circ W=\mathbb{C}L^3\oplus S_1\circ H$, and $r(W)=3$.
- $\mathcal{S}'$: the pencils $W=\langle L^4,\ L^3L_1+H(L,L_2)\rangle$, where L,L_1,L_2 is a basis of T_1, H is a binary quartic in L and L_2, and $r(W)=3$; equivalently, the derivative of H with respect to L_2 is not a multiple of L^3.

Lemma A.6. *If $r(W)=2$, then $W=\nu_4(\Gamma)$ and $S_1\circ W=\nu_3(\Gamma)$ for a subscheme $\Gamma\subset\mathbb{P}^2$ of length two. If $r(W)=3$, then W belongs to $\mathcal{V}$, $\mathcal{S}$ or $\mathcal{S}'$.*

Proof. The case $r(W) = 2$ is Lemma 2.15. Let $r(W) = 3$.

A member with three essential variables. Suppose $F \in W$ is concise, so that $S_1 \circ W = S_1 \circ F$, and let $G \in W$ be independent of F. Since $v \mapsto v \circ F$ is injective on S_1, there is a linear map $\varphi\colon S_1 \to S_1$ with $v \circ G = \varphi(v) \circ F$, and φ is not a multiple of the identity. Since $u \circ (v \circ G) = v \circ (u \circ G)$, the quadrics $u\,\varphi(v) - v\,\varphi(u)$ annihilate F for all $u, v \in S_1$. Conversely G is determined by φ and F. Adding a multiple of the identity to φ changes G by a multiple of F, and rescaling φ rescales G, so after a change of coordinates we may bring φ to one of the following Jordan forms, with respect to the basis x, y, z of S_1, and solve for F (script `pencils13632.sage`).

(i) φ diagonal with distinct eigenvalues. Then xy, xz, yz annihilate F, so $F \in \langle X^4, Y^4, Z^4\rangle$, and W lies in ν_4 of the three coordinate points: family $\mathcal{V}$.

(ii) $\varphi = \operatorname{diag}(0, 0, 1)$. Then xz, yz annihilate F, so $F = H(X, Y) + \gamma Z^4$, and $W = \langle H, Z^4\rangle$: family $\mathcal{S}$.

(iii) $\varphi(y) = x$, $\varphi(x) = 0$, $\varphi(z) = z$. Then x^2, xz, yz annihilate F, so $F \in \langle Y^4, XY^3, Z^4\rangle$, which is ν_4 of a double point at $[Y]$ together with the point $[Z]$: family $\mathcal{V}$.

(iv) $\varphi(y) = x$, $\varphi(x) = \varphi(z) = 0$. Then x^2, xz annihilate F, so $F = H(Y, Z) + \beta XY^3$ with $\beta \neq 0$, and G is a multiple of Y^4: family $\mathcal{S}'$, with $L = Y$, $L_1 = X$, $L_2 = Z$.

(v) $\varphi(y) = x$, $\varphi(z) = y$, $\varphi(x) = 0$. Then $x^2, xy, y^2 - xz$ annihilate F, so $F \in \langle Z^4, YZ^3, XZ^3 + \frac{3}{2}Y^2Z^2\rangle$. This is the span of the first three coefficients of $(Z + sY + s^2X)^4$ in s, which is ν_4 of a curvilinear scheme of length three: family $\mathcal{V}$.

No member with three essential variables. Then a general member F of W has exactly two essential variables, in a plane $\Pi_F \subset T_1$. If Π_F does not depend on F, then W and $S_1 \circ W$ consist of binary forms in Π_F, say in X, Y; the three-dimensional space $S_1 \circ W$ is annihilated by a binary cubic q, then so is W, and $W \subseteq \nu_4(\Gamma)$ for $\Gamma = V(z, q)$: family $\mathcal{V}$. Otherwise choose general members $F \in \mathbb{C}[X, Y]$ and $G \in \mathbb{C}[X, Z]$. Every member $F + tG$ has at most two essential variables, so its Hessian determinant vanishes:

$$t\,G_{ZZ}\,(F_{XX}F_{YY} - F_{XY}^2) + t^2\,F_{YY}\,(G_{XX}G_{ZZ} - G_{XZ}^2) = 0.$$

The binary Hessians of F and G are nonzero, because a binary form with vanishing Hessian is a power of a linear form.[2] Hence $G_{ZZ} = F_{YY} = 0$, so $F, G \in X^3T_1$, and $W \subseteq X^3T_1 = \nu_4(\Gamma)$ for the subscheme Γ of $\mathbb{P}^2$ defined by the square of the ideal of $[X]$: family $\mathcal{V}$. □

The pencils in $\mathcal{S}'$ are limits of pencils in $\mathcal{S}$, and the algebras over them are limits too.

[2]The Hessian of $F \in \mathbb{C}[X, Y]$ is the Jacobian determinant of F_X and F_Y. If it vanishes, these two forms of equal degree are algebraically dependent, hence proportional, and F is annihilated by a nonzero linear form.

Lemma A.7. *Let $W = \langle X^4,\ X^3Y + H(X,Z)\rangle \in \mathcal{S}'$, and for $t \in \mathbb{C}$ let*

$$b_1(t) = (X+tY)^4, \qquad b_2(t) = \frac{(X+tY)^4 - X^4 + 4tH(X,Z)}{4t}.$$

These are polynomials in t with $b_1(0) = X^4$ and $b_2(0) = X^3Y + H(X,Z)$, and for small $t \neq 0$ the pencil $W_t = \langle b_1(t), b_2(t)\rangle$ belongs to $\mathcal{S}$. The space $S_1 \circ W_t$ has, for all t near 0, the polynomial basis $x \circ b_2(t)$, $y \circ b_2(t) = (X+tY)^3$ and $z \circ b_2(t) = H_Z$. Consequently, every local algebra with $H_A(2) = 6$, $\mathfrak{m}^5 \subseteq I$ and $D_4 = W$ is a flat limit of local algebras with the same Hilbert function and $D_4 \in \mathcal{S}$.

Proof. For $t \neq 0$, $W_t = \langle (X+tY)^4,\ X^4 - 4tH(X,Z)\rangle$, and $S_1 \circ W_t = \langle (X+tY)^3,\ X^3 - tH_X,\ tH_Z\rangle$, which contains the three displayed polynomials and is spanned by them; at $t = 0$ they are $3X^2Y + H_X$, X^3 and H_Z, a basis of $S_1 \circ W$. So they are independent near $t = 0$. As $H_Z \notin \mathbb{C}X^3$, the quartic $X^4 - 4tH(X,Z)$ has two essential variables for small $t \neq 0$, so $W_t \in \mathcal{S}$ with $L = X + tY$ and $\Pi = \langle X, Z\rangle$. Now let $M = T_{\leq 2} \oplus D_3 \oplus \{F + \tau(F) : F \in W\}$ as in Lemma 2.7, and write $D_3 = S_1 \circ W \oplus C$ for a subspace C. Then

$$M_t = T_{\leq 2} \oplus (S_1 \circ W_t \oplus C) \oplus \langle b_1(t) + \tau(b_1(0)),\ b_2(t) + \tau(b_2(0))\rangle$$

is closed under differentiation by Lemma 2.7, it has the polynomial basis given above near $t = 0$, and $M_0 = M$. By Lemma 2.3(2) these spaces form a flat family. □

A.8. For pencils with $r(W) = 4$ we use the *contraction kernel* of a basis F_1, F_2 of W,

$$K = \{(v_1, v_2) \in S_1^2 :\ v_1 \circ F_1 + v_2 \circ F_2 = 0\},$$

which has dimension $6 - r(W) = 2$. Up to the action of GL_3 on S_1 and of GL_2 on the pair (F_1, F_2), a two-dimensional subspace of $S_1^2 = S_1 \otimes \mathbb{C}^2$ is one of the following six:

$$\langle (x,y),(y,z)\rangle, \quad \langle (x,0),(0,z)\rangle, \quad \langle (x,0),(y,z)\rangle,$$
$$\langle (x,0),(y,x)\rangle, \quad \langle (x,0),(0,x)\rangle, \quad \langle (x,0),(y,0)\rangle.$$

Indeed, think of K as a pencil of 3×2 matrices. If all its members have rank one, then either all of them have the same column space or all the same row space, which gives the last two. Otherwise its rank-one members are the common zeros of three binary quadrics, the 2×2 minors, so there are at most two of them. Two distinct rank-one members give the second form; a single one, $(a,0)$ say, completes to a basis $(a,0),(b,c)$ of K with $c \neq 0$, and the minors of $s(a,0) + t(b,c)$ are the coefficients of $t\,(sa + tb) \wedge c$. If $b \in \mathbb{C}a$, we may take $b = 0$, and then $(0,c)$ is a second rank-one member unless $c \in \mathbb{C}a$, in which case all members have rank one. Otherwise the rank-one member is simple if a, b, c are independent (third form) and double if c is a multiple of a (fourth form), while $c \in \langle a, b\rangle \setminus \mathbb{C}a$ gives a second rank-one member. Finally, suppose no member has rank one. The four columns of the two basis matrices span S_1: if they spanned a plane, the determinant of the resulting pencil of

2×2 matrices would vanish somewhere on $\mathbb{P}^1$. Their unique linear relation, written as a 2×2 matrix of coefficients, has rank two, since a relation of rank one would exhibit a member of the pencil annihilating a nonzero vector of $\mathbb{C}^2$, hence of rank at most one. Changing the bases of K and of $\mathbb{C}^2$, the relation becomes "the second column of the first matrix equals the first column of the second"; the three remaining columns are independent, and calling them x, y, z gives the first form.

The pairs $(F_1, F_2) \in T_4^2$ with $v_1 \circ F_1 + v_2 \circ F_2 = 0$ for all $(v_1, v_2) \in K$ form a linear space, which has dimension 10 for the first five subspaces K above and 16 for the last one (script `kernels13642.sage`). The last subspace means that a two-dimensional space of linear forms annihilates F_1, that is, F_1 is a fourth power.

Lemma A.9. *The pencils W with $r(W) = 4$ lie in two irreducible families: the family $\mathcal{P}$ of pencils $\langle L^4, G\rangle$, and the family $\mathcal{N}$ of pencils whose contraction kernel is not of the last type. In $\mathcal{N}$, the general member has the property that $S_1 \circ W$ contains no cube of a linear form; every member $W = \langle L^4, G\rangle$ of $\mathcal{P}$ with $r(W) = 4$ satisfies $L^3 \notin N_L \circ W$.*

Proof. The family $\mathcal{P}$ is parametrized by pairs (L, G), and it is irreducible. The subspaces K of the first five types form an open subset of the Grassmannian $\mathrm{Gr}(2, S_1^2)$, the complement of the closed subset of the last type, and over it the solution spaces (F_1, F_2) form a vector bundle of rank 10. Its total space is irreducible, and the pairs with $r(\langle F_1, F_2\rangle) = 4$ form an open subset, which parametrizes $\mathcal{N}$. For the assertion on $\mathcal{N}$ it suffices to give one example. Let $H = XY^4 + 6X^2Y^2Z + 2X^3Z^2$, which is annihilated by $y^2 - xz$. The pencil $W = \langle XY^3 + 3X^2YZ,\ Y^4 + 12XY^2Z + 6X^2Z^2\rangle$ has the basis $F_1 = y \circ H$, $F_2 = x \circ H$, and its contraction kernel is $\langle (x, -y), (y, -z)\rangle$, of the first type. The space $S_1 \circ W = \langle X^2Y,\ XY^2 + X^2Z,\ Y^3 + 6XYZ,\ Y^2Z + XZ^2\rangle$ has dimension four, and it is annihilated by x^3, z^3 and $y^3 - xyz$, which have no common zero; so it contains no cube. In $\mathcal{P}$, let $W = \langle L^4, G\rangle$ and $l \in S_1$ with $l \circ L = 1$. As $N_L \circ L^4 = 0$, we have $N_L \circ W = N_L \circ G$ and $S_1 \circ W = \mathbb{C}L^3 + N_L \circ G + \mathbb{C}\,(l \circ G)$; if L^3 lay in $N_L \circ W$, then $S_1 \circ W$ would have dimension at most three, contradicting $r(W) = 4$. □

A.10. The next lemma describes four-dimensional spaces of cubic forms; we write $D' = \{F \in T_4 : S_1 \circ F \subseteq D\}$ as in Lemma 2.16.

Lemma A.11. *Let $D \subseteq T_3$ be a subspace of dimension four, and let $\mathfrak{B} \subset \mathbb{P}^2$ be the subscheme defined by the cubics in $D^\perp$. Then either $D = \mathbb{C}[L, M]_3$ for two linear forms L, M, or $\mathfrak{B}$ is finite of length at most four. If $\mathfrak{B}$ has length four, then $D = \nu_3(\mathfrak{B})$ and $D' = \nu_4(\mathfrak{B})$.*

Proof. Let K be the ideal generated by $D^\perp$. As in the proof of Lemma 2.16, $H_{S/K}(3) = 4$ and $H_{S/K}(4) = \dim D'$, and Macaulay's bound gives $\dim D' \leq 4^{\langle 3\rangle} = 5$. If $\dim D' = 5$, Gotzmann's persistence theorem [1, Theorem 4.3.3] gives $H_{S/K}(j) = j + 1$ for all $j \geq 3$. Then K defines a subscheme of $\mathbb{P}^2$

with Hilbert polynomial $t+1$, which is a line $V(l)$, and $K_3 \subseteq lS_2$; both have dimension six, so $D = (lS_2)^\perp = \mathbb{C}[L,M]_3$, where L, M span the forms annihilated by l. If $\dim D' \leq 4$, then $H_{S/K}(j) \leq 4$ for all $j \geq 4$, because $h^{\langle j \rangle} = h$ for $h \leq j$, so $\mathfrak{B}$ is finite of length at most four. If its length is four, then $(I_{\mathfrak{B}})_j$ has codimension four in S_j for $j \geq 3$ by Lemma 2.14; it contains K_j, which has codimension at most four for $j = 3, 4$. So $K_3 = (I_{\mathfrak{B}})_3$ and $K_4 = (I_{\mathfrak{B}})_4$, which gives $D = \nu_3(\mathfrak{B})$ and $D' = \nu_4(\mathfrak{B})$. □

The next lemma is the analogue of Lemma A.11 for five cubics.

Lemma A.12. *Let $D \subseteq T_3$ be a subspace of dimension five, K the ideal generated by $D^\perp$, and $\mathfrak{B} \subset \mathbb{P}^2$ the subscheme it defines. Then one of the following holds.*

(a) *$\mathfrak{B}$ is finite of length at most four.*
(b) *$\mathfrak{B}$ is finite of length five, $D = \nu_3(\mathfrak{B})$ and $D' = \nu_4(\mathfrak{B})$.*
(c) *$\mathfrak{B}$ is a line together with a point, possibly embedded in the line, $D = \nu_3(\mathfrak{B})$ and $D' = \nu_4(\mathfrak{B})$.*
(d) *$\mathfrak{B}$ is a line Λ, $D = \nu_3(\Lambda) \oplus \mathbb{C}G$ for a cubic $G \notin \nu_3(\Lambda)$, and $D' = \nu_4(\Lambda)$.*

Proof. As in the proof of Lemma A.11, $H_{S/K}(3) = 5$ and $H_{S/K}(4) = \dim D' \leq 5^{\langle 3 \rangle} = 6$. If $\dim D' = 6$, Gotzmann's persistence theorem gives $H_{S/K}(j) = j+2$ for $j \geq 3$, so $\mathfrak{B}$ has Hilbert polynomial $t+2$: it is a line together with a point, isolated or embedded. Its Hilbert function is $j+2$ for $j \geq 1$, so $K_3 = (I_{\mathfrak{B}})_3$ and $K_4 = (I_{\mathfrak{B}})_4$, which is (c). If $\dim D' \leq 4$, then $H_{S/K}(j) \leq 4$ for $j \geq 4$, because $h^{\langle j \rangle} = h$ for $h \leq j$, and $\mathfrak{B}$ is finite of length at most four, which is (a).

Let $\dim D' = 5$. Macaulay's bound now only gives $H_{S/K}(5) \leq 5^{\langle 4 \rangle} = 6$. If $H_{S/K}(5) = 6$, the bound is attained, and since K is generated in degree three, Gotzmann's persistence theorem gives $H_{S/K}(j) = j+1$ for $j \geq 4$. Then $\mathfrak{B}$ has Hilbert polynomial $t+1$, so it is a line $\Lambda = V(l)$, with homogeneous ideal (l). The ideal K lies in its saturation (l), and K_4 and $(l)_4$ both have codimension five, so $K_4 = (l)_4$ and $D' = \nu_4(\Lambda)$. Moreover $K_3 \subseteq (l)_3$ gives $D \supseteq \nu_3(\Lambda)$, a space of dimension four; this is (d). If $H_{S/K}(5) \leq 5$, then $H_{S/K}(j) \leq 5$ for all $j \geq 4$, so $\mathfrak{B}$ is finite of length at most five. If its length is at most four, this is (a). If its length is five, then $(I_{\mathfrak{B}})_4$ has codimension five by Lemma 2.14, so $K_4 = (I_{\mathfrak{B}})_4$. Moreover $h_{\mathfrak{B}}(3) = 5$: otherwise the first differences of $h_{\mathfrak{B}}$ would be $(1,1,1,1,1)$, $\mathfrak{B}$ would lie on a line; but every cubic through a collinear scheme of length five contains the line, so $\mathfrak{B}$ would not be finite. So $K_3 = (I_{\mathfrak{B}})_3$, which is (b). □

Case (d) occurs. For $D = \langle Y^3, Y^2Z, YZ^2, Z^3, XYZ \rangle$ the ideal K is $(x^3, x^2y, x^2z, xy^2, xz^2)$, with Hilbert function $(1,3,6,5,5,6,7,\dots)$ and saturation (x); here $D' = \mathbb{C}[Y,Z]_4$. The value $H_{S/K}(4) = 5$ does not bound the later values, because $5^{\langle 4 \rangle} = 6$. Both alternatives after $\dim D' = 5$ occur: five general points give $H_{S/K} = (1,3,6,5,5,5,\dots)$ (script `l19_fivecubics_line.sage`).

Cases (c) and (d) are told apart by one quadric. In both, $\mathfrak{B}$ contains a line $V(l)$, so $K \subseteq (l)$, and $D^\perp = lV$ for a five-dimensional space V of quadrics, and $D = \nu_3(V(l)) \oplus \mathbb{C}G$; the space $V^\perp \subseteq T_2$ is spanned by the quadric $l \circ G$. If $l \circ G = P^2$ is a square, then V consists of all quadrics vanishing at $[P]$, and $\mathfrak{B}$ is the line together with $[P]$: case (c). Otherwise no nonzero cubic has all its first derivatives in $\mathbb{C}\, l \circ G$, since such a cubic is a cube M^3, whose derivatives span $\mathbb{C}M^2$; so $S_1V = S_3$, $K_4 = lS_3$, and we are in case (d). In the example, $x \circ XYZ = YZ$ is not a square. Script `l19_fivecubics_line2.sage` checks the criterion on examples.

A.13. Rays found by operators of high order survive when the inverse system grows in low degree.

Lemma A.14. *Let M' be the inverse system of a local algebra with $H(2) = 6$, and let $G \in T_3$ be a cubic not in the space $D_3(M')$ of top forms of degree three of M'. Then $M = M' \oplus \mathbb{C}G$ is the inverse system of a local algebra whose Hilbert function is that of M' with $H(3)$ increased by one, and every element of $\operatorname{Ann}(M')$ of order at least four lies in $\operatorname{Ann}(M)$. In particular, if (b, c) is a ray of $\operatorname{Ann}(M')$ given by Lemma 5.4 with $k \geq 3$, then it is a ray of $\operatorname{Ann}(M)$.*

Proof. The derivatives of G lie in $T_{\leq 2} \subseteq M'$, so M is closed under differentiation, and $\operatorname{Ann}(M) = \operatorname{Ann}(M') \cap G^\perp$ contains every element of $\operatorname{Ann}(M')$ of order at least four. The element $\Psi \in M'$ of Lemma 5.4 lies in M, and $a^k b$, $a^k c$ have order $k + 1 \geq 4$. □

If M' belongs to a family as in Lemma 2.3(2), all of whose members have the Hilbert function of M', in which a dense set of members has rays of this kind, then so does M: over the open subset of the base where $G \notin D_3(M'_u)$, the spaces $M'_u \oplus \mathbb{C}G$ form such a family. The hypothesis on the Hilbert function is needed: the polynomial family $M'_t = \langle 1, X, Y, Z, X^2, XY, XZ, YZ, Z^2, Y^2 + tX^3 \rangle$ has $M'_0 = T_{\leq 2}$, but $M'_t \oplus \mathbb{C}Y^3$ is not closed under differentiation for $t \neq 0$.

Conversely, one can often remove a cubic. Let A be a local algebra with $H_A(2) = 6$ and inverse system M, and let $N = \mathfrak{m} \circ M + T_{\leq 2}$ be the span of $T_{\leq 2}$ and of the derivatives of the elements of M. If $M = M' \oplus \mathbb{C}G$ with M' as in Lemma A.14, we call the algebra A' with inverse system M' a *cubic reduction* of A.

Lemma A.15. *The algebra A has a cubic reduction if and only if $D_3 \not\subseteq N$, that is, some cubic of D_3 is not a derivative of an element of M modulo $T_{\leq 2}$. A cubic reduction A' has the Hilbert function of A with $H(3)$ decreased by one.*

Proof. If $M = M' \oplus \mathbb{C}G$ with $M' \supseteq T_{\leq 2}$ closed under differentiation and $G \in D_3$, then M/M' is a one-dimensional S-module, so it is annihilated by $\mathfrak{m}$; hence $\mathfrak{m} \circ M \subseteq M'$, so $N \subseteq M'$, and $G \notin N$. Conversely, if $G \in D_3 \setminus N$, let M' be a hyperplane of M that contains N but not G. It is closed under

differentiation, as it contains $\mathfrak{m} \circ M$, and $M = M' \oplus \mathbb{C}G$. The elements of M of degree other than three have the same top forms as elements of M', and the cubic top forms of M' form the hyperplane $D_3 \cap M'$ of D_3; Lemma 2.5 gives the Hilbert function. □

So when $D_3 \not\subseteq N$, the algebra A is obtained from a shorter algebra A' by adding a cubic, and Lemma A.14 transfers rays given by operators of order at least four, and limits, from A' to A. When every cubic of D_3 is a derivative, a direct argument is needed. Adding a cubic to the inverse system removes a cubic from the tangent cone; the base scheme of the cubics usually does not change.

Lemma A.16. *Let $U \subseteq S_3$ be a space of cubics of dimension at least four with finite base scheme $\mathfrak{B}' \subset \mathbb{P}^2$, and suppose that the ideal of $\mathfrak{B}'$ is generated at each of its points by fewer than $\dim U$ elements. Then a general hyperplane of U has the base scheme $\mathfrak{B}'$.*

Proof. Let $U_0 \subset U$ be a hyperplane. At each point p of $\mathfrak{B}'$, the elements of U generate the local ideal $\mathfrak{b}_p$ of $\mathfrak{B}'$, and by Nakayama's lemma so do those of U_0, unless U_0 contains the kernel of the map from U onto $\mathfrak{b}_p/\mathfrak{m}_p\mathfrak{b}_p$. This kernel is nonzero by hypothesis, so this excludes a proper closed subset of hyperplanes. If U_0 has a base point q outside $\mathfrak{B}'$, then $U_0 = U \cap (I_q)_3$; these hyperplanes form a family of dimension at most two in the projective space of hyperplanes of U, which has dimension at least three. So a general U_0 has the base scheme $\mathfrak{B}'$. □

By Lemma A.4, an ideal of colength at most four in $\mathbb{C}[[u, v]]$ is generated by at most three elements, so the condition on generators holds when $\mathfrak{B}'$ has length at most four.

Appendix B. Socle degree four

B.1. In this appendix $H_A(2) = 6$ and $\mathfrak{m}^5 \subseteq I$, so that A is determined by D_3, D_4 and an arbitrary map τ (Lemma 2.7). The cubic base scheme decides the method: when D_3 is free of cubes, Theorem 3.4 links A to a shorter scheme; when the cubes in D_3 come from a plane scheme with independent cubes, Proposition 5.7 gives rays, and when they are the cubes of a line that carries D_4, step (d) of the proof of Theorem B.3 gives rays along points of the line; and when the base scheme is finite and short, Theorem 3.12 measures the excess. In length nineteen, socle degree four occurs for five Hilbert functions with $H(2) = 6$. For $(1, 3, 6, 8, 1)$ and $(1, 3, 6, 4, 5)$ the algebras form one irreducible family, and a complete intersection of length 36 links one member to a scheme of length 17 (the case $c = 8$ of Theorem 3.8, proved here, and Theorem B.4). For $(1, 3, 6, 6, 3)$, every algebra lies in an irreducible family whose general member is smoothable by global linkage: a member with the same graded data lies in a global complete intersection of length 36 (Proposition 3.18, Theorem B.5). For $(1, 3, 6, 7, 2)$ the general algebra is linked through three cubics (Proposition B.2), and $(1, 3, 6, 5, 4)$ is treated by

excess linkage and rays (Theorem B.3). The pencil D_4 in the first result is organized by the number $r(W) = \dim S_1 \circ W$ of Appendix A.

Proposition B.2. *Let $3 \le c \le 7$, and suppose that every finite subscheme of $\mathbb{A}^3$ of length less than $12 + c$ is smoothable. Then every local algebra with Hilbert function $(1, 3, 6, c, 2)$ is smoothable.*

Proof. By Lemma 2.7, the algebras are parametrized by the pairs (W, D_3) of a pencil $W = D_4$ and a c-dimensional space $D_3 \supseteq S_1 \circ W$, together with τ. These pairs form a variety $\mathcal{J}_c \subseteq \mathrm{Gr}(2, T_4) \times \mathrm{Gr}(c, T_3)$, the zero set of the section of the bundle $\mathrm{Hom}(S_1 \otimes \mathcal{W}, T_3/\mathcal{D})$ of rank $6(10 - c)$ induced by the contraction, where $\mathcal{W}$ and $\mathcal{D}$ are the tautological bundles. By Lemma 2.17, every irreducible component of $\mathcal{J}_c$ has dimension at least $26 + c(10 - c) - 6(10 - c) = 26 - (6 - c)(10 - c)$, which is 21, 26 and 29 for $c = 5, 6, 7$. We stratify $\mathcal{J}_c$ by $r = r(W) \le \min(c, 6)$; then $D_3 = S_1 \circ W \oplus C$ for a subspace C of dimension $c - r$, a point of a Grassmannian of dimension $(c - r)(10 - c)$. By Lemmas A.6 and A.9, the strata are as follows.

$r(W)$	pencils W	dim	how the algebras are smoothed
2	$\nu_4(\Gamma)$, $\Gamma \in \mathrm{Hilb}_2(\mathbb{P}^2)$	4	rays (Lemma 5.2), limits
3	$\mathcal{V}$, $\mathcal{S}$, $\mathcal{S}'$	8	rays, limits (Lemma A.7)
4	$\mathcal{P}$	15	rays
4	$\mathcal{N}$	14	linkage if $c = 4$; no component if $c \ge 5$
5	contraction kernel of dimension one	21	linkage if $c = 5$; no component if $c \ge 6$
6	an open subset of $\mathrm{Gr}(2, T_4)$	26	linkage ($c \ge 6$)

The dimensions are those of the strata of pencils in $\mathrm{Gr}(2, T_4)$, from the proofs of Lemmas A.6 and A.9: $\mathcal{V}$ is a $\mathbb{P}^2$-bundle over $\mathrm{Hilb}_3(\mathbb{P}^2)$, $\mathcal{S}$ has dimension $2 + 2 + 4$ and contains $\mathcal{S}'$ in its closure, $\mathcal{P}$ is parametrized by pairs $(L, [G])$ with $G \in \mathbb{P}(T_4/\mathbb{C}L^4)$, and $\mathcal{N}$ by a vector bundle of rank 10 over an open subset of $\mathrm{Gr}(2, S_1^2)$, of dimension 8, modulo the four-dimensional choice of a basis. If $r = 5$, the contraction kernel of a basis F_1, F_2 of W is spanned by one pair $(v_1, v_2) \ne 0$; for fixed (v_1, v_2), the pairs (F_1, F_2) with $v_1 \circ F_1 + v_2 \circ F_2 = 0$ form a space of dimension $30 - 10 = 20$, since contraction with a nonzero linear form maps T_4 onto T_3. So the triples $([v_1 : v_2], F_1, F_2)$ form a vector bundle of rank 20 over $\mathbb{P}^5$, and the pencils with $r = 5$ form an irreducible variety of dimension $25 - 4 = 21$.

Rays. In the strata with $r \le 3$ and in $\mathcal{P}$, the proofs do not depend on C. If $W = \nu_4(\Gamma)$ with $\Gamma = \{X, Y\}$, then $X^3 \in D_3$ and $N_X \circ W = \mathbb{C}Y^3$, so Lemma 5.2 gives a ray along the line of X; these data are dense in the stratum $r = 2$, a bundle over the irreducible $\mathrm{Hilb}_2(\mathbb{P}^2)$. If $W \in \mathcal{V}$ and Γ consists of three points P_1, P_2, P_3, Lemma 5.2 applies to P_1, because $N_{P_1} \circ W \subseteq \langle P_2^3, P_3^3 \rangle$; these data are dense in $\mathcal{V}$. If $W = \langle L^4, H \rangle \in \mathcal{S}$, Lemma 5.2 applies to $P = L$, because $N_L \circ W = N_L \circ H$ consists of binary cubics in Π. If $W \in \mathcal{S}'$,

the algebra is a limit of algebras over $\mathcal{S}$ by Lemma A.7, in which C stays complementary to $S_1 \circ W_t$ for t near 0. If $W \in \mathcal{P}$, then $L^3 \notin N_L \circ W$ by Lemma A.9, and Lemma 5.2 gives a ray.

Linkage. Let $c = 4$. In the remaining stratum, $W \in \mathcal{N}$ and $D_3 = S_1 \circ W$, an irreducible family whose general member has no cube in D_3, by Lemma A.9; so it is linked to a scheme of length at most $27 - 16 = 11$ by Theorem 3.4. In other words, the tangent cone of a general member lies on three cubic cones without a common line. Let $5 \le c \le 7$, and $r_0 = \min(c, 6)$. By the table, the strata with $r < r_0$ have dimension $4 + (c-2)(10-c)$, $8 + (c-3)(10-c)$, at most $15 + (c-4)(10-c)$, and, for $c \ge 6$, $21 + (c-5)(10-c)$, all less than the lower bound for the components. So $\mathcal{J}_c$ is irreducible, the closure of the stratum $r = r_0$. Its general member has no cube in D_3, since the following members have none: for $c = 5$, the ideal $(x^3, y^3, z^3, xz^2, yz^2, x^2y^2) + \mathfrak{m}^5$, with Hilbert function $(1, 3, 6, 5, 2)$ and pencil $D_4 = \langle X^2YZ, XY^2Z\rangle$; for $c = 6$, a pencil with $r = 6$ whose four cubics $(S_1 \circ W)^\perp$ have no common zero (script `l18_socle4.sage`); for $c = 7$, the space D_3 spanned by the cubic monomials other than X^3, Y^3, Z^3, with $D_4 = \langle X^2Y^2, X^2Z^2\rangle$ (script `l19_13672.sage`). In the first and the last example, x^3, y^3, z^3 annihilate D_3 and have no common zero. So the general algebra is linked to a scheme of length $27 - (12 + c)$ by Theorem 3.4.

In all cases Proposition 2.2(1) concludes. □

The proposition covers the Hilbert functions $(1, 3, 6, c, 2)$, $3 \le c \le 7$, of lengths fifteen to nineteen.

Theorem B.3. *Every local algebra with Hilbert function $(1, 3, 6, c, k)$, where $c \in \{4, 5\}$ and $k \in \{3, 4\}$, is smoothable.*

Proof. The cubic base scheme $\mathfrak{B}$ is short, or the graded data are those of c points of the plane, or the cubics of D_3 include those of a line and the quartics of D_4 are quartics of that line; in the last case a point leaves along the line of a point of it. Note that $D_4 \subseteq D_3'$, and that every point of $\mathfrak{B}$ has weight $k(p) - 3 = 1$ in Theorem 3.12, as $\mathfrak{m}^5 \subseteq I$.

(a) If $\mathfrak{B}$ is finite of length at most $c - 1$, then $e(\mathfrak{B}) \le c$, and Theorem 3.12 applies, as $27 + c < 2(10 + c + k)$.

(b) If $D_3 = \nu_3(\Gamma)$ and $D_4 \subseteq \nu_4(\Gamma)$ for a subscheme $\Gamma \subset \mathbb{P}^2$ of length c with $\dim \nu_3(\Gamma) = c$, Proposition 5.7 applies.

For $c = 4$, Lemma A.11 with $D = D_3$ shows that (a) or (b) holds: if $\mathfrak{B}$ has length four, then (b) holds with $\Gamma = \mathfrak{B}$; and if $D_3 = \mathbb{C}[X, Y]_3$ in suitable coordinates, then D_4 is a subspace of $D_3' = \mathbb{C}[X, Y]_4$ of dimension at most four, orthogonal to a binary quartic $g \ne 0$, and for the collinear scheme $\Gamma = V(z, g)$ of length four we have $(I_\Gamma)_3 = zS_2$ and $(I_\Gamma)_4 = zS_3 + \mathbb{C}g$, so $\nu_3(\Gamma) = \mathbb{C}[X, Y]_3 = D_3$ and $\nu_4(\Gamma) \supseteq D_4$: (b) holds.

For $c = 5$ we apply Lemma A.12 to $D = D_3$. Its case (a) is our (a), and its case (b) is our (b) with $\Gamma = \mathfrak{B}$. In its case (d), $D_3 \supseteq \nu_3(\Lambda)$ and $D_4 \subseteq D_3' = \nu_4(\Lambda)$ for a line Λ, which is our case (d) below. In its case (c),

$\mathfrak{B}$ is a line $V(l)$ with a point q, possibly embedded; in both cases the ideal of $\mathfrak{B}$ is $l\,\mathfrak{m}_q$, where $\mathfrak{m}_q$ is the ideal of q (in the embedded case, $I_{\mathfrak{B}} \subseteq (l)$ gives $I_{\mathfrak{B}} = l\,(I_{\mathfrak{B}} : l)$, and $(I_{\mathfrak{B}} : l)$ is saturated with Hilbert polynomial one, so it is the ideal of a point), and $D_3 = \nu_3(\mathfrak{B})$, $D_3' = \nu_4(\mathfrak{B})$, of dimension six. The classes of the quartics g modulo $(I_{\mathfrak{B}})_4$ with $g \perp D_4$ form a space $\mathcal{G}$ of dimension $6-k$. Let $\operatorname{ev}_q(g) = g(q)$ if $q \notin V(l)$, and let $\operatorname{ev}_q(g)$ be the value at q of the restriction of g to $V(l)$ if $q \in V(l)$; this is a linear form on $S_4/(I_{\mathfrak{B}})_4$. If g does not vanish identically on $V(l)$, then $\Gamma = \mathfrak{B} \cap V(g)$ has length five exactly when $\operatorname{ev}_q(g) = 0$ (in the embedded case, if the restriction of g vanishes at q to order $j \geq 1$, the length of Γ at q is $j+1$, and $4-j$ elsewhere). The classes vanishing identically on $V(l)$ form the line spanned by lu, where $u \in S_3$ and $u(q) \neq 0$.

(c1) If $\mathcal{G} \cap \ker \operatorname{ev}_q$ contains a class g that does not vanish identically on $V(l)$, then $\Gamma = \mathfrak{B} \cap V(g)$ has length five and does not lie on a line, $(I_\Gamma)_4 = (I_{\mathfrak{B}})_4 + \mathbb{C}g$ by Lemma 2.14, so $D_4 \subseteq \nu_4(\Gamma)$, and $D_3 = \nu_3(\mathfrak{B}) = \nu_3(\Gamma)$, as both have dimension five ($h_\Gamma(3) = 5$ because Γ does not lie on a line, as in the proof of Lemma A.12): (b) holds. This is always the case if $k = 3$, since then $\mathcal{G} \cap \ker \operatorname{ev}_q$ has dimension at least two.

(c2) Otherwise $k = 4$, and $\mathcal{G} \cap \ker \operatorname{ev}_q$, which is nonzero, is the line of lu; then $\operatorname{ev}_q(lu) = 0$, so $q \in V(l)$. In coordinates with $l = x$ and $q = [Z]$, we have $I_{\mathfrak{B}} = (x^2, xy)$, $\nu_4(\mathfrak{B}) = \mathbb{C}[Y,Z]_4 \oplus \mathbb{C}XZ^3$ and $lu \equiv xz^3$, so $D_4 \perp xz^3$ gives $D_4 \subseteq \mathbb{C}[Y,Z]_4 = \nu_4(V(x))$. As $D_3 \supseteq \mathbb{C}[Y,Z]_3 = \nu_3(V(x))$, (d) applies with $\Lambda = V(x)$.

(d) Suppose that $D_3 \supseteq \nu_3(\Lambda)$ and $D_4 \subseteq \nu_4(\Lambda)$ for a line $\Lambda \subset \mathbb{P}^2$; this happens in (c2) and in case (d) of Lemma A.12. In coordinates, $\Lambda = V(x)$ and $\nu_j(\Lambda) = \mathbb{C}[Y,Z]_j$, and the binary quartics $g \in \mathbb{C}[y,z]_4$ orthogonal to D_4 form a space of dimension $5-k \geq 1$. We first show:

(11)

if such a quartic $g \neq 0$ has a simple root P on Λ, then L_P is a ray of I.

Write $g = ha$ with $h \in \mathbb{C}[y,z]_1$ vanishing at P and $a \in \mathbb{C}[y,z]_3$ not vanishing at P. Then $N_P = \langle x, h\rangle$ and $x \circ D_4 = 0$, so $N_P \circ D_4 = h \circ D_4$. For $u \in D_4$ we have $\langle a, h \circ u\rangle = \langle g, u\rangle = 0$, while $\langle a, P^3\rangle = 6a(p) \neq 0$. So $P^3 \notin N_P \circ D_4$, and $P^3 \in \nu_3(\Lambda) \subseteq D_3$; Lemma 5.2 gives (11).

If $k = 3$, the quartics g form a pencil, and every pencil of binary forms has a member with a simple root. Indeed, write its members as $h_0 \cdot (sf_1 + tf_2)$, where h_0 is the greatest common divisor and f_1, f_2 are coprime of positive degree. The map $f_1/f_2 \colon \mathbb{P}^1 \to \mathbb{P}^1$ is not constant, so it is unramified at all but finitely many points; at such a point p that is not a root of h_0, the member vanishing at p has a simple root. So every algebra in (d) with $k = 3$ has a ray. If $k = 4$, then D_4 consists of the binary quartics orthogonal to a quartic g, unique up to scalar. For fixed Λ and D_3, the algebras form a vector bundle of maps $\tau \colon D_4 \to T_3/D_3$ over the projective space $\mathbb{P}(\mathbb{C}[y,z]_4) \cong \mathbb{P}^4$ of the quartics g (Lemma 2.7), an irreducible family that contains A. The quartics without a simple root are the squares of binary

quadrics, so those with a simple root form a dense open subset, and (11), Theorem 2.11(1) and Proposition 2.2(1) conclude. (Scripts `l18_socle4.sage` and `l19_13654.sage` check (c1) on examples with $q \notin V(l)$ and $q \in V(l)$; `l19_fivecubics_line.sage` and `l19_fivecubics_line2.sage` check the rays of (d) for $k = 3, 4$, including quartics g with multiple roots.) □

Proof of Theorem 3.8 *for* $c = 8$. We show that the algebras with Hilbert function $(1, 3, 6, 8, 1)$ form one irreducible family and link one member to a scheme of length 17. By Lemma 2.7, $M = T_{\leq 2} \oplus D_3 \oplus \mathbb{C}(F + \tau(F))$ with $D_4 = \mathbb{C}F$, $S_1 \circ F \subseteq D_3$, $\dim D_3 = 8$ and $\tau(F) \in T_3/D_3$ free.

The graded data. The pairs $(D_3, [F]) \in \mathrm{Gr}(8, T_3) \times \mathbb{P}(T_4)$ with $S_1 \circ F \subseteq D_3$ form the zero set $\mathcal{J}$ of a section of a bundle of rank $3 \cdot 2 = 6$ on a smooth irreducible variety of dimension $16 + 14 = 30$, so every irreducible component of $\mathcal{J}$ has dimension at least 24 (Lemma 2.17). We stratify by the number r of essential variables of F. If $r = 3$, then $S_1 \circ F$ has dimension three and D_3 is a point of $\mathrm{Gr}(5, T_3/S_1 \circ F)$: an irreducible stratum of dimension $14 + 10 = 24$. If $r = 2$, then F is a binary quartic in the two linear forms of a plane of T_1, and $D_3 \in \mathrm{Gr}(6, T_3/S_1 \circ F)$: dimension $2 + 4 + 12 = 18$. If $r = 1$, then F is a fourth power and $D_3 \in \mathrm{Gr}(7, T_3/S_1 \circ F)$: dimension $2 + 14 = 16$. So $\mathcal{J}$ is the closure of the stratum $r = 3$, and together with the tails, a vector bundle of rank two, the algebras with this Hilbert function form one irreducible family (Lemma 2.3(2)).

A linkage anchor. Let M_0 be spanned by $T_{\leq 2}$, the eight cubic monomials other than X^3 and Y^3, and X^2Y^2. It is closed under differentiation, with Hilbert function $(1, 3, 6, 8, 1)$, and its annihilator contains x^3, y^3 and z^4. The complete intersection $V(x^3, y^3, z^4)$ is supported at the origin and has length $36 < 38$; the residual scheme has length 17 and Hilbert function $(1, 3, 6, 7)$ (script `l19_13681.sage`). Proposition 2.24 concludes. □

Here I_3^* is a pencil, so three cubics are not available, and the third equation of the complete intersection has degree four, as in Lemma 3.17.

Theorem B.4. *Every local algebra with Hilbert function* $(1, 3, 6, 4, 5)$ *is smoothable.*

Proof. The graded data are unique up to coordinates, so the algebras form an affine space, and one member lies in a complete intersection of length 36. Let K be the ideal generated by $D_3^{\perp}$. Then $H_{S/K}(3) = 4$ and $H_{S/K}(4) = \dim D_3' \geq \dim D_4 = 5 = 4^{\langle 3 \rangle}$, so Macaulay's bound is attained, and Gotzmann's persistence theorem, applied as in the proof of Lemma 2.16, shows that K has the Hilbert polynomial $t+1$ of a line. So $\mathfrak{B}$ is not finite, and by Lemma A.11, $D_3 = \mathbb{C}[Y, Z]_3$ in suitable coordinates, and $D_4 = \mathbb{C}[Y, Z]_4$, which is the space of all quartics whose derivatives lie in $\mathbb{C}[Y, Z]_3$. So $I_3^* = xS_2$ and $I_4^* = xS_3$, and by Lemma 2.7 the algebras are parametrized by the affine space of the maps $\tau \colon \mathbb{C}[Y, Z]_4 \to T_3/\mathbb{C}[Y, Z]_3$, of dimension 30. The base scheme is the line $x = 0$, so Theorems 3.4 and 3.12 do not apply, and Lemma 5.2 fails for every point of the line, as $N_P \circ \mathbb{C}[Y, Z]_4 = \mathbb{C}[Y, Z]_3$.

Every element of I of order three has leading form xq with $q \in S_2$, and I contains an element $xq + v_q$ with $v_q \in \mathbb{C}[y,z]_4$ determined by τ. For the map τ given by

$$\begin{aligned}
\tau(Y^4) &= 3XZ^2 + XYZ + X^2Z - XY^2 + 3X^2Y + X^3,\\
\tau(Y^3Z) &= 2XYZ + 3X^2Z + 2XY^2 - 2X^2Y + 2X^3,\\
\tau(Y^2Z^2) &= -XZ^2 + 3XYZ + X^2Z + 3XY^2 + 2X^2Y - X^3,\\
\tau(YZ^3) &= -2XZ^2 + 2XYZ + 2X^2Z + 3XY^2 + 2X^2Y - X^3,\\
\tau(Z^4) &= 2XZ^2 + 2XYZ - 3X^2Z + XY^2 + 2X^2Y - X^3,
\end{aligned}$$

the three elements

$$\begin{aligned}
f_1 &= x^3 - \tfrac{1}{4}y^4 - 2y^3z + \tfrac{3}{2}y^2z^2 + yz^3 + \tfrac{1}{4}z^4,\\
f_2 &= xy^2 + \tfrac{1}{12}y^4 - \tfrac{2}{3}y^3z - \tfrac{3}{2}y^2z^2 - yz^3 - \tfrac{1}{12}z^4,\\
f_3 &= xz^2 - \tfrac{1}{4}y^4 + \tfrac{1}{2}y^2z^2 + \tfrac{2}{3}yz^3 - \tfrac{1}{6}z^4
\end{aligned}$$

of I generate an ideal J with $\dim_{\mathbb{C}} S/(J + \mathfrak{m}^N) = 36$ for $N = 9, 10$, so the local ring of $V(J)$ at the origin has length $36 < 38$; the residual scheme has length 17 and Hilbert function $(1,3,6,4,3)$ (script `l19_13645.sage`, exact computation over $\mathbb{Q}$). Lemma 3.3 does not apply here, because the leading forms vanish on the line $x = 0$; the length is computed directly. Proposition 2.24 concludes. □

When the cubics of I_3^* have a common factor, the algebra can often be deformed to one where they do not. We show this on the Hilbert function $(1,3,6,6,3)$, where an expected-dimension count suffices.

Theorem B.5. *Suppose that every finite subscheme of $\mathbb{A}^3$ of length at most eighteen is smoothable. Then every local algebra with Hilbert function $(1,3,6,6,3)$ is smoothable.*

Proof. Here $\mathfrak{m}^5 \subseteq I$ and $U = I_3^*$ has dimension four. If the cubics of U have no common factor, Proposition 3.18 applies. Otherwise the common factor is linear, since a common factor q of degree two would give $U \subseteq qS_1$, of dimension three. So $U = lV$ with $l \in S_1$ and $V \subseteq S_2$ of dimension four.

We show that these algebras are limits of algebras of the first kind. Let $\mathcal{J}$ be the variety of pairs (D_3, D_4) in $\mathrm{Gr}(6, T_3) \times \mathrm{Gr}(3, T_4)$ with $S_1 \circ D_4 \subseteq D_3$. It is the zero set of the section of the bundle $\mathrm{Hom}(S_1 \otimes \mathcal{D}_4, T_3/\mathcal{D}_3)$, of rank $3 \cdot 3 \cdot 4 = 36$, induced by differentiation, where $\mathcal{D}_3$ and $\mathcal{D}_4$ are the tautological bundles; the ambient variety has dimension $24 + 36 = 60$, so every irreducible component of $\mathcal{J}$ has dimension at least 24 (Lemma 2.17). The pairs with $D_3^\perp = lV$ have dimension at most 22: the pair (l, V) has at most $2 + 8$ parameters, and D_4 lies in the space of quartics all of whose derivatives lie in D_3, the orthogonal complement of S_1U in T_4. Since $H_{S/(V)}(2) = 2$, Macaulay's bound gives $H_{S/(V)}(3) \leq 2^{\langle 2 \rangle} = 2$, so $\dim S_1U = \dim S_1V \geq 8$, and D_4 moves in a Grassmannian $\mathrm{Gr}(3,7)$ of dimension at most 12. The

common-factor locus is closed, and by this count it contains no component of $\mathcal{J}$. So every point of it lies on a component of $\mathcal{J}$ whose general point has no common factor. Over this component, with its reduced structure, the maps τ form a vector bundle, and local frames give, by Lemma 2.3(2), an irreducible family containing A whose general member is smoothable by Proposition 3.18. Proposition 2.2(1) concludes. □

Script `l19_13663.sage` checks the construction of Proposition 3.18 on two examples; the proof above needs no computer.

Appendix C. Two points of the plane

C.1. In this appendix the graded data are governed by a subscheme $\Gamma \subset \mathbb{P}^2$ of length two: $D_k = \nu_k(\Gamma)$ in a range of degrees (Lemma 2.16 or Lemma 2.15), possibly with extra cubics. The models are Theorems 5.10 and 5.21. Corollary 3.14 settles most of these algebras by a count: those whose cubic base scheme $\mathfrak{B}$ is finite and, near the support of Γ, coincides with Γ or is curvilinear of length three, as long as the bounds of the corollary hold. Where the corollary is used, the proofs below treat only the remaining algebras. Over two points there are rays along lines, parabolas (Lemma 5.11) or curves (Corollary 5.5), often found by Lemma 5.13 (Proposition C.6); over the double point $\Gamma_0 = V(z, y^2)$ at $[X]$ we use the collision frame and Lemma 5.19 or Lemma 5.16. There the scheme $\mathfrak{B}$ is longer than Γ_0 near $[X]$ exactly when D_3 contains X^2Z or a cubic $XY^2 + aX^2Z$ with $a \in \mathbb{C}$. Indeed, the subschemes $\Gamma' \supset \Gamma_0$ of length three are the fat point and the curvilinear schemes $V(z - ay^2, y^3)$ in the chart $x = 1$; their spaces $\nu_3(\Gamma')$ are $\nu_3(\Gamma_0) + \mathbb{C}X^2Z$ and $\nu_3(\Gamma_0) + \mathbb{C}(XY^2 + aX^2Z)$; and $\nu_3(\Gamma') \subseteq D_3$ whenever $\Gamma' \subseteq \mathfrak{B}$ (script `excess_planeschemes.sage`).

In socle degree five we write the inverse system as in (8): for $H_A(5) = 1$,

$$M = T_{\leq 2} \oplus D_3 \oplus \{Q + \tau(Q) : Q \in D_4\} \oplus \mathbb{C}(F + R + E),$$

where F spans D_5, and closedness under differentiation is the condition (9). For $H_A(5) = 2$ the last summand is replaced by $\{F + R(F) + E(F) : F \in D_5\}$ with R and E linear in F, subject to (9) for every $F \in D_5$. The condition (9) does not change if τ is changed by a map into D_3, or R by an element of D_4. So we may count τ in $\operatorname{Hom}(D_4, T_3/D_3)$ and R in T_4/D_4, using local frames of the bundles involved.

When $H_A(2) = 6$, a ray detected by operators of order three needs no condition on the inverse system, since $T_{\leq 2} \subseteq M$; only the operators must lie in I. We use this in the following form.

Lemma C.2. *Let $A = S/I$ be a local algebra with $H_A(2) = 6$, and let $v \in S_1$ and $P \in T_1$ with $v \circ P \neq 0$. If $v^2 \circ M$ is contained in $\mathbb{C} \oplus \mathbb{C}P$, then L_P is a ray of I.*

Proof. Complete $a = v$ by a basis b, c of N_P to linear coordinates. Then $(a^2 b) \circ M = b \circ (v^2 \circ M) = 0$, since $b \circ P = 0$, so $a^2 b \in I$, and similarly

$a^2c \in I$. The element P^2 of $T_2 \subseteq M$ satisfies $b \circ P^2 = c \circ P^2 = 0$ and $\langle a^2, P^2\rangle = 2(v \circ P)^2 \neq 0$, so Lemma 5.4 applies. □

Theorem C.3. *Every local algebra with Hilbert function* $(1,3,6,3,2,2)$ *is smoothable.*

Proof. By Lemma 2.16 with $k = 4$, $D_4 = \nu_4(\Gamma)$ and $D_5 = \nu_5(\Gamma)$ for Γ of length two, and $D_3 = \nu_3(\Gamma) \oplus \mathbb{C}U$ for a cubic U. Here $c = 3$, $s = 5$ and $d = 17$, and $\mathfrak{B}$ is finite of length at most three. If $\mathfrak{B}$ coincides with Γ near the support of Γ, or is curvilinear of length three there, Corollary 3.14 applies, with the bounds 32 and 33, both less than 34. Otherwise $\mathfrak{B}$ has a non-curvilinear point of length three at a point of Γ; as $\ell(\mathfrak{B}) \leq 3$, this happens only when Γ is a double point and $\mathfrak{B}$ is the fat point containing it. In coordinates, $\Gamma = \Gamma_0$ and $U = X^2Z$, and the weight $2 \cdot 4$ of the fat point is too large.

The remaining algebras have fixed graded data: $D_3 = \langle X^3, X^2Y, X^2Z\rangle$, $D_4 = \langle X^4, X^3Y\rangle$ and $D_5 = \langle X^5, X^4Y\rangle$. As explained at the beginning of this appendix, the algebras with these graded data are given by their tails, subject to the linear conditions (9), so they form a vector space, and by Proposition 2.24 it suffices to link one of them to a short scheme. Let M_0 be spanned by $T_{\leq 2}$, the cubics X^3, X^2Y, X^2Z, the quartics $X^4 + 12XY^2$, $X^3Y + Y^3$ and the quintics $X^5 + 30X^2Y^2 + 20Y^3$, $X^4Y + 4XY^3$. It is closed under differentiation and has these graded data, and its ideal contains

$$K = (xyz,\ xz^2 + y^3 - x^3y - x^5,\ xy^2 + z^3 - x^4).$$

For an ideal J and elements a, b, multiplication by b maps $S/(J + (a))$ onto the kernel of $S/(J + (ab)) \to S/(J + (b))$, so the local length of $J + (ab)$ is at most the sum of those of $J + (a)$ and $J + (b)$. Hence the local length of K at the origin is at most $9 + 11 + 13 = 33$: modulo x the last two generators of K become y^3 and z^3; modulo y they become $x(z^2 - x^4)$ and $z^3 - x^4$, with local length at most $3 + 8$; modulo z they become $y^3 - x^3y - x^5$ and $x(y^2 - x^3)$, with local length at most $3 + 10$ (script `anchor136322.sage`). As $33 < 34$, Proposition 2.24 concludes. □

Theorem C.4. *Every local algebra with Hilbert function* $(1,3,6,3,2,2,1)$ *is smoothable.*

Proof. By Lemma 2.16 with $k = 4$, $D_4 = \nu_4(\Gamma)$, $D_5 = \nu_5(\Gamma)$ and $D_6 = \mathbb{C}F$ with $F \in \nu_6(\Gamma)$ for $\Gamma \subset \mathbb{P}^2$ of length two, and $D_3 = \nu_3(\Gamma) \oplus \mathbb{C}G$ for a cubic G. Let $\Phi = F + R_5 + R_4 + R_3$ be the lift of F.

Two points. Let $\Gamma = \{X, Y\}$, so $D_k = \langle X^k, Y^k\rangle$ for $k = 4, 5$. Suppose first that $F = X^6$. The elements of M of degree five are, modulo lower degrees, $x \circ \Phi/6$ and a lift $\Psi = Y^5 + U_4 + U_3$ of Y^5. Since $y \circ \Phi$ and $z \circ \Phi$ have degree at most four, $y \circ R_5, z \circ R_5 \in D_4$, so $R_5 = c_1X^5 + c_2X^4Y + c_3X^4Z + c_4Y^5$. With $\beta_2 = c_2/30$ and $\gamma_2 = c_3/30$, the operators $x^4(y - \beta_2x^2)$ and $x^4(z - \gamma_2x^2)$ kill Φ, since $x^4y \circ \Phi = 24c_2$ and $x^6 \circ \Phi = 720$; they kill Ψ, since x^4y and x^6 kill Y^5; and they kill $x \circ \Phi$ and $T_{\leq 4}$. The element $x^2 \circ \Phi/30$ has top form X^4 and cubic

part $\frac{2}{3}c_1X^3+12X^2(\beta_2Y+\gamma_2Z)$, so $E_4(\Lambda)\in M$ by (7), and Corollary 5.5 gives a ray, whatever G is. Now let $F=\alpha X^6+\beta Y^6$ with $\alpha\beta\neq 0$. Then $D_5=S_1\circ F$ and $D_4=S_2\circ F$, so $z\circ\Phi$ and $xy\circ\Phi$ have degree at most four, respectively three, and $z\circ R_5\in D_4$, $xy\circ R_5\in D_3\cap\mathbb{C}[X,Y]_3$ (the first condition makes R_5 binary up to terms X^4Z, Y^4Z). If $D_3\cap\mathbb{C}[X,Y]_3=\langle X^3,Y^3\rangle$, this gives $R_5\in\langle X^5,X^4Y,XY^4,Y^5,X^4Z,Y^4Z\rangle$, and the same parabola, with β_2, γ_2 the coefficients of X^4Y, X^4Z in R_5 divided by 30α, gives a ray, because x^2 kills XY^4, Y^5 and Y^4Z. If instead $G=gX^2Y+g'XY^2$ is binary, then $D_3=S_3\circ F+\mathbb{C}G$ and $M=T_{\leq 2}+D_3+S\circ\Phi$, by dimensions. An element $h\circ\Phi$ with $h\in\mathfrak{m}$ has degree at most four exactly when the linear part of h is a multiple μz of z, and at most three when moreover its quadratic part h_2 satisfies $h_2\circ F=-\mu\, z\circ R_5$. Following the parts of degree four and three of these elements, M has the Hilbert function $(1,3,6,3,2,2,1)$ if and only if $z\circ R_5\in D_4$, $xy\circ R_5\in D_3$ and $z\circ R_4-q\circ R_5\in D_3$, where $q\in\mathbb{C}[x,y]_2$ is any quadric with $q\circ F=z\circ R_5$. The computation of Section 5.17, applied to the automorphism with $\varphi(z)=z+\mu_1x^2+\mu_2y^2$ and to $F=\alpha X^6+\beta Y^6$, shows that this coordinate change changes R_5 by $30\mu_1\alpha X^4Z+30\mu_2\beta Y^4Z$ and nothing else in degree five, so we may assume $z\circ R_5=0$. Then $z\circ R_4\in D_3$ is binary, so z^2 kills R_4, R_5 and F, and $z^2\circ M\subseteq\mathbb{C}\oplus\mathbb{C}\,(z^2\circ R_3)$. If the coefficient of Z^3 in R_3 is nonzero, Lemma C.2 gives a ray. These normal forms, with (α,β), $[g:g']$, R_5 in a vector bundle of rank five, R_4 in an affine bundle of rank eight and R_3 free, form an irreducible family in which this coefficient is nonzero on a dense subset. (Script `l1363221_rays.sage` checks all identities with the tails as indeterminates.) So every algebra over two points is smoothable.

Double point. Let $\Gamma=\Gamma_0$; after a substitution $Y\mapsto Y+cX$, which fixes Γ_0 and every $\nu_k(\Gamma_0)$, D_6 is spanned by X^6 or by X^5Y. In the collision frame $\Gamma_t=\{X,X+tY\}$ of Section 5.17, with $D_3=\langle X^3,G_3(t),G\rangle$ and D_6 spanned by X^6 or by $G_6(t)$, which is X^5Y at $t=0$, the computation of Section 5.18 with all tails free shows, in eight charts for G and for both choices of D_6, that the ideal of the remaining equations is saturated with respect to t (script `l1363221_sat.sage`; for $D_6=\mathbb{C}G_6(t)$ the equations are solved by constant pivots in four of the charts). So every algebra over the double point is a limit of algebras over two points, and Proposition 2.2(1) concludes. □

Theorem C.5. *Every local algebra with Hilbert function* $(1,3,6,4,2,2)$ *is smoothable.*

Proof. By Lemma 2.16 with $k=4$, $D_4=\nu_4(\Gamma)$ and $D_5=\nu_5(\Gamma)$ for $\Gamma\subset\mathbb{P}^2$ of length two, and $D_3=\nu_3(\Gamma)\oplus K$ with K a two-dimensional subspace of $W=T_3/\nu_3(\Gamma)$. The pairs (Γ,K) form a $\mathrm{Gr}(2,8)$-bundle $\mathcal{M}$ over $\mathrm{Hilb}_2(\mathbb{P}^2)$, of dimension 16. As in (8), the inverse system is determined by linear maps $\tau\colon D_4\to T_3/D_3$, $R\colon D_5\to T_4/D_4$ and $E\colon D_5\to T_3/D_3$, with E free, and it is closed under differentiation if and only if $v\circ R(F)-\tau(v\circ F)\in D_3$ for $v\in S_1$, $F\in D_5$. Since $S_1\circ D_5=D_4$, this determines τ from R, and leaves the conditions coming from the kernel N of $S_1\otimes D_5\to D_4$, of rank four: the

linear map

$$\delta_R\colon N \to W, \qquad \textstyle\sum v_i \otimes F_i \mapsto \sum v_i \circ R(F_i) \bmod \nu_3(\Gamma),$$

must have its image in K. So the algebras form the zero set $\mathcal{Z}$ of a section of the rank-24 bundle $\operatorname{Hom}(N, W/K)$ on the total space of $\operatorname{Hom}(D_5, T_4/D_4) \oplus \operatorname{Hom}(D_5, T_3/D_3)$ over $\mathcal{M}$, which is smooth of dimension $16 + 26 + 12 = 54$; by Lemma 2.17 every component of $\mathcal{Z}$ has dimension at least 30, and local frames give a morphism $\mathcal{Z} \to \operatorname{Hilb}_{18}(\mathbb{A}^3)$. We show that every component contains a dense set of smoothable algebras, by listing the strata of $\mathcal{Z}$ of dimension at least 30.

Two points. Let $\Gamma = \{X, Y\}$ and $R_X = R(X^5)$, $R_Y = R(Y^5)$. The kernel N is spanned by $N_X \otimes X^5$ and $N_Y \otimes Y^5$, so δ_R consists of $\delta_X = (y \circ R_X, z \circ R_X)$ and $\delta_Y = (x \circ R_Y, z \circ R_Y)$, taken modulo $\nu_3(\Gamma)$; here $\delta_Y = 0$ exactly when $R_Y \in \langle Y^4, XY^3, Y^3Z, X^4 \rangle$. Two kinds of rays occur.

(a) If $\delta_Y = 0$, then R_Y has no terms X^3Y, X^3Z, so $x^3y \circ R_Y = x^3z \circ R_Y = 0$, and Lemma 5.11(2) applies with $a, b, c = x, y, z$ and $P = X$, for every K and all tails: one parabola tangent to the line of X serves the lifts of X^5 and Y^5. Symmetrically, $\delta_X = 0$ gives a ray along a parabola tangent to the line of Y.

(b) If δ_X and δ_Y have rank one with the same image, with kernels spanned by $w_1 = y - sz$ and $w_2 = x - s'z$, then $L = s'X + sY + Z$ is a third point and $L^3 \in D_3$: writing $\delta_X = (\alpha, \beta)\, l$ with $w_1 = \alpha z - \beta y$, the equality $z \circ (y \circ R_X) = y \circ (z \circ R_X)$ gives $w_1 \circ l \in \mathbb{C}Y^2$, and likewise $w_2 \circ l \in \mathbb{C}X^2$; these ten linear conditions on $l \in W$ have the solution space $\mathbb{C}L^3$ for all s, s' (script `l136422_strata.sage`). Moreover $L^3 \notin N_L \circ M$: the generators $v \circ m$ of $N_L \circ M$ with $v \in \{w_1, w_2\}$ have degree at most three except $w_2 \circ \Phi_X$ and $w_1 \circ \Phi_Y$, whose top forms $5X^4$, $5Y^4$ are independent, and the cubic parts of the others lie in $\nu_3(\Gamma) = \langle X^3, Y^3 \rangle$, which does not contain L^3. Theorem 2.11(2) gives a ray along the line of L.

Let $\mathcal{V}_1$ be the set of algebras with δ_X of rank two and $0 \neq \operatorname{im} \delta_Y \subseteq \operatorname{im} \delta_X$, and $\mathcal{V}_2$ its mirror image. Writing $x \circ R_Y \equiv a\, y \circ R_X + b\, z \circ R_X$ and $z \circ R_Y \equiv c\, y \circ R_X + d\, z \circ R_X$, the existence of R_Y amounts to $h \circ R_X \in \mathbb{C}X^2$ for $h = ayz + bz^2 - cxy - dxz \neq 0$, a space of R_X of dimension 8 modulo D_4; so $\mathcal{V}_1$ is the image of an irreducible variety of dimension $4 + 8 + 2 + 4 + 12 = 30$. The algebra with $R_X = R_Y = (X + Y)Z^3$, $K = \langle Z^3, (X + Y)Z^2 \rangle$ and $E = 0$ lies in $\mathcal{V}_1 \cap \mathcal{V}_2$, and three elements of order three of its ideal define a complete intersection of local length 33 at the origin, with residual scheme of length 15 (script `l136422_anchor.sage`); by Proposition 2.24, all algebras of $\mathcal{V}_1 \cup \mathcal{V}_2$ are smoothable.

The strata. We stratify $\mathcal{Z}$ by the ranks of δ_R. Over two points these are the ranks of δ_X and δ_Y. Over the double point we use the collision frame: there N has the basis $y \otimes X^5$, $z \otimes X^5$, $z \otimes G_5$, $x \otimes X^5 + 5t\, x \otimes G_5 - 5\, y \otimes G_5$, and we also record the rank of the pair $(c_1, c_2) = (y \circ R_X, z \circ R_X)$ modulo $\nu_3(\Gamma)$. The following table lists all strata; the dimensions add 4 for Γ over two points, 3 for

the double point, and 12 for E, and count R modulo D_4 and K in $\mathrm{Gr}(2, W)$ (scripts `l136422_strata.sage` and `l136422_doublepoint.sage`).

Γ	stratum	dim	smoothed by
two points	$\delta_R = 0$	32	$\mathcal{F}_0$, rays (a)
	$\delta_X = 0$, $\operatorname{rank}\delta_Y = 2$, or symmetrically	31	$\mathcal{F}_1$, $\mathcal{F}_1'$, rays (a)
	$\delta_X = 0$, $\operatorname{rank}\delta_Y = 1$, or symmetrically	30	rays (a)
	ranks $1,1$, same image, kernels $\neq \mathbb{C}z$	30	rays (b)
	ranks $1,1$, same image, a kernel $= \mathbb{C}z$	≤ 29	not a component
	ranks $1,1$, different images	28	not a component
	a rank 2, both nonzero	≤ 30	$\mathcal{V}_1 \cup \mathcal{V}_2$, linkage
double point	$\delta_R = 0$	31	$\subseteq \mathcal{F}_0$
	$\operatorname{rank}\delta_R = 1$	≤ 29	not a component
	$\operatorname{rank}\delta_R = 2$, $(c_1, c_2) = 0$	30	$\subseteq \mathcal{F}_1$
	$\operatorname{rank}\delta_R = 2$, $\operatorname{rank}(c_1, c_2) = 1$	28	not a component
	$\operatorname{rank}\delta_R = 2$, (c_1, c_2) independent	29	not a component

The list is exhaustive: over two points, if one of δ_X, δ_Y has rank two and the other is nonzero, then K is the image of the first and contains the image of the second, which is $\mathcal{V}_1$ or $\mathcal{V}_2$. The dimensions are counted as follows. If $\delta_R = 0$, then R lies in a space of dimension four and K is free; these algebras form a vector bundle $\mathcal{F}_0$ of rank $4 + 12$ over $\mathcal{M}$, in which two points are dense. If $\delta_X = 0$ and δ_Y has rank two, then $K = \operatorname{im}\delta_Y$; the data (R, E) with $\delta_X = 0$ form a vector bundle of rank $15 + 12$ over the marked pairs of Lemma A.2, and the algebras with $\delta_X = 0$ and δ_Y of rank two form an open subset $\mathcal{F}_1$ of it, in which two points are dense; symmetrically $\mathcal{F}_1'$. If $\delta_X = 0$ and δ_Y has rank one, then R_Y lies in a locus of dimension 6, R_X in a space of dimension 2, and $K \ni \operatorname{im}\delta_Y$ in a $\mathbb{P}^6$. If both have rank one with the same image, the ten conditions of (b) leave a space of dimension one for each pair (s, s'), giving 30; when one kernel is $\mathbb{C}z$, one of the parameters s, s' is lost and the space has dimension one, and when both are, both parameters are lost and the space has dimension two; this gives at most 29. Over the double point, the locus of R modulo D_4 with $\delta_R = 0$ has dimension 4, the locus with $\operatorname{rank}\delta_R \leq 1$ has dimension 8 (with K containing the image, ≤ 6 further dimensions), and the loci with $\operatorname{rank}\delta_R = 2$ and $\operatorname{rank}(c_1, c_2) = 1$, respectively 2, have dimensions 13 and 14: in the latter case the other two components of δ_R are combinations of c_1 and c_2 with four coefficients a, b, c, d, and the existence of $R(G_5)$ with these derivatives (the equality of mixed derivatives) reads $h \circ R_X \in \mathbb{C}X^2$ with $h = xz - (c + 5b)yz - dz^2 - 5ay^2$, which leaves 8 dimensions for R_X and 2 for $R(G_5)$.

Hence every component of $\mathcal{Z}$ meets one of the following in a dense set: the families $\mathcal{F}_0$, $\mathcal{F}_1$, $\mathcal{F}_1'$, irreducible with a dense set of algebras over two points

having rays (a); the strata of dimension 30 with rays (a) or (b); or $\mathcal{V}_1 \cup \mathcal{V}_2$. By Theorem 2.11(1) and Proposition 2.2(1), all algebras are smoothable. □

The proof of Lemma 5.14 extends to two points of the plane.

Proposition C.6 (Quartic jets)**.** *Let $A = S/I$ be a local algebra with $H_A(2) = 6$ and socle degree $s \in \{6, 7\}$, and let X, Y, Z be a basis of T_1. Suppose that one of the following holds:*

(a) *$s = 7$, $D_7 = \mathbb{C}X^7$, $D_6 = \mathbb{C}X^6$ and $D_5 \subseteq \langle X^5, Y^5\rangle$;*
(b) *$s = 6$, $X^6 \in D_6 \subseteq \langle X^6, Y^6\rangle$, $D_5 \subseteq \langle X^5, Y^5\rangle$ and $D_4 \subseteq \langle X^4, Y^4\rangle$.*

Then I has a ray along a curve $y = \beta_2 x^2 + \beta_3 x^3$, $z = \gamma_2 x^2 + \gamma_3 x^3$, tangent to the line of X, with $\beta_3 = \gamma_3 = 0$ in case (b)*. In particular, A is smoothable if all finite subschemes of length $\ell(A) - 1$ are.*

Proof. We apply Lemma 5.13 with $k = 4$; the space D_3 plays no role. In each case $X^4 \in D_4$ and $X^3 \in D_3$. Let F be X^7 in case (a) and X^6 in case (b), let $\Phi = F + R_{s-1} + \cdots + R_3$ be the element of M with top form F (its lift), and let Σ be spanned by Φ and, if $s = 7$, by $x \circ \Phi$. The forms $\langle x^j, \cdot\rangle$, $6 \le j \le s$, are linearly independent on Σ, as $\langle x^s, \Phi\rangle = s!$ and $\langle x^7, x\circ\Phi\rangle = 0 \neq \langle x^6, x\circ\Phi\rangle$. If $Y^6 \in D_6$, let $\Psi = Y^6 + V_5 + \dots$ be the lift of Y^6, and let $N = M_{\le 5} \oplus \mathbb{C}\Psi$, where $M_{\le 5}$ is the space of elements of M of degree at most five; otherwise $N = M_{\le 5}$. Then $M = N \oplus \Sigma$, and no element of N contains a monomial X^j with $j \ge 6$.

The parts of degree five. Those of the elements of $M_{\le 5}$ are zero or top forms in $D_5 \subseteq \langle X^5, Y^5\rangle$. The elements $x \circ \Psi$ and $z \circ \Psi$ of M have degree at most four, so $x \circ V_5$ and $z \circ V_5$ lie in $D_4 \subseteq \langle X^4, Y^4\rangle$; since x and z map distinct monomials divisible by X, respectively Z, to distinct monomials, $V_5 \in \langle X^5, XY^4, Y^5, Y^4Z\rangle$. So no part of degree five of an element of N contains X^4Y or X^4Z, and Lemma 5.13 gives $x^4(y - \beta_2 x^2 - \beta_3 x^3)$ and $x^4(z - \gamma_2 x^2 - \gamma_3 x^3)$ in I, with $\beta_3 = \gamma_3 = 0$ if $s = 6$. Evaluating $\langle x^4 y, \cdot\rangle = \beta_2\langle x^6, \cdot\rangle + \beta_3\langle x^7, \cdot\rangle$ at $x^{s-6}\circ\Phi$ gives $(s-2)!\, r_Y = s!\, \beta_2$, where r_Y and r_Z are the coefficients of $X^{s-2}Y$ and $X^{s-2}Z$ in R_{s-1}; so $\beta_2 = r_Y/(s(s-1))$, and likewise $\gamma_2 = r_Z/(s(s-1))$.

The tail R_{s-1}. We claim that the only monomials of R_{s-1} divisible by X^{s-4} are X^{s-1}, $X^{s-2}Y$ and $X^{s-2}Z$. If $s = 7$, then $y\circ\Phi$ and $z\circ\Phi$ have degree at most five, as $y \circ F = z \circ F = 0$, so $y \circ R_6$ and $z \circ R_6$ lie in $D_5 \subseteq \langle X^5, Y^5\rangle$. Monomial by monomial, this gives $R_6 \in \langle X^6, X^5Y, X^5Z, Y^6\rangle$. If $s = 6$, then $y \circ \Phi$ and $z \circ \Phi$ have degree at most four, so $y \circ R_5$ and $z \circ R_5$ lie in $D_4 \subseteq \langle X^4, Y^4\rangle$, and $R_5 \in \langle X^5, X^4Y, X^4Z, Y^5\rangle$.

The jet. The element $\frac{4!}{s!}\, x^{s-4} \circ \Phi$ of M has top form X^4, and by the claim its cubic part is $aX^3 + \frac{4!}{s!} \cdot \frac{(s-2)!}{2} X^2(r_Y\, Y + r_Z\, Z) = aX^3 + 12X^2(\beta_2 Y + \gamma_2 Z)$ for some $a \in \mathbb{C}$. By (7), $E_4(\Lambda)$ differs from it by $-aX^3$ modulo $T_{\le 2}$, so $E_4(\Lambda) \in M$, as $X^3 \in D_3$. Lemma 5.13 gives the ray, and Theorem 2.11(1) the last assertion. □

Lemma 5.14 is case (a) with $D_5 = \mathbb{C}X^5$. The proposition never looks at D_3. Over a double point it fails: if Γ is the double point at $[X]$ with tangent direction Y, then $X^3Y \in \nu_4(\Gamma)$, and the lift of $X^4Y \in \nu_5(\Gamma)$ is an element of N whose part of degree five contains X^4Y. Such algebras are treated as limits. The scripts `119_13632211.sage` and `119_1363222.sage` check cases (a) and (b), with all tails as indeterminates and an extra cubic in each of eight charts.

Theorem C.7. *Every local algebra with Hilbert function* $(1,3,6,3,2,2,2)$ *is smoothable.*

Proof. By Lemma 2.16 with $k = 4$, $n = 2$, there is $\Gamma \subset \mathbb{P}^2$ of length two with $D_k = \nu_k(\Gamma)$ for $k = 4, 5, 6$, and $D_3 = \nu_3(\Gamma) \oplus \mathbb{C}G$ for a cubic G. Here $c = 3$, $s = 6$ and $d = 19$, and $\mathfrak{B}$ has length at most three. Corollary 3.14 applies, with the bounds 34 and 36, both less than 38, unless $\mathfrak{B}$ has a non-curvilinear point at a point of Γ, that is, unless $\Gamma = \Gamma_0$ in suitable coordinates and $G = X^2Z$ modulo $\nu_3(\Gamma_0)$. For these algebras we use the collision frame of Section 5.17 with $D_k = \langle X^k, G_k(t)\rangle$ for $k = 4, 5, 6$ and $D_3 = \langle X^3, G_3(t), G\rangle$, where $G = X^2Z + \sum_j h_j m_j$ with free coefficients h_j on the seven cubic monomials m_j other than X^3, X^2Y and X^2Z. The ideal of the equations of Section 5.18 that remain after the eliminations by constant pivots is saturated with respect to t (script `119_1363222.sage`, chart X^2Z, an exact Gröbner basis computation over $\mathbb{Q}$). By Lemma 5.19, every algebra over Γ_0 with $G = X^2Z$ is a limit of algebras over two points, which are smoothable by Corollary 3.14, and Proposition 2.2(1) concludes. □

Theorem C.8. *Every local algebra with Hilbert function* $(1,3,6,3,2,2,1,1)$ *is smoothable.*

Proof. Over two points Proposition C.6(a) gives a ray; over a double point, Corollary 3.14 leaves two strata of the extra cubic, one settled by a collision, the other by a collision off one component C and a straight ray on C. By Lemma 2.16 with $k = 4$, $D_4 = \nu_4(\Gamma)$ and $D_5 = \nu_5(\Gamma)$ for Γ of length two, and $D_3 = \nu_3(\Gamma) \oplus \mathbb{C}G$. As in the proof of Theorem 5.21, $D_7 = \mathbb{C}X^7$ and $D_6 = \mathbb{C}X^6$, and $X^5 \in \nu_5(\Gamma)$ puts $[X]$ in Γ. Let $\Phi = X^7 + R_6 + \cdots + R_3$ be the lift of X^7.

Two points. If $\Gamma = \{X, Y\}$, Proposition C.6(a) applies, whatever G is.

Double points. Let $\Gamma = \Gamma_0$, so that $D_3 = \langle X^3, X^2Y, G\rangle$, where G has no terms X^3, X^2Y. Here $c = 3$, $s = 7$ and $d = 19$, and Corollary 3.14(1) applies, with the bound $36 < 38$, unless $\mathfrak{B}$ is longer than Γ_0 near $[X]$, that is, unless $G \in \langle XY^2, X^2Z\rangle$. Up to scaling, $G = XY^2$ or $G = X^2Z + hXY^2$. We run the collision frame Γ_t of Section 5.17, with $D_3 = \langle X^3, G_3(t), G\rangle$, $D_k = \nu_k(\Gamma_t)$ for $k = 4, 5$, $D_6 = \mathbb{C}X^6$, $D_7 = \mathbb{C}X^7$ and all tails free, through the computation of Section 5.18 (script `119_13632211.sage`), for $G = XY^2$ and for the family $G = X^2Z + h_6Y^3 + h_7XY^2$, which contains the cubics $X^2Z + hXY^2$.

For $G = XY^2$ the remaining equations have the form gf_i, where g is a polynomial not divisible by t and the f_i are solved by constant pivots. Then (f_i) is the prime ideal $\mathfrak{P}$ of a graph, $g \notin \mathfrak{P}$, and the ideal is $g\mathfrak{P} = (g) \cap \mathfrak{P}$, whose associated primes do not contain t; so it is t-saturated, and every algebra over Γ_0 with $G = XY^2$ is a limit of algebras over two points (Lemma 5.19).

For $G = X^2Z + h_6Y^3 + h_7XY^2$ the ideal $\mathfrak{J}$ of the remaining equations is not t-saturated. A tail coefficient u singled out by the remaining equations splits the family as $V(\mathfrak{J}) = V(\mathfrak{J} : u^\infty) \cup V(\mathfrak{J} + (u))$; here $\mathfrak{J} + (u)$ is t-saturated, and $\mathfrak{J} : u^\infty$ is prime and contains t: it defines an irreducible family C of algebras over Γ_0, the graph of a polynomial map over an affine space (script `l19_13632211_comp.sage`). By Section 5.17, the changes of coordinates $y \mapsto y + \varepsilon x^2$, $z \mapsto z + \varepsilon x^2$ and $z \mapsto z + \varepsilon x^3$ move the coefficients of X^5Y, X^5Z in R_6 by 42ε and that of X^4Z in R_5 by 210ε, and they preserve the graded data; the connected group they generate preserves the algebras over Γ_0 in $V(\mathfrak{J})$, hence each irreducible component of this set, in particular C. So every member of C is isomorphic to a member of the slice $C_0 \subseteq C$ where these three coefficients vanish, which is again an affine space. On C_0 we have $X^4 \in M$, and at the generic point of C_0, $X^4 \notin N_X \circ M$ (an exact rank computation over the function field of C_0). So on a dense open subset of C_0, Theorem 2.11(2) with $P = X$ and $k = 4$ gives a ray along the line of X, and all members of C_0, hence of C, are smoothable.

So every algebra over a double point is smoothable, and Proposition 2.2(1) concludes. □

At the generic point of C the cubic G is a derivative of M. The straight ray over the double point does not contradict the failure of Proposition C.6 there, which concerns the operators $x^4(y - \beta(x))$; it uses the special shape of C.

Theorem C.9. *Every local algebra with Hilbert function* $(1, 3, 6, 4, 2, 2, 1)$ *is smoothable.*

Proof. This is Theorem C.4 with one more cubic. Over two points we use parabolas (Proposition C.6(b)) or straight rays; over a double point, excess linkage (Corollary 3.14) in fifteen of the 28 cells of the extra cubics, and collisions and a cubic reduction in the other thirteen. By Lemma 2.16 with $k = 4$, $D_4 = \nu_4(\Gamma)$, $D_5 = \nu_5(\Gamma)$ and $D_6 = \mathbb{C}F$ with $F \in \nu_6(\Gamma)$ for Γ of length two, and $D_3 = \nu_3(\Gamma) \oplus K$ with $\dim K = 2$. Let $\Phi = F + R_5 + R_4 + R_3$ be the lift of F.

Two points, F a sixth power. Let $\Gamma = \{X, Y\}$ and $F = X^6$. Proposition C.6(b) applies.

Two points, F of rank two. After scaling, $F = X^6 + Y^6$. Since $z \circ F = 0$ and $xy \circ F = 0$, the elements $z \circ \Phi$ and $xy \circ \Phi$ have degree at most four and three, so $z \circ R_5 \in D_4$ and $xy \circ R_5 \in D_3$. Hence $R_5 = P + Z(aX^4 + bY^4)$ with $P = \sum_i p_i X^{5-i} Y^i$ binary, and $xy \circ P = 4p_1X^3 + 6p_2X^2Y + 6p_3XY^2 + 4p_4Y^3$ lies in $\Delta = D_3 \cap \mathbb{C}[X, Y]_3$, which contains $\nu_3(\Gamma) = \langle X^3, Y^3 \rangle$. Here $M = M_{\leq 5} \oplus \mathbb{C}\Phi$ and the parts of degree five of $M_{\leq 5}$ lie in $D_5 = \langle X^5, Y^5 \rangle$, so

Lemma 5.13 with $k = 4$ gives $x^4(y - \beta_2 x^2)$ and $x^4(z - \gamma_2 x^2)$ in I with $\beta_2 = p_1/30$ and $\gamma_2 = a/30$. The element $x^2 \circ \Phi/30$ has top form X^4 and cubic part $\frac{2}{3}p_0X^3 + \frac{2}{5}X^2(p_1Y + aZ) + \frac{1}{5}p_2XY^2 + \frac{1}{15}p_3Y^3$, and $12\beta_2 = \frac{2}{5}p_1$, $12\gamma_2 = \frac{2}{5}a$. By (7), $E_4(\Lambda) \in M$, and there is a ray along a parabola tangent to the line of X, as soon as

$$3p_2XY^2 + p_3Y^3 \in D_3;$$

symmetrically, there is one tangent to the line of Y as soon as $p_2X^3 + 3p_3X^2Y \in D_3$. If $\Delta = \nu_3(\Gamma)$, then $p_2 = p_3 = 0$, and the first condition holds. If $\dim \Delta = 3$, then $\Delta = \nu_3(\Gamma) \oplus \mathbb{C}(uX^2Y + vXY^2)$ and $(p_2, p_3) = \frac{\lambda}{6}(u, v)$ for some $\lambda \in \mathbb{C}$; the first condition holds when $\lambda u = 0$, and the second when $\lambda v = 0$. If $\Delta = \mathbb{C}[X, Y]_3$, both conditions hold. There remains the case $\dim \Delta = 3$ with $u \neq 0$. We normalize $u = 1$, and a fourth basis vector G of D_3 to have, among the binary monomials, at most the term XY^2, and its first nonzero coefficient among $Z^3, YZ^2, XZ^2, Y^2Z, XYZ, X^2Z$ equal to one, with free coefficients on the later monomials and on XY^2. In each of these six families of graded data, all equations of Section 5.18 are solved by constant pivots, so the algebras form an affine space, and at its generic point $X^3 \notin N_X \circ M$ (script `l19_1364221_b3x.sage`). As $X^3 \in D_3$, Theorem 2.11(2) with $P = X$ and $k = 3$ gives a ray along the line of X on a dense open subset.

Double points: normal forms. Let $\Gamma = \Gamma_0$, the double point at $[X]$ with ideal (z, y^2). After a substitution $Y \mapsto Y + cX$, D_6 is spanned by X^6 or by X^5Y. Let $N = \mathfrak{m} \circ M + T_{\leq 2}$. If $D_3 \not\subseteq N$, Lemma A.15 gives $M = M' \oplus \mathbb{C}G$ with M' the inverse system of an algebra with Hilbert function $(1, 3, 6, 3, 2, 2, 1)$ over Γ_0. By the proof of Theorem C.4, M' lies in a family as in Lemma 2.3(2), over an irreducible base, with constant Hilbert function $(1, 3, 6, 3, 2, 2, 1)$, in which the algebras over two points are dense. Over the open subset where $G \notin D_3(M'_u)$, the spaces $M'_u \oplus \mathbb{C}G$ form a family through M (Lemma A.14) whose members over two points have Hilbert function $(1, 3, 6, 4, 2, 2, 1)$ and are smoothable by the first two steps; so A is smoothable. We may therefore assume $D_3 \subseteq N$ whenever this helps, but the computations below cover all K.

Write K modulo $\nu_3(\Gamma_0)$ in reduced echelon form with respect to the order $m_0, \ldots, m_7 = Z^3, YZ^2, XZ^2, Y^2Z, XYZ, X^2Z, Y^3, XY^2$ of the cubic monomials other than X^3, X^2Y; its pivots (i, k), $i < k$, define 28 Schubert cells. If $K \cap \langle X^2Z, XY^2 \rangle = 0$, the cubic base scheme $\mathfrak{B}$ coincides with Γ_0 near $[X]$ (see the introduction of this appendix), and it is finite, since otherwise $D_3 = \mathbb{C}[X, Y]_3$ and $XY^2 \in K$; so Corollary 3.14(1) applies, with the bound $35 < 38$. A nonzero element of $K \cap \langle X^2Z, XY^2 \rangle$ is a combination of the two echelon basis vectors of K; if it involves the first, then $i = 5$, and otherwise the second lies in $\langle X^2Z, XY^2 \rangle$, and $k \in \{5, 7\}$. So only the thirteen cells $(i, 5)$ and $(i, 7)$ with $i \leq 4$, and $(5, 6)$, $(5, 7)$, $(6, 7)$ remain. The substitutions $Z \mapsto Z + aX + bY$ and, if $F = X^6$, $Y \mapsto Y + cX$ fix Γ_0, every $\nu_k(\Gamma_0)$ and F, and they act on each cell by unipotent lower-triangular

transformations; solving for them, one sets up to three cell coordinates to zero (script `l19_1364221_nf.sage`). For each normalized remaining cell we run the collision frame Γ_t, with $D_3 = \nu_3(\Gamma_t) \oplus K$ for constant K and D_6 spanned by X^6, respectively $G_6(t)$, through the computation of Section 5.18.

$F = X^5Y$. In the cell $(5,6)$ the ideal of the remaining equations is t-saturated, and in the other twelve cells all equations are solved by constant pivots.

$F = X^6$. In the cells $(1,7)$ and $(6,7)$ all equations are solved by constant pivots, and in the cells $(0,7)$ and $(3,7)$ the ideal is t-saturated. In the nine cells $(0,5)$, $(1,5)$, $(2,5)$, $(3,5)$, $(4,5)$, $(2,7)$, $(4,7)$, $(5,6)$, $(5,7)$ we compute the minimal primes of the ideal that contain t, that is, the components of the family lying over Γ_0; each is the graph of a polynomial or rational map over an affine space (script `l19_1364221_comp.sage`). As in the proof of Theorem C.8, the substitutions $z \mapsto z + \varepsilon_1 x^2$ and $z \mapsto z + \varepsilon_2 x^3$ preserve each component; they move the coefficient of X^4Z in R_5 by $30\varepsilon_1$ and that of X^3Z in R_4 by $120\varepsilon_2$ plus terms in ε_1, so each component is isomorphic to $\mathbb{A}^2$ times its slice where these two coefficients vanish, which is therefore irreducible, and every member is isomorphic to a member of the slice. At the generic point of the slice, $X^k \in M$ and $X^k \notin N_X \circ M$ for $k = 3$ or 4, so Theorem 2.11(2) gives a ray along the line of X on a dense open subset, for every component except one in each of the cells $(0,5)$, $(1,5)$, $(2,5)$. On these three components, at the generic point, $\dim N = 16 < 17 = \dim(N + D_3)$. Since $N + D_3 = \mathfrak{m} \circ M + (M \cap T_{\leq 3})$ has dimension 17 on all algebras with $F = X^6$, and $D_3 \subseteq N$ is the open condition $\dim N \geq 17$, we have $D_3 \not\subseteq N$ on the whole component, and the cubic reduction above applies.

Hence every algebra over a double point is smoothable, and Proposition 2.2(1) concludes. □

The straight rays over two points for $\dim \Delta \geq 3$ replace the transverse ray of Theorem C.4, which is of order three and does not survive the extra cubic.

The following Hilbert function shows that collisions can often be avoided altogether (Section 5.22).

Theorem C.10. *Suppose that every finite subscheme of $\mathbb{A}^3$ of length at most eighteen is smoothable. Then every local algebra with Hilbert function $(1,3,6,5,2,2)$ is smoothable.*

Proof. We show that all these algebras form one irreducible family, and that one member lies in a complete intersection of length $32 < 38$.

The incidence. By Lemma 2.16 with $k = 4$ and $n = 2$, $D_4 = \nu_4(\Gamma)$ and $D_5 = \nu_5(\Gamma)$ for a unique $\Gamma \in \mathrm{Hilb}_2(\mathbb{P}^2)$. Then $\nu_3(\Gamma) = S_1 \circ D_4$ is a plane in D_3, and D_3 is determined by the three-dimensional subspace $D_3/\nu_3(\Gamma)$ of $T_3/\nu_3(\Gamma)$. Since $S_1 \circ D_5 = D_4$, the elements of M with top forms in D_4 are, up to elements of degree at most three, derivatives of the elements with top forms in D_5. Let F_1, F_2 be a basis of D_5 and $\Phi_i = F_i + R_i + E_i$ elements of M with tails $R_i \in T_4$ and $E_i \in T_3$. The kernel of the map $S_1 \otimes D_5 \to D_4$, $v \otimes F \mapsto v \circ F$, has dimension $6 - 2 = 4$, and for a relation $\sum_k v_k \otimes F_k$ in

it, the element $\sum_k v_k \circ \Phi_k$ of M has degree at most three. So the space spanned by $T_{\leq 2}$, D_3 and the Φ_i and their derivatives is the inverse system of an algebra with Hilbert function $(1,3,6,5,2,2)$ if and only if, for each of the four relations, the cubic $\sum_k v_k \circ R_k$, which we call a *column*, lies in D_3; the cubic tails E_i are free. For two points $\Gamma = \{X, Y\}$, with $F_1 = X^5$ and $F_2 = Y^5$, the columns are $c_1 = y \circ R_1$, $c_2 = z \circ R_1$, $c_3 = x \circ R_2$ and $c_4 = z \circ R_2$; for the double point at $[X]$ tangent to $[Y]$, with $F_1 = X^5$ and $F_2 = X^4 Y$, they are $y \circ R_1$, $z \circ R_1$, $z \circ R_2$ and $y \circ R_2 - \frac{1}{5} x \circ R_1$.

The data $(\Gamma, D_3/\nu_3(\Gamma), R_1, R_2, E_1, E_2)$, with R_i taken modulo D_4 and E_i modulo D_3, form a smooth irreducible variety of dimension $4+15+26+10 = 55$, a tower of bundles over $\mathrm{Hilb}_2(\mathbb{P}^2)$ (Lemma 2.14). The condition that the four columns lie in D_3 is the vanishing of a section of a vector bundle of rank $4 \cdot 5 = 20$, with fibers $\mathrm{Hom}(\mathbb{C}^4, T_3/D_3)$. By Lemma 2.17, every component of the zero set $\mathcal{Y}$ has dimension at least 35, and $\mathcal{Y}$ maps onto the set of algebras with this Hilbert function, by Lemma 2.3(2).

The strata. Let $r \leq 3$ be the rank of the four columns modulo $\nu_3(\Gamma)$. Three facts are used, for both forms of Γ (script `119_136522.sage`, part (A)): (i) differentiation by a nonzero linear form maps T_4 onto T_3; (ii) the maps $R_1 \mapsto (y \circ R_1, z \circ R_1)$ and $R_2 \mapsto (x \circ R_2, z \circ R_2)$, respectively $(z \circ R_2, y \circ R_2)$, from T_4/D_4 to $(T_3/\nu_3(\Gamma))^2$ have kernels of dimension two, by Lemma A.3; (iii) for every nonzero vector $\kappa \in \mathbb{C}^4$, the map sending (R_1, R_2) to the combination of the columns with coefficients κ is onto $T_3/\nu_3(\Gamma)$, by (i). If $r = 3$, then $D_3/\nu_3(\Gamma)$ is the span of the columns, and there is a unique $[\kappa] \in \mathbb{P}^3$ with vanishing combination; by (iii), the pairs (R_1, R_2) with a given κ form a space of dimension $26 - 8 = 18$. So the data with $r = 3$ over two points form an open subset $\mathcal{U}$ of a vector bundle of rank $18 + 10$ over the product of $\mathbb{P}^3$ with the four-dimensional open subset of reduced Γ; it is irreducible of dimension 35, and nonempty by the example below. Over the three-dimensional locus of double points, the same count gives 34. For $r \leq 2$ the space $D_3/\nu_3(\Gamma)$ contains the span of the columns and moves in a Grassmannian $\mathrm{Gr}(3-r, 8-r)$, of dimension $5(3-r)$, while the pairs (R_1, R_2) are constrained more. We count them for two points; let a be the rank of the first two columns modulo $\nu_3(\Gamma)$. If $r = 2$ and $a = 2$, then $x \circ R_2 \equiv \alpha\, y \circ R_1 + \beta\, z \circ R_1$ and $z \circ R_2 \equiv \gamma\, y \circ R_1 + \delta\, z \circ R_1$ modulo $\nu_3(\Gamma)$; since $z \circ (x \circ R_2) = x \circ (z \circ R_2)$, $x \circ \nu_3(\Gamma) = \mathbb{C}X^2$ and $z \circ \nu_3(\Gamma) = 0$, the quadric $D = \alpha yz + \beta z^2 - \gamma xy - \delta xz$ satisfies $D \circ R_1 \in \mathbb{C}X^2$. If $D \neq 0$, differentiation by D maps T_4/D_4 onto $T_2/\mathbb{C}X^2$, so R_1 moves in dimension $13 - 5 = 8$, and R_2 in dimension 2 by (ii): at most $4 + 8 + 2 = 14$ parameters. If $D = 0$, then R_2 lies in the kernel of (ii) and R_1 is free: $2 + 13 = 15$. If $a = 1$, some $d \in \langle y, z \rangle$ has $d \circ R_1 \in \nu_3(\Gamma)$, which leaves $1 + 5$ parameters for $([d], R_1)$ by (i); as $r = 2$, the last two columns satisfy a relation $\lambda_3 c_3 + \lambda_4 c_4 \equiv \mu c$ modulo $\nu_3(\Gamma)$, where c is a column among c_1, c_2 that is nonzero modulo $\nu_3(\Gamma)$ and $(\lambda_3, \lambda_4) \neq 0$; this gives $1 + 1$ parameters, and then R_2 lies in an affine space of dimension $13 - 8 = 5$ by (iii): at most 13. If $a = 0$, then R_1 is in a kernel and R_2 free: 15. If $r = 1$ and $a = 1$, the last two columns are

multiples modulo $\nu_3(\Gamma)$ of the same nonzero column c (2 parameters), and R_2 is determined up to the kernel of (ii): $6+2+2=10$; if $r=1$ and $a=0$, then R_1 lies in the kernel of (ii), the last two columns satisfy one relation, and R_2 lies in an affine space of dimension 5: $2+1+5=8$. If $r=0$, both R_1 and R_2 lie in kernels: $2+2=4$. Over the double point the same bounds hold: the fourth column differs by $-\frac{1}{5}x\circ R_1$, which shifts the conditions on R_2 by a translation, and the compatibility $y\circ(z\circ R_2)=z\circ(y\circ R_2)$ gives $D\circ R_1\in\mathbb{C}X^2$ with a quadric D that contains the term $-\frac{1}{5}xz$ and is never zero. So the strata with $r=2,1,0$ have dimension at most

$$4+15+5+10=34,\qquad 4+10+10+10=34,\qquad 4+4+15+10=33,$$

and one less over double points. All of them are smaller than 35. By Lemma 2.18, $\mathcal{Y}$ is irreducible, the closure of $\mathcal{U}$.

A member linked to a short scheme. Let M_0 be spanned by $T_{\le 2}$, the cubics X^3, Y^3, Z^3, X^2Y+YZ^2, X^2Z+Y^2Z, the elements $X^4+\frac{2}{5}XY^2+\frac{2}{5}XZ^2$ and Y^4, and the elements

$$X^5+X^2Y^2+X^2Z^2+Y^2Z^2,\qquad Y^5+Z^4-\tfrac{1}{2}X^2Y.$$

It is closed under differentiation, with Hilbert function $(1,3,6,5,2,2)$, $\Gamma=\{X,Y\}$ and $r=3$. The polynomials

$$x(y^2+yz+z^2)-\tfrac{1}{15}x^4,\qquad y(x^2-z^2)+\tfrac{1}{120}y^5,\qquad z(x^2-y^2)+\tfrac{1}{120}z^5$$

lie in its ideal and define a complete intersection of local length 32 at the origin, with a residual scheme of length 13 (script `119_136522.sage`, part (B)). As $32<38$, Proposition 2.24 shows that all algebras of the family are smoothable. □

The cubic base scheme of M_0 consists of the three coordinate points, with $k=5,5,4$, and $32=27+2+2+1$ is the bound of Theorem 3.12.

Appendix D. Three points of the plane

D.1. Here Lemma 2.16 gives a subscheme $\Gamma\subset\mathbb{P}^2$ of length three with $D_3=\nu_3(\Gamma)$ and $D_4=\nu_4(\Gamma)$. The model is Corollary 5.8. When the length is large compared with the socle degree, excess linkage settles everything at once: the Hilbert functions $(1,3,6,3,3,2)$, $(1,3,6,3,3,3)$ and $(1,3,6,3,3,2,1)$ are covered by Corollaries 3.15 and D.4. For $(1,3,6,3,3,1)$ and $(1,3,6,3,3,1,1)$ the numerical test fails for some Γ, and rays along parabolas or cubic curves over three non-collinear points take over, together with families of constant rank over the strata where the points collide. In the last theorem of this appendix, $H_A(3)=4$, and an incidence of plane schemes of length four appears.

We first refine Theorem 3.12. When the top form is a power X^s, the closedness of M often leaves some quartic tails free. Adding such a tail to the lift of X^s moves the values of degree $s-1$ at every point of $\mathfrak{B}$, and lowers by one the weight $s-3$ of every point with $k(p)=s$.

Remark D.2. Step 2 of the proof of Theorem 3.12 does not use how the elements f_i^ε were constructed. The conclusion of Step 3 holds whenever, for all but finitely many ε, the ideal of some member of an irreducible family through A contains elements f_i^ε of the form of Step 2 that satisfy its hypotheses, and $27 + \sum_p (k_p - 3)e_p < 2d$. The vector θ may depend on the w_i, and one then checks that the cubics $h_1 = w_1 - \frac{\theta_1}{\theta_3} w_3$, $h_2 = w_2 - \frac{\theta_2}{\theta_3} w_3$ and w_3 satisfy them; this holds when $\theta_3 \neq 0$ and (h_1, h_2, w_3) is a general triple of cubics of I_3^*. In Lemma D.3 and Corollary D.4 the values of degree $s-1$ are moved by a quartic tail in this way, which lowers the weight $(s-3)e_p$ of a point with $k(p) = s$ to $(s-4)e_p$; in socle degree six, $2e_p$ instead of $3e_p$.

Lemma D.3 (Free quartic tails)**.** *Let $A = S/I$ be a local algebra of length d with $H_A(2) = 6$, socle degree $s \geq 6$ and $D_s = \mathbb{C}X^s$, and suppose that every finite subscheme of $\mathbb{A}^3$ of length less than d is smoothable. Suppose that $\dim I_3^* \geq 3$ and that $\mathfrak{B}$ is finite. Let $K = \{R \in T_4 : N_X \circ R \subseteq D_3\}$, and suppose that the cubics $u \circ R$, $R \in K$, span a space of dimension at least three modulo D_3, where $u \in S_1$ is a linear form with $u \circ X = 1$. If*

$$27 + \sum_{p \in \mathfrak{B}} \left(\min(k(p), s-1) - 3 \right) e_p < 2d,$$

then A is smoothable.

The hypothesis does not depend on u, since $N_X \circ R \subseteq D_3$ for $R \in K$.

Proof. Choose u not vanishing at the finitely many points p of $\mathfrak{B}$ with $k(p) = s$. Let $\Phi \in M$ have top form X^s, and let $M'' = \{m \in M : \langle u^{s-1}, m\rangle = \langle u^s, m\rangle = 0\}$. As $\langle u^s, \Phi\rangle = \langle u^{s-1}, u \circ \Phi\rangle = s!$ and $\langle u^s, u \circ \Phi\rangle = 0$, we have $M = M'' \oplus \mathbb{C}\, u \circ \Phi \oplus \mathbb{C}\Phi$. An element of M'' has degree at most $s-1$, since otherwise its top form would be a nonzero multiple of X^s. Hence M'' is closed under differentiation: for $m \in M''$ and $v \in S_1$, $\langle u^{s-1}, v \circ m\rangle = \langle u^{s-1}v, m\rangle = 0$ by degrees, and likewise for u^s. Moreover M'' contains $T_{\leq 2} \oplus D_3$, the elements $v \circ \Phi$ with $v \in N_X$, as $\langle u^{s-1}v, \Phi\rangle = \langle u^{s-1}v, X^s\rangle = 0$, and $u^2 \circ \Phi$.

The line. For $R \in K$ and $\lambda \in \mathbb{C}$ let $M_\lambda = M'' \oplus \mathbb{C}\,(u \circ \Phi + \lambda\, u \circ R) \oplus \mathbb{C}\,(\Phi + \lambda R)$. It is closed under differentiation: $v \circ (\Phi + \lambda R) = v \circ \Phi + \lambda\, v \circ R$ lies in M'' for $v \in N_X$, as $v \circ R \in D_3$; $u \circ (\Phi + \lambda R)$ is a basis element; and the derivatives of $u \circ \Phi + \lambda\, u \circ R$ differ from those of $u \circ \Phi$, which lie in M'', by quadrics. An element of M_λ outside M'' has the same parts of degrees $s-1$ and s as the corresponding element of M, so M_λ has the spaces D_k of M. By Lemma 2.3(2), the ideals I_λ of the spaces M_λ form a family over the line with $I_0 = I$.

Elements of I_λ. Let $f \in I$ have leading form $w \in I_3^*$ and quartic part $f^{(4)}$. Then $f + \lambda(\theta u^{s-1} + \theta' u^s) \in I_\lambda$ for $\theta = -\langle uw, R\rangle/s!$ and $\theta' = -(\langle f^{(4)}, R\rangle + \theta\, \langle u^{s-1}, \Phi\rangle)/s!$: both u^{s-1} and u^s pair to zero with M'', and the pairings with the two other basis elements are $\lambda(\langle w, u \circ R\rangle + \theta\, s!) = 0$ and $\lambda(\langle f^{(4)}, R\rangle + \theta\langle u^{s-1}, \Phi\rangle + \theta' s!) = 0$, the terms with λ^2 vanishing by degrees. Elements of

I of order at least five lie in I_λ, and if $g \in I$ has order four and leading form q, then $g + \lambda a u^s \in I_\lambda$ for $a = -\langle q, R\rangle/s!$.

Steps 2 and 3. Let $w_1, w_2, w_3 \in I_3^*$, let $f_i \in I$ have leading forms w_i, and let θ_i, θ_i' be as above. For $4 \le j \le s-1$ let q_j be a general element of I_j^* and $g_j \in I$ an element with leading form q_j ($g_j = 0$ if $I_j^* = 0$), and let $a = -\langle q_4, R\rangle/s!$, as above. For $\varepsilon \neq 0$ and $\lambda = \varepsilon^{-(s-4)}$ the polynomials

$$f_i^\varepsilon = f_i + \theta_i\Big(\sum_{4\le j\le s-2} \varepsilon^{-(j-3)} g_j + \varepsilon^{-(s-4)}\big(g_{s-1} + u^{s-1}\big)\Big) + \varepsilon^{-(s-4)}\theta_i' u^s + \varepsilon^{-(s-3)}\theta_i a u^s$$

lie in I_λ, and they have the form of Step 2 of the proof of Theorem 3.12, with the forms $q_4, \dots, q_{s-2}$, $q_{s-1} + u^{s-1}$ and au^s in the degrees $4, \dots, s$. At $[X]$ all q_j vanish, as $X^j \in D_j$, and $q_{s-1} + u^{s-1}$ does not. At another point p of $\mathfrak{B}$, the first of these forms that does not vanish at p has degree $\min(k(p), s-1)$: if $k(p) \le s-1$, it is $q_{k(p)}$ or $q_{s-1} + u^{s-1}$, by generality; if $k(p) = s$, then $q_{s-1}(p) = 0 \neq u(p)$. The numbers θ_i are the values at w_i of the linear form $w \mapsto -\langle w, u \circ R\rangle/s!$ on I_3^*, which depends on $u \circ R$ modulo $D_3 = (I_3^*)^\perp$. By hypothesis these linear forms span a space V of dimension at least three. So the map $(w_1, w_2, w_3, R) \mapsto (h_1, h_2, w_3)$, with $h_i = w_i - \frac{\theta_i}{\theta_3} w_3$ for $i = 1, 2$, defined where $\theta_3 \neq 0$, is dominant: given h_1, h_2, some nonzero form of V, given by some $R \in K$, vanishes on h_1 and h_2, and for every w_3 outside its kernel, (h_1, h_2, w_3, R) maps to (h_1, h_2, w_3). Hence, for general w_1, w_2, w_3 and R, the triple (h_1, h_2, w_3) is a general triple of cubics of I_3^*, and Step 2 of the proof of Theorem 3.12 applies with $k_p = \min(k(p), s-1)$ (Remark D.2). As in Step 3 there, for all but finitely many ε, a complete intersection of length at most $27 + \sum_p (\min(k(p), s-1) - 3)e_p < 2d$ contains $V(I_\lambda)$, and Proposition 2.24, applied to the whole line, concludes. □

The hypothesis on K holds, for instance, when D_3 contains X^2T_1, the space ν_3 of the fat point at $[X]$, but no nonzero element of $X \cdot \mathbb{C}[Y, Z]_2$, in coordinates with $u = x$: then K contains $X^2\mathbb{C}[Y, Z]_2$, whose derivatives $2X\mathbb{C}[Y, Z]_2$ by x are independent modulo D_3.

The next result applies Lemma D.3 to the fat point.

Corollary D.4. *Let $A = S/I$ be a local algebra of length $d \ge 18$ with $H_A(2) = 6$, $H_A(3) = H_A(4) = 3$, socle degree six and $H_A(6) = 1$, and suppose that every finite subscheme of $\mathbb{A}^3$ of length less than d is smoothable. If the subscheme Γ of Lemma 2.16 with $D_3 = \nu_3(\Gamma)$ is the fat point $V(\mathfrak{m}_p^2)$, then A is smoothable.*

Proof. Let $p = [X]$; then $\mathfrak{B} = \Gamma$, with $e_p = 4$, $D_3 = X^2T_1$, $D_4 = X^3T_1$, and $D_6 = \mathbb{C}F$ with $F \in \nu_6(\Gamma) = X^5T_1$ (Lemma 2.16). If F is a sixth power, then $F = X^6$, and Lemma D.3 applies by the remark after it, with the bound $27 + 2 \cdot 4 = 35 < 36 \le 2d$.

Otherwise $F = X^5Z$ in suitable coordinates, and we run the proof of Lemma D.3 with $z \circ \Phi$, x^5, $x^5 z$ in place of $u \circ \Phi$, u^{s-1}, u^s, and with

$R = Z^4$. Here $\langle x^5 z, \Phi\rangle = \langle x^5, z\circ\Phi\rangle = 120$, the space $M'' = \{m \in M : \langle x^5, m\rangle = \langle x^5 z, m\rangle = 0\}$ contains $x\circ\Phi$ and $y\circ\Phi$, and $x\circ R = y\circ R = 0$; so $M_\lambda = M'' \oplus \mathbb{C}\,(z\circ\Phi + \lambda\, z\circ R) \oplus \mathbb{C}\,(\Phi + \lambda R)$ is a line of algebras through A with the Hilbert function of A, and for $f_i \in I$ with leading forms w_i, the elements $f_i + \lambda(\theta_i x^5 + \theta_i' x^5 z)$ lie in I_λ for suitable θ_i' and $\theta_i = -\langle z w_i, Z^4\rangle/120$, which is $-\frac{1}{5}$ times the coefficient of z^3 in w_i. In the chart $x = 1$ at p, the quadratic part of a cubic $w \in (I_\Gamma)_3 = (y,z)^2 S_1$ is given by its coefficients of xy^2, xyz, xz^2, and the cubics without the term z^3 still have all binary quadrics as quadratic parts. Let $H \subset (I_\Gamma)_3$ be the space of cubics without the term z^3; its cubics have only the common zeros p and $[Z]$. The map $(w_1, w_2, w_3) \mapsto (h_1, h_2, w_3)$ with $h_i = w_i - \frac{\theta_i}{\theta_3} w_3$, $i = 1, 2$, maps the triples with $\theta_3 \neq 0$ onto $H^2 \times \{w_3 : \theta_3 \neq 0\}$, so for general w_1, w_2, w_3 the triple (h_1, h_2, w_3) is general there. Then h_1 and h_2 have no common factor, their quadratic parts at p form a pencil without common factor, so $V(h_1, h_2)$ has length $4 = e_p$ at p, and w_3 vanishes neither at $[Z]$, as $\theta_3 \neq 0$, nor at the other common zeros of h_1 and h_2. Step 2 of the proof of Theorem 3.12 with $k_p = 5$, $q_5 = x^5$ and $\lambda = \varepsilon^{-2}$, and the argument of Step 3, give a complete intersection of length at most $27 + 2\cdot 4 = 35 < 2d$ containing $V(I_\lambda)$, for all but finitely many λ, and Proposition 2.24 concludes. □

Theorem D.5. *Every local algebra with Hilbert function* $(1, 3, 6, 3, 3, 1)$ *is smoothable.*

Proof. By Lemma 2.16 with $n = k = 3$, there is $\Gamma \in \mathrm{Hilb}_3(\mathbb{P}^2)$ with $D_3 = \nu_3(\Gamma)$, $D_4 = \nu_4(\Gamma)$ and $F \in \nu_5(\Gamma)$. The pairs (Γ, F) with $F \neq 0$ (F framed) form the complement $\mathcal{M}$ of the zero section of a vector bundle of rank three over $\mathrm{Hilb}_3(\mathbb{P}^2)$; it is smooth and irreducible of dimension 9. Over $\mathcal{M}$, the tails (τ, R), with τ taking values in T_3/D_3 and R taken modulo D_4, lie in a vector bundle of rank $21+12 = 33$, and (9) consists of $3\cdot 7 = 21$ linear equations. Let $\mathcal{X}$ be the variety of solutions. By Lemma 2.17, every irreducible component of $\mathcal{X}$ has dimension at least $9 + 33 - 21 = 21$.

The fibers. Fix (Γ, F), let $\psi\colon S_1 \to D_4$ be the map $v \mapsto v\circ F$, and $N = \ker\psi$. Condition (9) says that $v \mapsto v\circ R$ modulo D_3 factors through ψ, followed by τ. So the solutions are the R with $N\circ R \subseteq D_3$, together with τ, which is determined on the image of ψ and arbitrary on a complement. If ψ has rank three, all R are allowed: the fiber has dimension 12. If ψ has rank two and $N = \mathbb{C}k$, the R with $k\circ R \in D_3$ form a space of dimension $5 + 3$, which is 5 modulo D_4, and τ contributes 7: again 12. If ψ has rank one, then F is a multiple of L^5 for a point $[L]$ of Γ; in coordinates with $L = X$, $N = \langle y, z\rangle$, and the R form the space $\mathcal{K}_4(\Gamma, X)$ of Lemma A.3, of dimension $3 + \sigma + 1$, where the socle dimension σ is one, unless Γ is the fat point at $[X]$, with ideal $(y,z)^2$, where it is two. So the R form a space of dimension 5, respectively 6, which is 2, respectively 3, modulo D_4, and τ contributes 14. The fiber has dimension 16, respectively 17. (Script `1136331.sage` checks these numbers on examples on each of the seven GL_3-orbits of $\mathrm{Hilb}_3(\mathbb{P}^2)$.)

The strata. Let $\mathcal{R} \subset \mathcal{M}$ be the locus where F is a fifth power, and $\mathcal{E} \subset \mathcal{R}$ the locus where moreover Γ is the fat point at $[L]$. Over $\mathcal{M} \setminus \mathcal{R}$, which is open in $\mathcal{M}$, the solutions form a vector bundle of rank 12, with irreducible total space of dimension 21. The locus $\mathcal{R}$ is a $\mathbb{C}^*$-bundle over the pairs $(\Gamma, [L])$ with $[L] \in \Gamma$, which form an irreducible variety of dimension six by Lemma A.2. Over $\mathcal{R} \setminus \mathcal{E}$ the solutions form a vector bundle of rank 16, with irreducible total space of dimension 23. Finally $\dim \mathcal{E} = 3$, and the solutions over $\mathcal{E}$ have dimension $3 + 17 = 20 < 21$, so they do not form a component. Hence every component of $\mathcal{X}$ is the closure of the part over $\mathcal{M} \setminus \mathcal{R}$ or over $\mathcal{R} \setminus \mathcal{E}$.

Rays. In both parts, the algebras with Γ three non-collinear points P_1, P_2, P_3 are dense. Write $F = \sum \alpha_i P_i^5$ with, say, $\alpha_1 \neq 0$. Let a be the linear form vanishing at P_2 and P_3 with $a \circ P_1 = 1$, and let b, c span the linear forms vanishing at P_1. Then $P_1^3 \in D_3$; the operators $a^3 b$ and $a^3 c$ annihilate P_i^4 and P_i^5 for all i, because $a \circ P_2 = a \circ P_3 = 0$ and $b \circ P_1 = c \circ P_1 = 0$; and $a^5 \circ F = 120\alpha_1 \neq 0$. Lemma 5.11 applies. Together with $E \in T_3$, which is arbitrary, local frames give families as in Lemma 2.3(2), and Proposition 2.2(1) concludes. □

Theorem D.6. *Every local algebra with Hilbert function* $(1, 3, 6, 3, 3, 1, 1)$ *is smoothable.*

Proof. By Lemma 2.16, $D_3 = \nu_3(\Gamma)$, $D_4 = \nu_4(\Gamma)$, $D_5 \subseteq \nu_5(\Gamma)$ and $D_6 \subseteq \nu_6(\Gamma)$ for $\Gamma \subset \mathbb{P}^2$ of length three. As in the proof of Theorem 5.21, $D_6 = \mathbb{C}X^6$ and $D_5 = \mathbb{C}X^5$ with $[X] \in \Gamma$, so $X^4 \in D_4$ and $X^3 \in D_3$. Let $\Phi = X^6 + R_5 + R_4 + R_3$ be the lift of X^6 and U' a complement of X^4 in D_4; then

$$M = T_{\leq 2} \oplus D_3 \oplus \{Q + \tau(Q) : Q \in U'\} \oplus \langle \Phi,\, x \circ \Phi,\, x^2 \circ \Phi \rangle,$$

and M is closed under differentiation if and only if $y \circ \Phi$ and $z \circ \Phi$ lie in M. With $N_X = \langle y, z \rangle$ and $d \circ R_5 = \alpha_d X^4 + \delta(d)$, $\delta(d) \in U'$, this says

$$N_X \circ R_5 \subseteq D_4, \qquad d \circ R_4 - \tau(\delta(d)) - \tfrac{\alpha_d}{30}\, x^2 \circ R_5 \in D_3 \quad (d \in N_X). \tag{12}$$

Since Γ has length three, either it is curvilinear at $[X]$, or $\Gamma = V(\mathfrak{m}_X^2)$ is the fat point, and then Corollary D.4 applies. So let Γ be curvilinear at $[X]$. Let $\mathcal{P}_5 = \mathcal{K}_5(\Gamma, X)$ and $\mathcal{K}_4 = \mathcal{K}_4(\Gamma, X)$ be the spaces of Lemma A.3, of the quintics and quartics R with $N_X \circ R$ in D_4, respectively D_3; by that lemma, $\dim \mathcal{P}_5 = 5$ and $\dim \mathcal{K}_4/D_4 = 2$. The space $\mathcal{P}_5$ contains $\mathcal{P}_5^0 = \langle X^5, X^4Y, X^4Z \rangle$, and $\delta = 0$ exactly when $R_5 \in \mathcal{P}_5^0$.

If $\delta \neq 0$, the linear map $(\tau, R_4) \mapsto \big(d \mapsto d \circ R_4 - \tau(\delta(d))\big)$, with values in $\operatorname{Hom}(N_X, T_3/D_3)$, is onto: if δ has rank two, $\tau \circ \delta$ is arbitrary; if $\delta(d_0) = 0 \neq \delta(d_1)$, first choose R_4 with $d_0 \circ R_4$ prescribed, then $\tau(\delta(d_1))$. So the solutions of the second condition in (12) form an affine space of dimension $26 - 14 = 12$. If $\delta = 0$, then τ is free and $R_4 \in R_4^0 + \mathcal{K}_4$ with $R_4^0 = X^2(c_2Y + c_3Z)^2/5$, where $R_5 = c_1X^5 + c_2X^4Y + c_3X^4Z$. Consequently these algebras form two irreducible families, affine bundles with R_3 free over the marked pairs $(\Gamma, [X])$ with Γ curvilinear at $[X]$, an open subset of the irreducible variety of Lemma A.2, and $R_5 \in \mathcal{P}_5 \setminus \mathcal{P}_5^0$, respectively $R_5 \in \mathcal{P}_5^0$.

Rays. The algebras with $\Gamma = \{X, Y, Z\}$ are dense in both families. Then $\mathcal{P}_5 = \langle X^5, X^4Y, X^4Z, Y^5, Z^5\rangle$; write $R_5 = c_1X^5 + c_2X^4Y + c_3X^4Z + \lambda Y^5 + \mu Z^5$, let ρ_Y, ρ_Z be the coefficients of X^3Y, X^3Z in R_4, and put $\beta_2 = c_2/30$, $\beta_3 = (6\rho_Y - 4c_1c_2)/720$, $b = y - \beta_2x^2 - \beta_3x^3$, and likewise γ_2, γ_3 and c. Then $x^3b \circ \Phi = 24c_2X + 6\rho_Y - \beta_2(720X + 120c_1) - 720\beta_3 = 0$ and $x^3b \circ (x \circ \Phi) = 24c_2 - 720\beta_2 = 0$, while x^3b kills $x^2 \circ \Phi$, the lifts of Y^4 and Z^4, and $T_{\leq 3}$; so $x^3b, x^3c \in I$, for every $R_5 \in \mathcal{P}_5$ and all tails. By (7), $E_3(\Lambda) \in X^3 + T_{\leq 2} \subseteq M$ for the curve $\Lambda\colon y = \beta_2x^2 + \beta_3x^3$, $z = \gamma_2x^2 + \gamma_3x^3$, and Corollary 5.5 gives a ray (script `l1363311_ray.sage`). □

Theorem D.7. *Every local algebra with Hilbert function* $(1, 3, 6, 4, 3, 1, 1)$ *is smoothable.*

Proof. As in the proof of Theorem D.6, $D_6 = \mathbb{C}X^6$ and $D_5 = \mathbb{C}X^5$ in suitable coordinates, so $X^4 \in D_4$ and $X^3 \in D_3$. Let $\Phi = X^6 + R_5 + R_4 + R_3$ be the lift of X^6; then $M = (M \cap T_{\leq 4}) \oplus \mathbb{C}\,x \circ \Phi \oplus \mathbb{C}\Phi$. Write $R_5 = c_0X^5 + X^4(r_Y\,Y + r_Z\,Z) + X^3\omega_2 + X^2\omega_3 + (\text{terms of degree at most one in } X)$, with binary forms ω_2, ω_3 in Y, Z of degrees two and three. The space $S_1 \circ D_4$ has dimension three or four, as $2^{\langle 3\rangle} = 2 < 3$.

Quartic operators. The elements of $(M \cap T_{\leq 4}) \oplus \mathbb{C}\,x \circ \Phi$ have degree at most five and contain neither X^4Y nor X^4Z, since the quintic part of $x \circ \Phi$ is $6X^5$; and $\langle x^6, \Phi\rangle = 720 \neq 0$. By Lemma 5.13 with $k = 4$ and the remark after it, $x^4(y - \beta_2x^2)$ and $x^4(z - \gamma_2x^2)$ lie in I, with $\beta_2 = r_Y/30$ and $\gamma_2 = r_Z/30$, for every algebra in question. The element $x^2 \circ \Phi/30$ of M has top form X^4 and cubic part $\frac{2}{3}c_0X^3 + \frac{2}{5}X^2(r_Y\,Y + r_Z\,Z) + \frac{1}{5}X\omega_2 + \frac{1}{15}\omega_3$, while $E_4(\Lambda) = X^4 + 12X^2(\beta_2Y + \gamma_2Z) + (\text{terms of degree at most two})$ for the curve Λ: $y = \beta_2x^2$, $z = \gamma_2x^2$, by (7). As $12\beta_2 = \frac{2}{5}r_Y$ and $X^3 \in D_3$, Lemma 5.13 gives a ray whenever

$$3X\omega_2 + \omega_3 \in D_3, \tag{13}$$

in particular whenever $\omega_2 = \omega_3 = 0$.

The case $\dim S_1 \circ D_4 = 3$. By Lemma 2.16 with $n = k = 3$, $D_4 = \nu_4(\Gamma)$ and $S_1 \circ D_4 = \nu_3(\Gamma)$ for a subscheme $\Gamma \subset \mathbb{P}^2$ of length three, which contains $[X]$, and $D_3 = \nu_3(\Gamma) \oplus \mathbb{C}G$ for a cubic G. The closedness conditions are (12), with this D_3, and we use the map δ of the proof of Theorem D.6. If $\delta = 0$, then $\omega_2 = \omega_3 = 0$, and there is a ray. If $\delta \neq 0$, the map $(\tau, R_4) \mapsto (d \mapsto d \circ R_4 - \tau(\delta(d)))$ is onto $\operatorname{Hom}(N_X, T_3/D_3)$, as in that proof, so for Γ curvilinear at $[X]$, respectively for Γ the fat point $V(\mathfrak{m}_X^2)$, the algebras form an irreducible family, an affine bundle over the data $(\Gamma, [X])$, $[G] \in \mathbb{P}(T_3/\nu_3(\Gamma))$ and $R_5 \in \mathcal{K}_5(\Gamma, X)$ with $\delta \neq 0$. In the first family the algebras with $\Gamma = \{X, Y, Z\}$ are dense, and for them $\mathcal{K}_5(\Gamma, X) = \langle X^5, X^4Y, X^4Z, Y^5, Z^5\rangle$, so $\omega_2 = \omega_3 = 0$ and there is a ray. In the second family the general member has a general cubic G; then $D_3 = X^2T_1 \oplus \mathbb{C}G$ contains no nonzero element of $X \cdot \mathbb{C}[Y, Z]_2$, its cubic base scheme, defined by the general hyperplane $(I_\Gamma)_3 \cap G^\perp$ of $(I_\Gamma)_3$, is Γ by Lemma A.16, and

Lemma D.3 applies, with the bound $27 + 2 \cdot 4 = 35 < 38$. In both cases Proposition 2.2(1) concludes.

The case $\dim S_1 \circ D_4 = 4$. Then $D_3 = S_1 \circ D_4$. By Lemma A.11, either the cubic base scheme $\mathfrak{B}$ is finite of length at most three, or $D_3 = \nu_3(\Gamma)$ and D_4 is a hyperplane of $\nu_4(\Gamma)$ for a subscheme $\Gamma \subset \mathbb{P}^2$ of length four: if $\mathfrak{B}$ has length four, $\Gamma = \mathfrak{B}$; if $D_3 = \mathbb{C}[X, L]_3$ is the space of binary cubics of a line, Γ is the collinear scheme $V(h, g)$, where h is the equation of the line and g a binary quartic apolar to a four-dimensional space V with $D_4 \subset V \subset \mathbb{C}[X, L]_4$, so that $\nu_4(\Gamma) = V$. As $X^3 \in D_3$ and $X^4 \in D_4$, the point $[X]$ lies in $\mathfrak{B}$, respectively in Γ. In Theorem 3.12, $k([X]) = 6$, and $k(p) \leq 5$ at the other points, as $P^5 \notin D_5 = \mathbb{C}X^5$. In the fat-point subcase of (a) and in (b) below we use Lemma D.3, which replaces $k([X]) = 6$ by 5; its hypothesis holds by the remark after it, as D_3 contains X^2T_1 but no nonzero element of $X \cdot \mathbb{C}[Y, Z]_2$, except in the last case of (b), where we check it directly.

(a) If $\mathfrak{B}$ is finite of length at most three and not the fat point $V(\mathfrak{m}_X^2)$, then $\mathfrak{B}$ is curvilinear, and $\sum_p (k(p) - 3)e_p \leq 9$, the worst case being a curvilinear point of length three at $[X]$; as $27 + 9 < 38$, Theorem 3.12 applies. If $\mathfrak{B} = V(\mathfrak{m}_X^2)$, then $D_3 = X^2T_1 \oplus \mathbb{C}G$ with $G \notin X^2T_1 + X \cdot \mathbb{C}[Y, Z]_2$, as otherwise $\mathfrak{B}$ would be longer at $[X]$, and Lemma D.3 gives the bound $27 + 2 \cdot 4 = 35 < 38$.

(b) Let Γ have length four, and let the socle of its local ring at $[X]$ have dimension two. Then Γ is not collinear, $\mathfrak{B} = \Gamma$, and in suitable coordinates the part of Γ at $[X]$ is $V(\mathfrak{m}_X^2)$ or has local ideal $(\eta^2, \eta\zeta, \zeta^3)$ in the chart $x = 1$, with $\eta = y/x$ and $\zeta = z/x$. In the first case, $\Gamma = V(\mathfrak{m}_X^2) \cup \{q\}$ and $D_3 = X^2T_1 \oplus \mathbb{C}Q^3$ for $q = [Q]$, and Lemma D.3 gives the weight $2 \cdot 4$ at $[X]$ and at most 2 at q: $27 + 10 = 37 < 38$. In the second case $e_{[X]} = 5$ and $D_3 = \nu_3(\Gamma) = \langle X^3, X^2Y, X^2Z, XZ^2 \rangle$; the space K of Lemma D.3 contains X^2Y^2, X^2YZ and XZ^3, whose derivatives by x are independent modulo D_3, and the lemma gives the weight $2 \cdot 5$ at $[X]$: $27 + 10 = 37 < 38$.

(c) Let Γ have length four and be Gorenstein at $[X]$, that is, let the socle of its local ring at $[X]$ have dimension one; this includes all collinear Γ. Write $U = D_4$, a hyperplane of $\nu_4(\Gamma)$ containing X^4. By dimensions,

$$M = T_{\leq 2} \oplus \nu_3(\Gamma) \oplus \{u + \tau(u) : u \in U\} \oplus \mathbb{C}\, x \circ \Phi \oplus \mathbb{C}\Phi$$

with $\tau \colon U \to T_3$, and, conversely, such a space is closed under differentiation if and only if

$$\begin{aligned} &y \circ R_5,\ z \circ R_5 \in U, && \tau(X^4) \equiv \tfrac{1}{30} x^2 \circ R_5, \\ &y \circ R_4 \equiv \tau(y \circ R_5), && z \circ R_4 \equiv \tau(z \circ R_5), \end{aligned} \tag{14}$$

where $\equiv$ means congruence modulo D_3: the derivatives of the lifts $u + \tau(u)$ lie in $D_3 + T_2$, as $S_1 \circ U \subseteq \nu_3(\Gamma)$, and the conditions say that $x^2 \circ \Phi$, $y \circ \Phi$ and $z \circ \Phi$ lie in the space. By the first condition, R_5 lies in $\mathcal{K}_5(\Gamma, X)$, of dimension 6 by Lemma A.3, which contains X^4T_1.

We show that all these algebras form one irreducible family. Let $\mathcal{H}$ be the set of $\Gamma \in \mathrm{Hilb}_4(\mathbb{P}^2)$ that contain $[X]$ and are Gorenstein there. The marked

pairs form an irreducible variety of dimension eight (Lemma A.2), and moving the marked point to $[X]$ by the linear maps $X \mapsto X+bY+cZ$ fixing Y, Z shows that the subschemes containing $[X]$ form an irreducible variety of dimension six; $\mathcal{H}$ is open in it, because $\dim \mathcal{K}_5(\Gamma, X) = 5 + \sigma$ is upper semicontinuous. Over $\mathcal{H}$ the spaces $\mathcal{K}_5(\Gamma, X)$ form a vector bundle of rank six, with total space $\mathcal{K}$ of dimension 12. Let $\mathcal{G} \to \mathcal{K}$ be the $\mathbb{P}^2$-bundle of the hyperplanes $U \subset \nu_4(\Gamma)$ that contain X^4, of dimension 14, and $\mathcal{I} \subseteq \mathcal{G}$ the subvariety where $y \circ R_5, z \circ R_5 \in U$, the zero set of a section of a bundle of rank two; every component of $\mathcal{I}$ has dimension at least 12. Let $\delta\colon N_X \to \nu_4(\Gamma)/\mathbb{C}X^4$, $d \mapsto d \circ R_5$. Where δ has rank two, $U = \langle X^4, y \circ R_5, z \circ R_5\rangle$ is determined, so this part of $\mathcal{I}$ is open in $\mathcal{K}$, irreducible of dimension 12, and nonempty (take $\Gamma = \{X, Y, Z, X + Y + Z\}$ and $R_5 = Y^5 + Z^5 + (X + Y + Z)^5$). Where $\delta = 0$, $R_5 \in X^4T_1$, and this part has dimension $6 + 3 + 2 = 11$. Where δ has rank one, write $R_5 = R^\circ + X^4 l$ with $l \in T_1$ and R° without the terms X^5, X^4Y, X^4Z. Some $h \in N_X$, unique up to scalar, has $h \circ R^\circ \in \mathbb{C}X^4$, and then $h \circ R^\circ = 0$, as R° has no terms X^4Y, X^4Z. The line $V(h)$ passes through $[X]$; let $L \in \langle Y, Z\rangle$ span the kernel of h, and let r be the length of $\Gamma \cap V(h)$. Then $R^\circ \in \mathbb{C}[X, L]_5$, and its derivative G along L is a nonzero element of $\nu_4(\Gamma) \cap \mathbb{C}[X, L]_4 = \nu_4(\Gamma \cap V(h))$, a space of dimension r; indeed the exact sequence $0 \to S/(I_\Gamma : h)(-1) \to S/I_\Gamma \to S/(I_\Gamma + (h)) \to 0$ and Lemma 2.14 show that $(I_\Gamma + (h))_4$ has codimension r, and it is contained in $(I_{\Gamma\cap V(h)})_4$, which has codimension r. As R° is determined by G, and G has no term X^4, the form R° has at most $r-1$ parameters, and $r \geq 2$. The pairs $(\Gamma, V(h))$ with $[X] \in V(h)$ and $\Gamma \cap V(h)$ of length at least r have at most $1 + (r-1) + 2(4-r) = 8 - r$ parameters: the line, a subscheme of length r of $V(h)$ through $[X]$, and then Γ, reached by $4 - r$ extensions of colength one, each in a family of dimension at most two (a new point, or a colength-one subideal of a local ideal with at most three generators). With l and $U \supseteq \langle X^4, G\rangle$, this part of $\mathcal{I}$ has dimension at most $(8-r)+(r-1)+3+1 = 11$. By Lemma 2.18, $\mathcal{I}$ is irreducible of dimension 12.

Over the open subset $\mathcal{I}^+$ of $\mathcal{I}$ where $\delta \neq 0$, the remaining conditions of (14) are linear in (τ, R_4), and the linear map $(\tau, R_4) \mapsto (\tau(X^4), y \circ R_4 - \tau(y \circ R_5), z \circ R_4 - \tau(z \circ R_5))$ to $(T_3/D_3)^3$ is onto: $\tau(X^4)$ is free, and the rest as in the case $\dim S_1 \circ D_4 = 3$. So the algebras over $\mathcal{I}^+$ form an affine bundle (Lemma 5.16), with R_3 free, over an irreducible base, and they are all the algebras in (c) with $\delta \neq 0$. In this family $\Gamma = \{X, L_1, L_2, L_3\}$ reduced is dense. Then $\mathcal{K}_5(\Gamma, X) = \nu_5(\Gamma) + X^4T_1$, so $R_5 = R' + X^4(aY + bZ)$ with $R' \in \nu_5(\Gamma)$. The coordinate change of Section 5.17 with $m = 2$, $a_2 = -a/30$ and $a_3 = -b/30$ replaces R_5 by R', and does not change D_3, U and Γ. Now $x^2 \circ R' \in \nu_3(\Gamma) = D_3$, so $\tau(X^4) \in D_3$ by (14), and $X^4 \in M$. Every element of $N_X \circ M$ has degree at most four, with quartic part in $\langle y \circ R', z \circ R'\rangle \subseteq \langle L_1^4, L_2^4, L_3^4\rangle$: the elements $N_X \circ (x \circ \Phi)$ have degree at most three, as $N_X \circ X^5 = 0$, and so have the derivatives of the lifts of U; since X^4, L_1^4, L_2^4, L_3^4 are linearly independent, $X^4 \notin N_X \circ M$, and Theorem 2.11(2)

gives a ray along the line of X. The same argument applies when $\delta = 0$, for every Γ: then R' is a multiple of X^5, and $N_X \circ R' = 0$. Proposition 2.2(1) concludes. □

In case (c), Γ is Gorenstein at $[X]$, which makes $\mathcal{K}_5$ a vector bundle by Lemma A.3; the schemes Γ whose socle at $[X]$ has dimension two are treated by excess linkage in (b). So no irreducibility is needed across the locus where the socle of Γ at $[X]$ jumps. The members of (c) are not difficult one by one, but for some of them, with Γ curvilinear of length three at $[X]$ and one further point, or of length four at $[X]$, the criterion of Theorem 2.11(2) fails along the line of X (and along the line of the further point), and the excess at $[X]$ is too large. But they lie in the small locus where δ has rank one, and the dimension count, not a local computation, shows that they are limits. The proof uses no computer. Script `l19_1364311_incidence.sage` checks the statements used in (c) on eleven types of Γ: the dimensions of $\mathcal{K}_5$, the dimensions r and $r-1$ in the rank-one locus, the rank of the map on the tails, closedness, and the ray after the coordinate change on random members.

Appendix E. Socle degree five

E.1. In this appendix $H_A(2) = 6$ and the socle degree is five. The models are Theorem 5.10, Lemma 5.11 and Theorem 5.21: rays along parabolas or cubic curves, chosen at a critical point of a quartic of D_4 when the top form is a power. New are Gorenstein members of the families (Proposition 2.20), and linkage, when the extra cubics make the tangent cone free of cubes or nearly so. In length nineteen, the cubic base scheme is often finite, and Theorem 3.12 applies with weights at most two; the other algebras are placed in irreducible families by dimension counts, and linked through a global complete intersection (Lemma E.8) or split by rays.

Theorem E.2. *Every local algebra with Hilbert function* $(1, 3, 6, 3, 2, 1)$ *is smoothable.*

Proof. Let F span D_5, and let r be its number of essential variables; $r \leq 2$, since $S_1 \circ F \subseteq D_4$.

The case $r = 2$. In suitable coordinates $F \in \mathbb{C}[X, Y]$, so $D_4 = S_1 \circ F$ and $z \circ F = 0$. Condition (9) for $v = x, y$ says that $\tau(x \circ F) \equiv x \circ R$ and $\tau(y \circ F) \equiv y \circ R$ modulo D_3, so we may take $\tau(v \circ F) = v \circ R$; for $v = z$ it says $z \circ R \in D_3$. So the algebras are parametrized by D_3, by R in the space $z^{-1}(D_3)$ of dimension $5 + 3 = 8$, and by $E \in T_3$. The space $S_2 \circ F \subseteq D_3$ has dimension 2 or 3.

If $S_2 \circ F$ has dimension three, then $D_3 = S_2 \circ F$ consists of binary forms, so $z \circ R \in D_3$ gives $z^2 \circ R = 0$. Then z^2 annihilates D_3 and the elements $v \circ F + v \circ R$, and $z^2 \circ (F + R + E) = z^2 \circ E$, so $z^2 \circ M \subseteq \mathbb{C} \oplus \mathbb{C}P$ with $P = z^2 \circ E$. If the coefficient of Z^3 in E is nonzero, then $z \circ P \neq 0$, and

Lemma C.2 gives a ray. These algebras are dense in the irreducible family of all algebras of this kind, so all of them are smoothable.

If $S_2 \circ F$ has dimension two, then a binary quadric annihilates F, so $F \in \nu_5(\Gamma)$ for a subscheme Γ of length two, and F is, in suitable coordinates, $X^5 + Y^5$ or X^4Y; moreover $D_3 = S_2 \circ F \oplus \mathbb{C}G$ for a cubic G. These data, together with R and E, form an irreducible family. It contains a Gorenstein algebra: the inverse system generated by the single polynomial

$$X^5 + Y^5 + X^2Y^2 + X^2YZ + XZ^3 + Y^2Z^2$$

has dimension 16 and Hilbert function $(1, 3, 6, 3, 2, 1)$ (script `1136321.sage`). By Proposition 2.20, all algebras in the family are smoothable.

The case $r = 1$. Then $F = X^5$ in suitable coordinates, and $D_4 = \langle X^4, Q\rangle$ for a quartic Q. Condition (9) says that $5\tau(X^4) \equiv x \circ R$ modulo D_3, and that R lies in $\mathcal{K} = \{R \in T_4 : y \circ R,\ z \circ R \in D_3\}$; $\tau(Q)$ and E are arbitrary. We use two kinds of rays, along curves Λ through the origin tangent to the line of X, with $b = y - \beta(x)$ and $c = z - \gamma(x)$ as before (7).

(a) *Parabola rays.* Suppose that $x^3y \circ Q = x^3z \circ Q = 0$. With $\beta_2 = x^3y \circ R/120$ and $\gamma_2 = x^3z \circ R/120$, the operators x^3b and x^3c annihilate M, and $E_3(\Lambda) \in M$, as $X^3 \in S_1 \circ D_4 \subseteq D_3$; so every algebra over (D_3, D_4) has a ray by Corollary 5.5.

(b) *Quartic rays.* For every algebra, x^4b and x^4c lie in I for every choice of β, γ: order-five operators annihilate $T_{\leq 4}$, $x^4y \circ X^5 = 0$, and $\mathfrak{m}^6 \subseteq I$. The elements of M with top form X^4 are $X^4 + \tau(X^4)$ modulo $D_3 + T_{\leq 2}$, so by (7) there is a curve with $E_4(\Lambda) \in M$ if and only if $x \circ R \in D_3 + X^2 \cdot \langle Y, Z\rangle$.

The coordinates x, y, z may be changed by $x \mapsto x + s_1y + s_2z$, which keeps X and replaces Y, Z by $Y - s_1X$, $Z - s_2X$. Condition (a) then has a simple meaning. Since $(x + s_1y + s_2z)^4 \circ Q = 24\, Q(1, s_1, s_2)$, where $Q(1, s_1, s_2)$ is the value of Q at the coefficient vector $(1, s_1, s_2)$, condition (a) holds for $x + s_1y + s_2z$ if and only if (s_1, s_2) is a critical point of the polynomial $q(s_1, s_2) = Q(1, s_1, s_2)$ of degree at most four. Such a point exists when the binary quartic $Q(0, Y, Z)$ has four distinct roots: then the cubic parts of the two derivatives of q, the derivatives of $Q(0, s_1, s_2)$, have no common zero, so the two derivatives have no common zero at infinity; hence they have no common component, and by Bézout's theorem they meet in nine points of the affine plane, counted with multiplicity. By Lemma A.6, applied to D_4, there are the following cases.

$r(D_4) = 2$. Then $D_4 = \nu_4(\Gamma)$ for Γ of length two containing $[X]$, and $D_3 = \nu_3(\Gamma) \oplus \mathbb{C}G$ for a cubic G. If $\Gamma = \{X, Y\}$, then $Q = Y^4$, $q = s_1^4$ has a critical point, and (a) applies. If Γ is a double point at $[X]$, consider for $v \in \langle y, z\rangle$ the map $\delta_R(v) = v \circ R$ modulo $\nu_3(\Gamma)$; its image lies in $\mathbb{C}G$. Either $\delta_R = 0$, which means that R lies in the space $\mathcal{K}_4(\Gamma, X)$, of dimension $2 + 1 + 1 = 4$ for both types of Γ by Lemma A.3, and G is arbitrary; or δ_R has rank one, with kernel spanned by some w, and then $R \in w^{-1}(\nu_3(\Gamma))$, a

space of dimension $5+2=7$, and G is determined by R modulo $\nu_3(\Gamma)$. In both cases, let $\Gamma_t = \{X, X+tY\}$, so that $\nu_k(\Gamma_t)$ has the polynomial basis X^k, $((X+tY)^k - X^k)/(kt)$. Keeping w fixed, the spaces in which R lies form vector bundles of constant rank over a neighborhood of $t=0$, so R extends to a family R_t, and G to G_t; the data $\tau(Q)$ and E stay fixed. By Lemma 2.3(2), the algebra is a limit of algebras over two points.

$r(D_4)=3$ and $D_4 \in \mathcal{S}$. The only fourth powers in $\langle L^4, H\rangle$ are multiples of L^4, so $L=X$, and in coordinates with $\Pi = \langle Y, Z\rangle$ we have $Q = H(Y,Z)$; $q = H(s_1, s_2)$ has a critical point at the origin, and (a) applies.

$r(D_4)=3$ and $D_4 \in \mathcal{S}'$. Then $D_4 = \langle X^4, X^3Y + H(X,Z)\rangle$, and $q = s_1 + H(1, s_2)$ has no critical point. After subtracting a multiple of X^4 and replacing Y by $Y + \text{const}\cdot Z$, we may take $H = \lambda_1 X^2Z^2 + \lambda_2 XZ^3 + \lambda_3 Z^4$. Writing $R = \sum_j Y^j R_j(X,Z)$ and comparing powers of Y in $y\circ R, z\circ R \in D_3 = \langle X^3,\ 3X^2Y + H_X,\ H_Z\rangle$, one finds that $R = R_0 + Y(\kappa X^3 + \rho H_X + \sigma H_Z) + \frac{3}{2}\rho X^2Y^2$ with

$$4\lambda_1\rho + 6\lambda_2\sigma = 0, \qquad 3\lambda_2\rho + 12\lambda_3\sigma = 0,$$

and with $\partial_Z R_0 \in \frac{2}{3}\lambda_1\sigma H_X + \langle X^3, H_Z\rangle$; the coefficient of H_X is that of $3X^2Y + H_X$ in $z\circ R$, which is forced by the part $Y(\rho H_{XZ} + \sigma H_{ZZ}) = 2\lambda_1\sigma X^2Y$ of $z\circ R$. If $8\lambda_1\lambda_3 \neq 3\lambda_2^2$, then $\rho=\sigma=0$, $\mathcal{K} = \langle X^4, X^3Y, X^3Z, H\rangle$, and $x\circ\mathcal{K} = \langle X^3, X^2Y, X^2Z, H_X\rangle \subseteq D_3 + X^2\langle Y,Z\rangle$, because $H_X \equiv -3X^2Y$ modulo D_3; so quartic rays (b) exist for every algebra. If $8\lambda_1\lambda_3 = 3\lambda_2^2$, the quartic $\frac{4}{3}\lambda_1^2X^3Z + 3\lambda_2 H$ (or H, if $\lambda_1=0$) is a third quartic, independent of D_4, all of whose derivatives lie in D_3 (script `1136321.sage`). So $\dim D_3' \geq 3$, and Lemma 2.16 gives Γ of length three with $D_3 = \nu_3(\Gamma)$ and $D_4 \subseteq D_3' = \nu_4(\Gamma)$: $D_4 \in \mathcal{V}$.

$r(D_4)=3$ and $D_4 \in \mathcal{V}$. Then $D_3 = \nu_3(\Gamma)$, and $[X] \in \Gamma$, because the forms of degree four in I_Γ vanish at $[X]$ and I_Γ is generated in degrees at most three. The local ring of Γ at $[X]$ has a socle of dimension one, since otherwise Γ would be the fat point $V(\mathfrak{m}_X^2)$ and $r(D_4)=2$. So $\mathcal{K} = \mathcal{K}_4(\Gamma, X)$ has dimension $3+1+1=5$ for all such Γ, by Lemma A.3. If $[X]$ is a reduced point of Γ, write $\Gamma = \{X\}\cup\Gamma'$, so that we may take $Q \in \nu_4(\Gamma')$. When $[X]$ is not on the line spanned by Γ', choose x vanishing on Γ'; then $x\circ Q = 0$, and (a) applies. When Γ lies on a line, $D_4 \subseteq \mathbb{C}[X,Y]_4$, and $Q = aP^4 + bP'^4$ for the two points $P = pX+Y$, $P' = p'X+Y$ of Γ', or $Q = aP^4 + bP^3X$ if Γ' is a double point at $[P]$. Then q depends only on s_1, and its derivative is $4\big(a(p+s_1)^3 + b(p'+s_1)^3\big)$, respectively $4a(p+s_1)^3 + 3b(p+s_1)^2$, a nonconstant polynomial; at a root, (a) applies. If $[X]$ is not a reduced point of Γ, then Γ is curvilinear at $[X]$, and it is the limit, along a curve, of schemes $\Gamma_t \ni [X]$ in which $[X]$ is reduced. Choose $Q_t \in \nu_4(\Gamma_t)$ depending polynomially on t with $Q_0 = Q$. Since $\nu_3(\Gamma_t)$, $\nu_4(\Gamma_t)$ and $\mathcal{K}(\Gamma_t)$ have constant dimension, R extends to a family R_t, while $\tau(Q)$ and E stay fixed, and by Lemma 2.3(2) the algebra is a limit of algebras with rays.

In all cases the algebra is smoothable. □

Theorem E.3. *Every local algebra with Hilbert function* $(1,3,6,4,2,1)$ *is smoothable.*

Proof. We follow the proof of Theorem E.2, whose notation we use; the difference is that D_3 now has one more dimension. Let F span D_5, with $r \leq 2$ essential variables.

The case $r = 2$. As before, $F \in \mathbb{C}[X,Y]$, $D_4 = S_1 \circ F$, τ is determined by R, and the only condition is $z \circ R \in D_3$, which defines a space of dimension $5 + 4 = 9$. If $\dim S_2 \circ F = 3$, then $D_3 = S_2 \circ F \oplus \mathbb{C}G$, and these data form an irreducible family. It contains a Gorenstein algebra: the inverse system generated by $X^3Y^2 + X^2Z^2 + YZ^3 + Z^4$, has Hilbert function $(1,3,6,4,2,1)$, and $\dim S_2 \circ X^3Y^2 = 3$ (script `1136421.sage`). Proposition 2.20 applies. If $\dim S_2 \circ F = 2$, then $F \in \nu_5(\Gamma)$ for a subscheme Γ of length two and $D_3 = \nu_3(\Gamma) \oplus K$ with $\dim K = 2$; these data form an irreducible family, a bundle over the pairs (Γ, F) with F not a fifth power. For $\Gamma = \{X, Y\}$ and $F = \alpha X^5 + \beta Y^5$ with $\alpha \neq 0$, Lemma 5.11 applies with $P = X$ and $a, b, c = x, y, z$.

The case $r = 1$. Now $F = X^5$, $D_4 = \langle X^4, Q\rangle$, $5\tau(X^4) \equiv x \circ R$ modulo D_3, and $R \in \mathcal{K} = \{R : y \circ R, z \circ R \in D_3\}$, while $\tau(Q)$ and E are arbitrary. Recall that condition (a) of the proof of Theorem E.2 gives a ray for all algebras over (D_3, D_4) when $q = Q(1, s_1, s_2)$ has a critical point. We distinguish $r(D_4) \in \{2,3,4\}$.

$r(D_4) = 4$. Then $D_3 = \mathbb{C}X^3 \oplus S_1 \circ Q$ with $\dim S_1 \circ Q = 3$. Write $y \circ R = \alpha X^3 + u \circ Q$ and $z \circ R = \beta X^3 + v \circ Q$ with $\alpha, \beta \in \mathbb{C}$ and $u, v \in S_1$, uniquely. Such a pair comes from some R, unique up to $\mathbb{C}X^4$, if and only if $z \circ (y \circ R) = y \circ (z \circ R)$, that is, $h \circ Q = 0$ for $h = zu - yv$. The quadrics $zu - yv$ form the space $W_0 = \langle xy, xz, y^2, yz, z^2\rangle$, and $(u, v) \mapsto zu - yv$ has kernel $\mathbb{C}(y, z)$. So $\mathcal{K}$ is the union of two kinds of spaces: $\mathcal{K}_0 = \langle X^4, X^3Y, X^3Z, Q\rangle$, where $h = 0$, and, for each $h \in W_0 \setminus 0$ with $h \circ Q = 0$, a space of dimension five where $zu - yv \in \mathbb{C}h$. Accordingly, the pairs (Q, R) lie in two irreducible families: a vector bundle of rank four over the open set of Q with $r(D_4) = 4$, and a vector bundle of rank five over the pairs $([h], Q)$ with $h \circ Q = 0$ and $r(D_4) = 4$, which form an open subset of a vector bundle of rank 9 over $\mathbb{P}(W_0)$, since contraction with h maps T_4 onto T_2. In both families, q has a critical point on the open subset where $Q(0, Y, Z)$ has four distinct roots (proof of Theorem E.2), and this subset is not empty: in the first family it contains a general quartic, and in the second $Q = Y^4 + Z^4 + XY^3 + XZ^3$, which is annihilated by $h = yz$ and has $r(D_4) = 4$. So the general algebra in each family has a ray.

$r(D_4) = 3$. Then $D_3 = S_1 \circ D_4 \oplus \mathbb{C}G$. For $R \in \mathcal{K}$, the map $\delta_R \colon \langle y, z\rangle \to D_3/S_1 \circ D_4 \cong \mathbb{C}$, $v \mapsto v \circ R$, is zero or of rank one. If $\delta_R = 0$, then R lies in the space $\mathcal{K}$ of the proof of Theorem E.2 for the same D_4, and that proof, in the case $r(D_4) = 3$, applies with $S_1 \circ D_4$ in place of D_3, carrying G along; G stays complementary to $S_1 \circ D_4$ in the limits used there. If δ_R has rank one, let w span its kernel. Then $R \in w^{-1}(S_1 \circ D_4)$, a space

of dimension $5+3=8$, and G is determined by R modulo $S_1 \circ D_4$. The pencils D_4 with $r(D_4)=3$ together with a fourth power $X^4 \in D_4$ form an irreducible variety, in which those of family $\mathcal{S}$ are dense. Indeed, by Lemma A.6 such a pencil lies in $\mathcal{S}$, $\mathcal{S}'$ or $\mathcal{V}$, with the fourth power marked; the marked pencils of $\mathcal{S}$ form an irreducible variety; the marked pencils of $\mathcal{V}$ form an irreducible variety, a bundle over the pairs of Lemma A.2, whose general member, $\langle P_1^4, \alpha P_2^4 + \beta P_3^4\rangle$ for three non-collinear points, lies in $\mathcal{S}$, and the marked pencils of $\mathcal{S}'$ are limits of marked pencils of $\mathcal{S}$ by Lemma A.7, composed with the linear change of coordinates $X + tY \mapsto X$, which tends to the identity and keeps X^4 marked. So all of them lie in the closure of the marked pencils of $\mathcal{S}$. Hence the data $(D_4, [w], R)$ form an irreducible family, a bundle of rank 8, whose general member has $D_4 = \langle X^4, H(Y,Z)\rangle \in \mathcal{S}$. There x^3y and x^3z annihilate D_4, and Lemma 5.11 gives a ray.

$r(D_4)=2$. Then $D_4 = \nu_4(\Gamma)$ with $[X] \in \Gamma$ of length two, and $D_3 = \nu_3(\Gamma) \oplus K$ with $\dim K = 2$. Let $\delta_R \colon \langle y, z\rangle \to T_3/\nu_3(\Gamma)$, $v \mapsto v \circ R$; the condition $R \in \mathcal{K}$ says that the image of δ_R lies in K. The data $(\Gamma, [X], R \bmod D_4, K)$ lie in a smooth variety of dimension $4+13+12=29$, a bundle over the pairs $(\Gamma, [X])$, which form a smooth irreducible variety, the blow-up of $\mathbb{P}^2 \times \mathbb{P}^2$ along the diagonal. The condition is given by $2 \cdot 6 = 12$ equations, so every component of the solution set has dimension at least 17 by Lemma 2.17. If $\delta_R = 0$, then R lies in $\mathcal{K}_4(\Gamma, X)$, of dimension four, two modulo D_4, for both types of Γ (Lemma A.3), and K is arbitrary: an irreducible family of dimension $4+2+12=18$. If δ_R has rank two, then K is its image, and R is general: an irreducible family of dimension $4+13=17$. If δ_R has rank one, with kernel spanned by w, then $R \in w^{-1}(\nu_3(\Gamma))$, of dimension $5+2=7$, and K contains the image: the dimension is at most $4+1+5+6=16$, so these data form no component. Hence every algebra is a limit of algebras in the first two families, where $\Gamma = \{X, Y\}$ is dense. There Lemma 5.11 applies with $a, b, c = x, y, z$.

In all cases, the algebras are limits of smoothable algebras, by Lemma 2.3(2) and Proposition 2.2(1). □

E.4. The next Hilbert function has a five-dimensional D_3 and is a relative of $(1,3,6,4,2,1)$; the extra cubic makes linkage available in the binary case and is harmless for the rays when the top form is a fifth power.

Theorem E.5. *Every local algebra with Hilbert function* $(1,3,6,5,2,1)$ *is smoothable.*

Proof. Let F span D_5, with $r \le 2$ essential variables, and let $\Phi = F + R + E$ be its lift.

The case $r=2$. In suitable coordinates $F \in \mathbb{C}[X,Y]_5$ and $D_4 = S_1 \circ F$, and as in the proof of Theorem E.2 the only condition on the tails is $z \circ R \in D_3$, a space of dimension 10 for every D_3. If $S_2 \circ F$ has dimension three, then $D_3 = S_2 \circ F \oplus K$ with $\dim K = 2$, and the algebras form an irreducible family, a bundle over the triples (plane of F, F, K). The space $S_2 \circ F$ is the

hyperplane of $\mathbb{C}[X,Y]_3$ orthogonal to the binary cubic annihilating F, so it contains the cubes of the three roots of that cubic; for a general member these are the only cubes in D_3, the base scheme of I_3^* consists of three reduced points, and none of them has its fourth power in D_4: so Theorem 3.12 applies to it, with $27+3<36$: its proof places a member I_λ of a line of algebras through it in a complete intersection of local length at most 30. Script `l136521.sage` exhibits a member of the family in a complete intersection of local length 30, with residual scheme of length 12. By Proposition 2.24, all members are smoothable. If $S_2\circ F$ has dimension two, then $F\in\nu_5(\Gamma)$ for Γ of length two on the line $z=0$, $D_4=\nu_4(\Gamma)$ and $D_3=\nu_3(\Gamma)\oplus K$ with $\dim K=3$; these data form an irreducible family in which $\Gamma=\{X,Y\}$ is dense, and there $F=\alpha X^5+\beta Y^5$ with $\alpha\beta\neq 0$, so Lemma 5.11 applies with $P=X$.

The case $r=1$. Then $F=X^5$, $D_4=\langle X^4,Q\rangle$, and the tails satisfy $5\tau(X^4)\equiv x\circ R$ modulo D_3 and $y\circ R, z\circ R\in D_3$, while $\tau(Q)$ and E are free, as in the proof of Theorem E.2. Let $C_0=S_1\circ D_4+\langle y\circ R, z\circ R\rangle\subseteq D_3$. If $\dim C_0\leq 4$, choose a four-dimensional C with $C_0\subseteq C\subseteq D_3$ and a cubic G with $D_3=C\oplus\mathbb{C}G$. Then

$$M'=T_{\leq 2}\oplus C\oplus\langle x\circ\Phi/5,\ Q+\tau(Q)\rangle\oplus\mathbb{C}\Phi$$

is closed under differentiation, hence the inverse system of an algebra A' with Hilbert function $(1,3,6,4,2,1)$, and $M=M'\oplus\mathbb{C}G$. In the proof of Theorem E.3, case $r=1$, every such A' is shown to be a limit, in a family as in Lemma 2.3(2), of algebras with rays given by Corollary 5.5 or Lemma 5.11 with $k\geq 3$. By Lemma A.14 and the remark following it, A is such a limit as well, and it is smoothable.

If $\dim C_0=5$, then $D_3=C_0$, and either $\dim S_1\circ D_4=4$ and the map $\delta_R\colon\langle y,z\rangle\to T_3/S_1\circ D_4$, $v\mapsto v\circ R$, has rank one, or $\dim S_1\circ D_4=3$ and δ_R has rank two. In the first case let w span the kernel of δ_R. The quartics Q with $\dim S_1\circ D_4=4$ form an open set, and for each of them and each $[w]\in\mathbb{P}^1$ the quartics R with $w\circ R\in S_1\circ D_4$ form a space of dimension 9; the triples $(Q,[w],R)$ with $\delta_R\neq 0$ thus form an irreducible variety, over which $D_3=C_0$ is determined and $\tau(Q)$, E are free. For general Q the polynomial $Q(1,s_1,s_2)$ has a critical point, and in the coordinates of the proof of Theorem E.2 the parabola rays (a) exist for every R. In the second case, the pencils $D_4\ni X^4$ with $\dim S_1\circ D_4=3$ form an irreducible variety in which the pencils $\langle X^4,H(Y,Z)\rangle$ of the family $\mathcal{S}$ are dense (proof of Theorem E.3), the quartics R with δ_R of rank two form an open subset, and $D_3=C_0$ is determined; for $D_4\in\mathcal{S}$ the parabola rays (a) again exist for every R. In both cases the algebras with rays are dense in an irreducible family, and Proposition 2.2(1) concludes. (Script `l136521.sage` checks the examples $Q=Y^4+Z^4+X^2Y^2+X^2Z^2+Y^2Z^2$, $R=Y^4$ and $Q=Y^4+Z^4$, $R=Y^2Z^2$.) □

Example E.6 (Second order)**.** Let A have Hilbert function $(1,3,6,4,3,1)$, and suppose that its cubic base scheme $\mathfrak{B}$ is finite of length at most three;

then $e(\mathfrak{B}) \le 4$. Here $s = 5$, so $k(p) \le 5$ at every point, and Theorem 3.12 applies, as $27 + 2 \cdot 4 = 35 < 36$. A point $p = [P]$ gets the weight $2e_p$ when $P^4 \in D_4$. The quintic terms that separate the surfaces at such a point come from the free cubic tail E of the lift $F + R + E$ of the top form (Step 1 of the proof of Theorem 3.12); if $P^5 \notin D_5$, they could also come from the tangent cone. For instance, let $D_3 = \langle X^3, Y^3, Z^3, G\rangle$ with a general cubic G, $D_4 = \langle X^4, Y^4, Z^4\rangle$ and $F = X^5$. Then $\mathfrak{B} = \{[X], [Y], [Z]\}$, every point has $k(p) = 5$, and the theorem bounds the length of a suitable complete intersection by $33 = 27 + 2 \cdot 3$. For general tails, the quartic parts of the elements of order three of I do not all vanish at $[Y]$ and $[Z]$ (at $[X]$ they are derivatives of the cubic parts in a fixed direction and do not separate the surfaces), and three general elements of order three in I already define a complete intersection of length $31 = 27 + 2 + 1 + 1$ (script `l136431_excess.sage`); the bound of the theorem is not always attained.

Theorem E.7. *Every local algebra with Hilbert function* $(1, 3, 6, 4, 3, 1)$ *is smoothable.*

Proof. Let F span D_5. As in Lemma 2.7, the inverse system is

$$M = T_{\le 2} \oplus D_3 \oplus \{u + \tau(u) : u \in D_4\} \oplus \mathbb{C}(F + R + E)$$

with $R \in T_4$, $E \in T_3$ and $\tau\colon D_4 \to T_3$; it is closed under differentiation if and only if $v \circ R - \tau(v \circ F) \in D_3$ for all $v \in S_1$, and the cubic tail E is free. Let $\mathfrak{B}$ be the cubic base scheme. We apply Lemma A.11 to $D = D_3$.

Finite base of length at most three. This is Example E.6.

The graded data in the other cases. If $\mathfrak{B}$ has length four, then $D_3 = \nu_3(\mathfrak{B})$ and D_4 is a hyperplane of $\nu_4(\mathfrak{B})$, and Lemma 2.16 with $k = n = 4$, applied to $D = \nu_4(\mathfrak{B})$, gives $F \in \nu_5(\mathfrak{B})$; put $\Gamma = \mathfrak{B}$. If $D_3 = \mathbb{C}[X, Y]_3$, then $D_4 \subseteq \mathbb{C}[X, Y]_4$ is annihilated by a binary quartic q, and with $\Gamma = V(z, q)$, collinear of length four, we have $D_3 = \nu_3(\Gamma)$, $D_4 \subseteq \nu_4(\Gamma)$ and $F \in \nu_5(\Gamma)$, because $z \circ F$ and $q \circ F$ have all their derivatives zero. So in both cases the graded data are a subscheme $\Gamma \in \mathrm{Hilb}_4(\mathbb{P}^2)$, a hyperplane $D_4 \subset \nu_4(\Gamma)$, and $F \in \nu_5(\Gamma)$ with $S_1 \circ F \subseteq D_4$.

Let $\pi\colon \mathcal{Z} \to \mathrm{Hilb}_4(\mathbb{P}^2)$ be the universal family and $\mathcal{A}_k = \pi_*\mathcal{O}_{\mathcal{Z}}(k)$, a vector bundle of rank four whose fiber at Γ is $A_k = S_k/(I_\Gamma)_k$ for $k \ge 3$ (Lemma 2.14). Then $\nu_k(\Gamma) = A_k^\vee$, a hyperplane D_4 of $\nu_4(\Gamma)$ is the orthogonal complement of a point $[u] \in \mathbb{P}(A_4)$, and $F \in A_5^\vee$ satisfies $S_1 \circ F \subseteq D_4$ if and only if F vanishes on the subspace S_1u of A_5, the image of $S_1 \otimes u$ under multiplication. So the graded data form the incidence variety $\mathcal{J}$ of pairs $([u], [F]) \in \mathbb{P}(\mathcal{A}_4) \times_{\mathrm{Hilb}_4(\mathbb{P}^2)} \mathbb{P}(\mathcal{A}_5^\vee)$ with $F|_{S_1u} = 0$. Locally over the base it is cut out by three equations in a smooth variety of dimension $8 + 3 + 3 = 14$, so every irreducible component of $\mathcal{J}$ has dimension at least 11 (Lemma 2.17). Let $r = \dim S_1u$. Where $r = 3$, the form F is determined by u, so this part of $\mathcal{J}$ is an open subset of $\mathbb{P}(\mathcal{A}_4)$, irreducible of dimension 11. We show that the parts with $r \le 2$ have dimension at most ten. Then no component of $\mathcal{J}$ lies in them, and $\mathcal{J}$ is irreducible, the closure of the part with $r = 3$.

To compute r, choose a linear form l vanishing at no point of Γ. Division by l^k identifies A_k with the coordinate ring $\mathcal{O}_\Gamma$ of Γ in the affine plane $\mathbb{P}^2 \setminus V(l)$, and $S_1 u$ with the span of $\bar{u}$, $y'\bar{u}$, $z'\bar{u}$, where $\bar{u} = u/l^4$ and $1, y', z'$ are the affine coordinates. If $r = 1$, then $\bar{u}$ spans an $\mathcal{O}_\Gamma$-submodule, so it lies in the socle of the local ring of Γ at one of its points. These $[u]$ form finitely many projective spaces of dimension at most one, and the dimension one occurs only when Γ has a point with a socle of dimension two, that is, a point at which Γ is $V(\mathfrak{m}_p^2)$ or $V(y^2, yz, z^3)$ in suitable coordinates, a locus of dimension at most four in $\mathrm{Hilb}_4(\mathbb{P}^2)$. With the $\mathbb{P}^2$ of possible $[F]$, the part with $r = 1$ has dimension at most $\max(8 + 0, 4 + 1) + 2 = 10$. If $r = 2$ and Γ is collinear, then Γ moves in a locus of dimension $2 + 4 = 6$, $[u]$ adds three and $[F]$ one: ten. If $r = 2$ and Γ is not collinear, then $1, y', z'$ are linearly independent in $\mathcal{O}_\Gamma$, so $r = 3$ when $\bar{u}$ is a unit; hence the $[u]$ with $r \leq 2$ form a proper closed subset of $\mathbb{P}(A_4) \cong \mathbb{P}^3$, of dimension at most two. For four points with no three collinear, $\mathcal{O}_\Gamma = \mathbb{C}^4$, and $r \leq 2$ exactly when $\bar{u}$ is supported at two of the points: six lines, of dimension one. By the upper semicontinuity of the dimension of the fibers of the proper map to $\mathrm{Hilb}_4(\mathbb{P}^2)$, the non-collinear Γ over which this locus has dimension two form a proper closed subset of dimension at most seven. So these pairs $(\Gamma, [u])$ form a set of dimension at most $\max(8 + 1, 7 + 2) = 9$, and $[F] \in \mathbb{P}^1$ adds one: ten.

F not a fifth power. The map $(\tau, R) \mapsto (\tau(v \circ F) - v \circ R)_v$ to $\mathrm{Hom}(S_1, T_3/D_3)$ is onto: if $S_1 \circ F$ has dimension three, τ alone is onto; if a linear form δ kills F, then τ covers the other two components and $\delta \circ R$ the third, because $\delta \circ T_4 = T_3$ and $\delta \circ D_4 \subseteq D_3$. So over the open subset of $\mathcal{J}$ where F is not a fifth power the tails form a vector bundle of rank $12 + 6$, and these algebras form an irreducible family. It contains the Gorenstein algebra generated by $X^5 + Y^5 + Z^5 + L^5 + XYZL$, $L = X + Y + Z$, which has Hilbert function $(1, 3, 6, 4, 3, 1)$: here Γ consists of the four points X, Y, Z, L, the element $\bar{u}$ takes the values $1, 1, 1, -1$ at them, and $F = X^5 + Y^5 + Z^5 + L^5$ (script `l136431_nonpure.sage`). Proposition 2.20 applies.

$F = X^5$. Then $[X] \in \Gamma$, $X^4 \in D_4$, $X^3 \in D_3$, $\tau(X^4) \equiv x \circ R/5$, and R lies in $\mathcal{K} = \{R \in T_4 : N_X \circ R \subseteq D_3\}$, while τ is free on a complement of X^4 and E is free. By Lemma A.3, $\dim \mathcal{K} = 5 + \sigma$, where σ is the dimension of the socle of the local ring of Γ at $[X]$: $\sigma = 1$ unless Γ is, near $[X]$, the fat point $V(\mathfrak{m}_X^2)$ (together with a fourth point) or the scheme $V(y^2, yz, z^3)$, where $\sigma = 2$. The marked pairs $(\Gamma, [X])$ with $\sigma = 1$ form an open subset of the irreducible variety of Lemma A.2, and the two types with $\sigma = 2$ form irreducible varieties of dimensions four and three. Over each of the three, $[u]$ moves in the $\mathbb{P}^2$ of elements of A_4 vanishing at $[X]$ (this is the condition $F|_{S_1 u} = 0$ for $F = X^5$), and R, τ, E in vector bundles; so these algebras form three irreducible families, of dimensions 31, 28 and 27. In each, an explicit member, with four reduced points, respectively with $\Gamma = V(\mathfrak{m}_X^2) \cup \{q\}$ and $\Gamma = V(y^2, yz, z^3)$, lies in a complete intersection of three explicit elements of its ideal of order three, of local length 32 at the origin (script `l136431_pure.sage`), and Proposition 2.24 applies. Here $32 = 27 + 2 + 3$, respectively $27 + 4 + 1$

and $27 + 5$. In the first family this is the bound of Theorem 3.12, with $k([X]) = 5$; in the other two the numbers agree with the weight e_p at $[X]$, while Theorem 3.12, with the weight $2e_p$, would only give a bound of at least 36.

In all cases the smoothability of all schemes of length at most seventeen, Theorem 2.22 and Proposition 2.2(1) conclude. □

Lemma E.8. *Let $A = S/I$ be a local algebra of length $d \geq 19$ with $H_A(2) = 6$ and socle degree five, and suppose that every finite subscheme of $\mathbb{A}^3$ of length less than d is smoothable. Write*

$$M = T_{\leq 2} \oplus D_3 \oplus \{u + \tau(u) : u \in D_4\} \oplus \{F + R(F) + E(F) : F \in D_5\}$$

with linear maps $\tau\colon D_4 \to T_3$, $R\colon D_5 \to T_4$ and $E\colon D_5 \to T_3$, and let $\mathcal{T}$ be the linear space of the pairs (τ, R) with $v \circ R(F) - \tau(v \circ F) \in D_3$ for all $v \in S_1$ and $F \in D_5$. Suppose that I_3^* contains w_1, w_2, w_3, with w_1, w_2 without common factor, such that the linear map

$$\Theta\colon \mathcal{T} \to \mathbb{C}^3 \otimes D_4^*, \qquad (\tau, R) \mapsto \big(u \mapsto \langle w_i, \tau(u)\rangle\big)_{i=1,2,3},$$

is onto. Then every algebra with the graded data (D_3, D_4, D_5) of A is smoothable.

Proof. For fixed graded data the algebras are parametrized by the affine space $\mathcal{T} \times \operatorname{Hom}(D_5, T_3/D_3)$, as the cubic tails E are free; by Proposition 2.24 it suffices to find one of them inside a complete intersection of length $36 < 2d$. As $\mathfrak{m}^6 \subseteq I$, a polynomial $h_3 + h_4 + h_5$ with $h_k \in S_k$ lies in I if and only if

$$h_3 \in I_3^*, \qquad \langle h_4, u\rangle = -\langle h_3, \tau(u)\rangle \ \ (u \in D_4),$$
$$\langle h_5, F\rangle = -\langle h_3, E(F)\rangle - \langle h_4, R(F)\rangle \ \ (F \in D_5).$$

Choose $L \in S_1$ vanishing at no point of $V(w_1, w_2) \subset \mathbb{P}^2$. Since Θ is onto, there is $(\tau, R) \in \mathcal{T}$ with $\langle w_1, \tau(u)\rangle = \langle w_2, \tau(u)\rangle = 0$ and $\langle w_3, \tau(u)\rangle = -\langle L^4, u\rangle$ for $u \in D_4$; in particular w_1, w_2, w_3 are linearly independent. As I_3^* is dual to T_3/D_3, there is E with $\langle w_1, E(F)\rangle = \langle w_2, E(F)\rangle = 0$ and $\langle w_3, E(F)\rangle = -\langle L^4, R(F)\rangle$ for $F \in D_5$. Then w_1, w_2 and $w_3 + L^4$ lie in I, and by Lemma 3.17 they define a finite complete intersection of length 36 in $\mathbb{A}^3$. Proposition 2.24 concludes. □

The surjectivity of Θ is easy to test in the cases we need. If $H_A(4) = 3$ and $D_5 = \mathbb{C}F$ with F concise, then $D_4 = S_1 \circ F$, the map τ is determined by $R = R(F)$ through $\tau(l \circ F) \equiv l \circ R$ modulo D_3, and $R \in T_4$ is free; so $\Theta(R) = (\langle lw_i, R\rangle)_{i,l}$, and Θ is onto if and only if the nine quartics lw_i, $l \in \{x, y, z\}$, are linearly independent, that is, if w_1, w_2, w_3 have no linear syzygy.

Theorem E.9. *Every local algebra with Hilbert function $(1, 3, 6, 4, 4, 1)$ is smoothable.*

Proof. If the cubic base scheme is finite, we use excess linkage; otherwise the graded data come from four points on a line, and we place the algebra in a family over $\mathrm{Hilb}_4(\mathbb{P}^2)$ whose general member has a finite base. Let F span D_5; as in the proof of Theorem E.7 (see (8) and (9)), the inverse system is given by τ, $R \in T_4$ and a free cubic tail E, subject to $v \circ R - \tau(v \circ F) \in D_3$ for $v \in S_1$. Apply Lemma A.11 to D_3.

Finite base. Then $\mathfrak{B}$ has length at most four and $\sum_p e_p \leq 5$. As the socle degree is five, every weight $k(p)-3$ is at most two, and Theorem 3.12 applies, as $27 + 10 < 38$.

A common line. Otherwise $D_3 = \mathbb{C}[X,Y]_3$ in suitable coordinates, so D_4 is a four-dimensional subspace of $D_3' = \mathbb{C}[X,Y]_4$, and $D_4 = \nu_4(\Gamma)$ for $\Gamma = V(z,g)$, where $g \in \mathbb{C}[x,y]_4$ spans the binary quartics orthogonal to D_4; then also $D_3 = \nu_3(\Gamma)$. The forms $z \circ F$ and $g \circ F$ have all their derivatives zero, so $F \in \nu_5(\Gamma)$. Conversely, for every $\Gamma \in \mathrm{Hilb}_4(\mathbb{P}^2)$ and nonzero $F \in \nu_5(\Gamma)$, the spaces $\nu_3(\Gamma)$, $\nu_4(\Gamma)$, $\mathbb{C}F$, of dimensions $4, 4, 1$, are the graded data of algebras with this Hilbert function (Lemma 2.14). If Γ consists of four general points, then $I_3^* = (I_\Gamma)_3$ has the finite base Γ, and the first step applies. We show that every algebra over a collinear Γ lies in an irreducible family in which these are dense.

If F is not a fifth power, the map $(\tau, R) \mapsto (\tau(v \circ F) - v \circ R)_v$ to $\mathrm{Hom}(S_1, T_3/D_3)$ is onto, as in the proof of Theorem E.7. So over the open subset of the projective bundle $\mathbb{P}(\nu_5)$ over $\mathrm{Hilb}_4(\mathbb{P}^2)$ where F is not a fifth power, the tails form a vector bundle, of rank $24 + 15 - 18 = 21$, plus the free E, and the algebras form one irreducible family. If $F = X^5$, then $[X] \in \Gamma$, the closedness condition determines $\tau(X^4)$ from $x \circ R$ and says $R \in \mathcal{K}_4(\Gamma, X)$, and τ is free on a complement of X^4. By Lemma A.3, $\dim \mathcal{K}_4(\Gamma, X) = 5 + \sigma$, where σ is the dimension of the socle of Γ at $[X]$. A collinear Γ is curvilinear, so $\sigma = 1$. The marked pairs $(\Gamma, [X])$ with $\sigma = 1$ form a dense open subset of the irreducible variety of Lemma A.2, over which the spaces $\mathcal{K}_4(\Gamma, X)$ form a vector bundle of rank six; so the algebras with $F = X^5$ form an irreducible family too. In both families the members with Γ four general points are dense, and Proposition 2.2(1) concludes. □

This is the proof of Theorem E.7 with D_4 the whole space $\nu_4(\Gamma)$. Script `119_136441.sage` computes, as a check, complete intersections of local length at most 33 for members with finite base, including the types $V(\mathfrak{m}_X^2) \cup \{q\}$ and $V(y^2, yz, z^3)$.

Theorem E.10. *Every local algebra with Hilbert function* $(1,3,6,4,3,2)$ *is smoothable.*

Proof. If the cubic base scheme is finite, we use excess linkage; otherwise the top forms are binary, and we use one linkage anchor or a ray along a parabola, according to the dimension of the space of their derivatives. The inverse system is as in Lemma E.8, with $\dim D_5 = 2$. Apply Lemma A.11 to D_3.

Finite base. As in the proof of Theorem E.9, $27 + \sum_p (k(p) - 3)e_p \leq 27 + 10 < 38$, and Theorem 3.12 applies.

A common line. Otherwise $D_3 = \mathbb{C}[Y, Z]_3$, so $I_3^* = xS_2$, and $D_4 \subseteq \mathbb{C}[Y, Z]_4$, $D_5 \subseteq \mathbb{C}[Y, Z]_5$. Let $r = \dim S_1 \circ D_5$, which is two or three. For $F \in D_5$, the closedness condition for $v = x$ gives $x \circ R(F) \in D_3$, so $R(F) = B_F + XC_F$ with $B_F \in \mathbb{C}[Y, Z]_4$, defined modulo D_4, and $C_F \in \mathbb{C}[Y, Z]_3$.

$r = 3$. Then $D_4 = S_1 \circ D_5$. For a fixed line, the pairs (D_4, D_5) with $S_1 \circ D_5 \subseteq D_4$ form the zero set of a section of a bundle of rank $2 \cdot 2 \cdot 2 = 8$ on $\mathrm{Gr}(3, \mathbb{C}[Y, Z]_4) \times \mathrm{Gr}(2, \mathbb{C}[Y, Z]_5)$, of dimension 14, so each of its components has dimension at least six. If the binary quartics orthogonal to D_4 have no common cubic factor, then $D_5 = \{F : S_1 \circ F \subseteq D_4\}$ is determined by D_4, which gives an irreducible part of dimension six; if they have one, c, they form the pencil $c\,\mathbb{C}[y, z]_1$, and D_5 moves in a $\mathbb{P}^2$: dimension $3 + 2 = 5$. So the pairs form an irreducible variety. On it, τ is determined by R, through $\tau(v \circ F) \equiv v \circ R(F)$, and the one linear relation $v_1 \circ F_1 + v_2 \circ F_2 = 0$ between the derivatives of a basis F_1, F_2 of D_5 imposes $v_1 \circ C_{F_1} + v_2 \circ C_{F_2} = 0$, three independent conditions. So the pairs (τ, R) form a space of dimension $2 \cdot 2 + 2 \cdot 4 - 3 = 9$, and with the cubic tails, of dimension 12, the algebras form a vector bundle of rank 21 over an irreducible base. One member, with inverse system spanned by $T_{\leq 2}$, $\mathbb{C}[Y, Z]_3$ and

$$YZ^3,\ Y^2Z^2,\ Y^3Z - 3XZ^2,\ Y^2Z^3,\ Y^3Z^2 - 2XZ^3,$$

lies in the complete intersection $V(x^3+y^4,\ xy^2+z^4,\ xz^2+y^3z)$, of local length 36 at the origin, with residual scheme of length 17 (script `l19_136432.sage`). Proposition 2.24 concludes.

$r = 2$. By Lemma 2.16 with $k = 4$, $n = 2$, applied to $D = S_1 \circ D_5$, $D_5 = \nu_5(\Gamma)$ for a subscheme Γ of length two of the line $x = 0$, and $D_4 = \nu_4(\Gamma) \oplus \mathbb{C}G$ with $G \in \mathbb{C}[Y, Z]_4$. If $\Gamma = \{Y, Z\}$, the closedness conditions $z \circ R(Y^5) \in D_3$ and $y \circ R(Z^5) \in D_3$ give $C_{Y^5} = \alpha Y^3$ and $C_{Z^5} = \beta Z^3$; the classes of B_{Y^5}, B_{Z^5} modulo D_4 are free, τ is determined on $\nu_4(\Gamma)$ and free on G, and E is free: an affine space of dimension $2+2+2+6+12 = 24$. If Γ is a double point, $D_5 = \langle Y^5, Y^4Z\rangle$, and the conditions give $C_{Y^5} = \alpha Y^3$ and $C_{Y^4Z} = \frac{3}{5}\alpha Y^2Z + \beta Y^3$: again 24. So these algebras form an affine bundle of rank 24 over the irreducible variety of triples (line, Γ, D_4), and those with Γ reduced are dense. Every algebra with Γ reduced is equivalent, under a linear change of coordinates, to one with $\Gamma = \{Y, Z\}$, and we show that a dense open subset of the latter has a ray. Write $B_Y = B_{Y^5}$ and $B_Z = B_{Z^5}$.

Consider the two conditions $\langle zh, G\rangle = \langle zh, B_Y\rangle = 0$ on $h = y^3 + \mu_1 y^2 z + \mu_2 yz^2$. The quartics zh with $h \in \langle y^3, y^2z, yz^2\rangle$ are orthogonal to Y^4 and Z^4, and the first condition makes zh orthogonal to G; so the two conditions, and $\langle zh, B_Z\rangle$, do not depend on the representatives of B_Y and B_Z modulo D_4. The conditions are two linear equations in μ_1 and μ_2, and they have a unique solution on a dense open subset: for example, $\mu_1 = \mu_2 = 0$ when $G = Y^2Z^2$

and $B_Y = YZ^3$. Where moreover $\beta \neq 0$, let

$$b = x - \tfrac{\alpha}{20}y^2, \qquad u = h - \frac{\langle zh, B_Z\rangle}{6\beta}\, xz^2.$$

Then ub and uz lie in I. Indeed, as $\mathfrak{m}^6 \subseteq I$ and ub, uz have order at least four, it suffices to pair them with the parts of degree four and five of the elements of M; the cubic tails do not enter. The quartic part ux of ub pairs to zero with every binary quartic, such as the elements of D_4, B_Y and B_Z. On the lifts of D_4, moreover, uz has no monomial y^4 or z^4, and $\langle uz, G\rangle = \langle zh, G\rangle = 0$. On the lift Φ_Y of Y^5, with quartic part $B_Y + \alpha XY^3$: $\langle uz, \Phi_Y\rangle = \langle zh, B_Y\rangle = 0$ and $\langle ub, \Phi_Y\rangle = \alpha\langle ux, XY^3\rangle - \frac{\alpha}{20}\langle uy^2, Y^5\rangle = 6\alpha - 6\alpha = 0$. On the lift Φ_Z of Z^5, with quartic part $B_Z + \beta XZ^3$: $\langle uz, \Phi_Z\rangle = \langle zh, B_Z\rangle - \frac{\langle zh, B_Z\rangle}{6\beta}\,\beta\langle xz^3, XZ^3\rangle = 0$, and $\langle ub, \Phi_Z\rangle = \beta\langle ux, XZ^3\rangle - \frac{\alpha}{20}\langle uy^2, Z^5\rangle = 0$, as u has no monomial z^3 and uy^2 is divisible by y^2.

Let $E_3(\Lambda) = Y^3 + \frac{3\alpha}{10}XY$ be the jet of the parabola Λ: $x = \frac{\alpha}{20}y^2$, $z = 0$ (as in (7), with the roles of x and y exchanged). It lies in M, $b \circ E_3(\Lambda) = z \circ E_3(\Lambda) = 0$, and $\langle u, E_3(\Lambda)\rangle = 6 \neq 0$. So $u \in (I : (b, z))$, while $\langle \cdot, E_3(\Lambda)\rangle$ vanishes on $I + (b, z)$; hence $u \notin I + (b, z)$, and (b, z) is a ray of I, as in the proof of Lemma 5.4. Script `119_136432_ray.sage` checks these identities with all data symbolic. Theorem 2.11(1) and Proposition 2.2(1) conclude. □

Theorem E.11. *Every local algebra with Hilbert function* $(1, 3, 6, 5, 3, 1)$ *is smoothable.*

Proof. We combine four methods: excess linkage when the cubic base scheme is short, the global complete intersection of Lemma E.8 when the top form is concise, an expected-dimension count otherwise, and a ray along a line for one stratum of exactly the expected dimension. Let F span D_5, write the inverse system as in Lemma E.8, and apply Lemma A.12 to D_3: the base scheme $\mathfrak{B}$ is (a) finite of length at most four, (b) finite of length five, with $D_3 = \nu_3(\mathfrak{B})$ and $D_3' = \nu_4(\mathfrak{B})$, (c) a line with a point, with the same two properties, or (d) a line Λ, with $D_3 = \nu_3(\Lambda) \oplus \mathbb{C}G$ and $D_3' = \nu_4(\Lambda)$.

Short base. If $\mathfrak{B}$ is finite and $\sum_p e_p \leq 5$, which holds in case (a) and when $\mathfrak{B}$ is curvilinear of length five, then Theorem 3.12 applies, as $27 + 2 \cdot 5 < 38$.

F concise. Then $D_4 = S_1 \circ F$, and the Gorenstein algebra of F has Hilbert function $(1, 3, h, h, 3, 1)$ with $h = \dim S_2 \circ F$; Macaulay's bound gives $h \geq 3$, and $h \leq 5$ as $S_2 \circ F \subseteq D_3$. The algebras with concise F form four irreducible families, vector bundles of tails (R, E), with $R \in T_4$ free, over the following graded data. $(h = 5)$ $D_3 = S_2 \circ F$, with F in an open subset of the image of the $\mathbb{P}^{10}$-bundle $\{(q, F) : q \circ F = 0\}$ over $\mathbb{P}(S_2)$. $(h = 4)$ The quadrics annihilating F form a pencil V, and $D_3 = S_2 \circ F \oplus \mathbb{C}G$ with $[G] \in \mathbb{P}(T_3/S_2 \circ F)$; either V has no common factor, V is open in $\mathrm{Gr}(2, S_2)$, and F lies in the $\mathbb{P}^3$ of quintics annihilated by V; or $V = l\langle l_1, l_2\rangle$, a $\mathbb{P}^2 \times \mathbb{P}^2$ of pencils, and F lies in a $\mathbb{P}^6$. $(h = 3)$ By Lemma 2.16 with $k = n = 3$ and $D = S_2 \circ F$, $F \in \nu_5(\Gamma)$ for a subscheme $\Gamma \subset \mathbb{P}^2$ of length three, not collinear as F is concise, and $D_3 = \nu_3(\Gamma) \oplus K$ with K of dimension two. In each

family, one member satisfies the hypothesis of Lemma E.8: the tops F are sums of five general fifth powers, of four, $X^5+Y^5+Z^5+(Y+Z)^5$, and $X^5+Y^5+Z^5$, and for random further data three cubics w_1, w_2, w_3 of I_3^* with w_1, w_2 coprime and without linear syzygy produce a complete intersection of length 36 containing the member, with residual scheme of length 17 (script `119_136531.sage`). By Lemma E.8 and Proposition 2.24, all algebras with concise F are smoothable.

F a fifth power or binary. For each kind of top form we write all these algebras, graded data and tails together, as the zero set of a section of a vector bundle on a smooth irreducible variety, so that each irreducible component has at least the expected dimension (Lemma 2.17). In these counts the quartic tail R is raw, in T_4, the cubic tails τ and E are classes modulo D_3, and single forms are projectivized (Table 4). We show that the algebras of type (c), those of type (b) for which $\mathfrak{B}$ is not curvilinear (type (b′)), and those of type (d) for the first and the third kind of top form, form sets of smaller dimension. Then the general member of every component has a short base or is of type (d) with a top form of the second kind; the latter algebras are smoothable by a ray (step (iv)), and Proposition 2.2(1) concludes. In case (b), $\mathfrak{B}$ has $h_{\mathfrak{B}}(3)=5$, so $I_{\mathfrak{B}}$ is generated in degrees at most four, and $D_4 \subseteq \nu_4(\mathfrak{B})$ gives $F \in \nu_5(\mathfrak{B})$; the same holds in case (c), where $I_{\mathfrak{B}} = l\,\mathfrak{m}_q$ is generated in degree two. In case (d), let $\Lambda = V(l)$; then $D_4 \subseteq \nu_4(\Lambda)$, so every derivative of $l \circ F$ lies in $l \circ \nu_4(\Lambda) = 0$, and $F \in \nu_5(\Lambda)$. So F is a fifth power of a point of Λ, or a binary form whose two essential variables span the plane of Λ. Families of finite schemes $\mathfrak{B}$ are counted by extensions of colength one, as in the proof of Theorem D.7: a finite scheme containing a scheme W is reached from W by extensions of colength one, and those of a fixed scheme of length at most four form a family of dimension at most two (a new point, or a colength-one subideal of a local ideal with at most three generators). Thus the schemes $\mathfrak{B}$ of length five containing a fat point $V(\mathfrak{m}_q^2)$ form a family of dimension at most $2+2+2=6$, those meeting a line in a scheme of length k one of dimension at most $2+k+2(5-k)$, and the pairs $(\mathfrak{B}, \Gamma)$ with $\Gamma \subset \mathfrak{B}$ of length two one of dimension at most $4+2\cdot 3 = 10$.

(i) $F = X^5$. The data are $[X] \in \mathbb{P}^2$, $D_4 \ni X^4$ in $\mathrm{Gr}(2, T_4/\mathbb{C}X^4)$, $D_3 \ni X^3$ in $\mathrm{Gr}(4, T_3/\mathbb{C}X^3)$, $R \in T_4$, τ on $D_4/\mathbb{C}X^4$ with values in T_3/D_3, and E, of dimensions $2+24+20+15+10+5 = 76$, and the conditions $S_1 \circ D_4 \subseteq D_3$ and $y \circ R, z \circ R \in D_3$ are $30+10$ equations; $\tau(X^4)$ is determined by $x \circ R$. The expected dimension is 36. In case (b′), $\mathfrak{B}$ contains a fat point $V(\mathfrak{m}_q^2)$, so the pairs $(\mathfrak{B}, [X])$ with $[X] \in \mathfrak{B}$ form a set of dimension at most six; $D_4/\mathbb{C}X^4$ lies in $\mathrm{Gr}(2, \nu_4(\mathfrak{B})/\mathbb{C}X^4)$, of dimension four; R lies in $\mathcal{K} = \{R : N_X \circ R \subseteq \nu_3(\mathfrak{B})\}$, of dimension $15 - \dim N_X(I_{\mathfrak{B}})_3 = 11 - h_{\mathfrak{B}'}(2) \leq 8$, where $\mathfrak{B}'$, the residual of $\mathfrak{B}$ with respect to the socle at $[X]$ (proof of Lemma A.3), has length at least three; and τ, E contribute 15. So (b′) has dimension at most $6+4+8+15 = 33$. In case (c), the line and the point give at most four parameters, and $D_4/\mathbb{C}X^4$ lies in $\mathrm{Gr}(2, \nu_4(\mathfrak{B})/\mathbb{C}X^4) = \mathrm{Gr}(2,5)$, of dimension

six; if $[X]$ is a point of the line other than the isolated point, it gives one more parameter and $\dim \mathcal{K} = 7$, and if $[X]$ is the point, $\dim \mathcal{K} = 8$; with τ and E, the dimension is at most $4+1+6+7+15 = 33$, respectively $4+6+8+15 = 33$. In case (d), $[X] \in \Lambda$, and in coordinates with $\Lambda = V(z)$ the data are Λ and $[X]$ ($2+1$ parameters), $[G] \in \mathbb{P}(T_3/\mathbb{C}[X,Y]_3)$ (five), $D_4/\mathbb{C}X^4 \in \mathrm{Gr}(2, \mathbb{C}[X,Y]_4/\mathbb{C}X^4)$ (four), $R \in \mathcal{K}$, and τ, E (fifteen). Here $\dim \mathcal{K} = 6 + j$, where j is the dimension of the space of pairs $(\alpha, \beta) \in \mathbb{C}^2$ with $\alpha\, z\circ G - \beta\, y \circ G \in \mathbb{C}[X,Y]_2$. Indeed, $R \mapsto (y\circ R, z \circ R)$ has kernel $\mathbb{C}X^4$, and its image consists of the pairs $(a,b) \in T_3^2$ with $z\circ a = y \circ b$, as both have dimension 14; for $a = a_0 + \alpha G$ and $b = b_0 + \beta G$ with $a_0, b_0 \in \mathbb{C}[X,Y]_3$ this condition reads $y \circ b_0 = \alpha\, z\circ G - \beta\, y \circ G$, and y maps $\mathbb{C}[X,Y]_3$ onto $\mathbb{C}[X,Y]_2$ with kernel $\mathbb{C}X^3$, so these pairs form a space of dimension $4+1+j$. In case (d), $j \le 1$: if $j = 2$, then $z\circ G$ and $y \circ G$ lie in $\mathbb{C}[X,Y]_2$, so $G \equiv aX^2Z$ modulo $\mathbb{C}[X,Y]_3$, with $a \neq 0$ as $G \notin \mathbb{C}[X,Y]_3$, and $z \circ G = aX^2$ is a square, which is case (c) by the criterion after Lemma A.12. So case (d) has dimension at most $3+5+4+7+15 = 34$ (script `l19_fivecubics_line.sage`: $\dim \mathcal{K} = 6$ for general G, at most 7 for every tested G of type (d), and 8 for $G = X^2Z$, which is of type (c)).

(ii) F binary with $\dim S_2 \circ F = 3$, so that $z \circ F = 0$ in suitable coordinates. The data are $[z]$, $[F] \in \mathbb{P}(\mathbb{C}[X,Y]_5)$, $D_3/S_2\circ F \in \mathrm{Gr}(2,7)$, $D_4 = S_1 \circ F \oplus \mathbb{C}u'$ with $[u'] \in \mathbb{P}(T_4/S_1\circ F)$, $R \in T_4$, $\tau(u')$ and E, of dimensions $2+5+10+12+15+5+5 = 54$, and the conditions $S_1 \circ u' \subseteq D_3$ and $z \circ R \in D_3$ are $15+5$ equations: expected dimension 34. In case (b), $F \in \nu_5(\mathfrak{B})$ is binary, so it lies in ν_5 of the part of $\mathfrak{B}$ on the line $z = 0$, whose length k is three or four, as $\dim S_2 \circ F = 3$ and $h_{\mathfrak{B}}(3) = 5$. The schemes $\mathfrak{B}$ meeting a line in a scheme of length k form a set of dimension at most 9, respectively 8; then $[F]$ gives $k-1$ parameters, $[u'] \in \mathbb{P}(\nu_4(\mathfrak{B})/S_1\circ F)$ two, and $R \in z^{-1}(D_3)$, $\tau(u')$, E give $10+5+5$: at most 33. In case (c), $\dim S_2 \circ F = 3$ forces the line of F to be the line of $\mathfrak{B}$, since otherwise $\mathfrak{B}$ meets the line $z = 0$ in a scheme of length at most two and $S_2 \circ F \subseteq \nu_3(\mathfrak{B}) \cap \mathbb{C}[X,Y]_3$ has dimension at most two; the count is $4+5+3+20 = 32$. In case (d), $\Lambda = V(z)$ is the line of F, and the data are $[z]$ and $[F]$ $(2+5)$, $[G] \in \mathbb{P}(T_3/\mathbb{C}[X,Y]_3)$ (five), $[u'] \in \mathbb{P}(\mathbb{C}[X,Y]_4/S_1\circ F)$ (two), and R, $\tau(u')$, E $(10+5+5)$: the count is 34, *equal* to the expected dimension. These algebras are treated in step (iv).

(iii) F binary with $\dim S_2 \circ F = 2$, so that $F \in \nu_5(\Gamma)$ for a subscheme Γ of length two of the line $z = 0$, and F is not a fifth power. The data are $[z]$, Γ, $[F]$, $D_3/\nu_3(\Gamma) \in \mathrm{Gr}(3,8)$, $[u']$, R, $\tau(u')$ and E, of dimensions $2+2+1+15+12+15+10 = 57$, with $15+5$ equations: expected dimension 37. In case (b), $\Gamma \subset \mathfrak{B}$; the pairs $(\mathfrak{B}, \Gamma)$ form a set of dimension at most ten, and the count is at most $10+1+2+20 = 33$; in case (c) it is at most $4+2+1+3+20 = 30$; in case (d), Λ is the line of Γ, and the count is $4+1+5+2+20 = 32$.

(iv) F binary with $\dim S_2 \circ F = 3$, of type (d). In the coordinates of (ii), $\Lambda = V(z)$, $\nu_j(\Lambda) = \mathbb{C}[X,Y]_j$, $D_3 = \mathbb{C}[X,Y]_3 \oplus \mathbb{C}G$ and $D_4 = S_1 \circ F \oplus \mathbb{C}u' \subseteq \mathbb{C}[X,Y]_4$. The data of (ii) with these properties form an irreducible variety $\mathcal{S}$

of dimension 34, the tower of bundles just described, and all its members are algebras with Hilbert function $(1,3,6,5,3,1)$. We show that a dense subset of $\mathcal{S}$ consists of algebras with a ray along the line of a point of Λ; then all of $\mathcal{S}$ is smoothable by Theorem 2.11(1) and Proposition 2.2(1).

Let $P \in \Lambda$. Suppose that $P^3 \in N_P \circ M$, say $P^3 = v_1 \circ m_1 + v_2 \circ m_2$ with a basis v_1, v_2 of N_P and $m_i \in M$, where m_i has the component $a_i(F+R+E)$ along the lift of F and the component $u_i + \tau(u_i)$, $u_i \in D_4$, along the lifts of quartics. The part of degree four is $w \circ F$ with $w = a_1 v_1 + a_2 v_2$, so $w \circ F = 0$, and as F has the two essential variables X, Y, $w \in \mathbb{C}z$. The part of degree three is $w \circ R + v_1 \circ u_1 + v_2 \circ u_2$. Hence

$$P^3 \in N_P \circ M \quad \Longrightarrow \quad P^3 \in N_P \circ D_4 + \mathbb{C}\, z \circ R.$$

Now $z \circ R \in D_3$, and $R \mapsto z \circ R$ maps $z^{-1}(D_3)$ onto D_3; so on a dense open subset of $\mathcal{S}$, $z \circ R \notin \mathbb{C}[X,Y]_3$. There, since P^3 and $N_P \circ D_4$ lie in $\mathbb{C}[X,Y]_3$ and $z \circ R$ does not, the right side reduces to $P^3 \in N_P \circ D_4$. Next, the binary quartics orthogonal to D_4 form a pencil, which has a member with a simple root P_1 on Λ, and then $P_1^3 \notin N_{P_1} \circ D_4$, as in (d) of the proof of Theorem B.3. So for every member of $\mathcal{S}$ with $z \circ R \notin \mathbb{C}[X,Y]_3$, a dense open subset, $P_1^3 \in D_3 \subseteq M$ and $P_1^3 \notin N_{P_1} \circ M$, and Theorem 2.11(2) gives a ray along the line of P_1. (Script `l19_fivecubics_line.sage` checks a member with random G, R, $\tau(u')$, E: Hilbert function $(1,3,6,5,3,1)$, cubic base scheme the line $z=0$, and $(I : L_{P_1}) \not\subseteq I + L_{P_1}$ for $P_1 = X + 2Y$.)

A second proof of step (iv) uses linkage. In coordinates with $\Lambda = V(x)$, the member

$$M_0 = T_{\le 2} \oplus \mathbb{C}[Y,Z]_3 \oplus \mathbb{C}XYZ \oplus \langle YZ^3, Y^2Z^2, Y^3Z - 3XZ^2 \rangle \oplus \mathbb{C}(Y^3Z^2 - 2XZ^3)$$

of $\mathcal{S}$, with $F = Y^3Z^2$, $G = XYZ$, $u' = YZ^3$ and $R = -2XZ^3$, lies in the complete intersection $V(x^3+y^4,\ xy^2+z^4,\ xz^2+y^3z)$ of local length 36 of the proof of Theorem E.10, with residual scheme of length 17, and Proposition 2.24 applies to $\mathcal{S}$ (script `l19_fivecubics_line2.sage`).

In summary, every component has at least the dimension in the second column, and the strata have at most the dimensions in the others:

top form	components	type (b′)	type (c)	type (d)
(i) $F = X^5$	≥ 36	≤ 33	≤ 33	≤ 34
(ii) binary, $\dim S_2 \circ F = 3$	≥ 34	≤ 33	≤ 32	34, step (iv)
(iii) binary, $\dim S_2 \circ F = 2$	≥ 37	≤ 33	≤ 30	≤ 32

So the sets of type (b′) and (c), and in cases (i) and (iii) those of type (d), contain no component. Hence the general member of every component has a short base, or is of type (d) in case (ii) and is smoothable by step (iv). Every nonzero quintic has one, two or three essential variables, so the steps cover all algebras. □

In the concise case, one member per irreducible family suffices, by Proposition 2.24. The counts in the last step are upper bounds by parametrizations

of the strata; script `l19_136531_strata.sage` checks the dimensions of the spaces $\mathcal{K}$ on representatives.

Theorem E.12. *Every local algebra with Hilbert function* $(1,3,6,6,2,1)$ *is smoothable.*

Proof. We follow the proof of Theorem E.5, with D_3 one dimension larger. Let F span D_5, with $r \leq 2$ essential variables, and write the inverse system as in (8), with tails τ, R and E.

$r = 2$. In suitable coordinates $F \in \mathbb{C}[X,Y]_5$, $D_4 = S_1 \circ F$, τ is determined by R, and the only condition is $z \circ R \in D_3$, a space of dimension $5 + 6 = 11$ for every D_3. If $\dim S_2 \circ F = 3$, then $D_3 = S_2 \circ F \oplus K$ with $\dim K = 3$, and these algebras form an irreducible family. For one member, with $F = X^5 + Y^5 + (X+Y)^5$ and K, R, E chosen with a fixed random seed, the cubics of I_3^* vanish at the three roots X, Y, $X+Y$ of the cubic annihilating F, none of whose fourth powers lies in D_4, and three elements of I of order three define a complete intersection of local length $30 = 27 + 3$, with residual scheme of length 11 (script `l19_136621.sage`). Proposition 2.24 concludes. If $\dim S_2 \circ F = 2$, then $F \in \nu_5(\Gamma)$ for Γ of length two on the line $z = 0$, $D_4 = \nu_4(\Gamma)$, and $D_3 = \nu_3(\Gamma) \oplus K$ with $\dim K = 4$; these data form an irreducible family in which $\Gamma = \{X, Y\}$ is dense, and there $F = \alpha X^5 + \beta Y^5$ with $\alpha\beta \neq 0$, so Lemma 5.11 applies with $P = X$.

$r = 1$. Then $F = X^5$, $D_4 = \langle X^4, Q\rangle$, $5\tau(X^4) \equiv x \circ R$ modulo D_3, $y \circ R, z \circ R \in D_3$, and $\tau(Q)$ and E are free. Let $C_0 = S_1 \circ D_4 + \langle y \circ R, z \circ R\rangle \subseteq D_3$. If $\dim C_0 \leq 5$, choose a five-dimensional C with $C_0 \subseteq C \subseteq D_3$ and G with $D_3 = C \oplus \mathbb{C}G$; then $M = M' \oplus \mathbb{C}G$, where M' is the inverse system of an algebra with Hilbert function $(1,3,6,5,2,1)$ and top form X^5. By the proof of Theorem E.5, case $r = 1$, every such algebra lies in a family as in Lemma 2.3(2) in which the algebras with rays given by Corollary 5.5 or Lemma 5.11, with operators of order at least four, are dense; by Lemma A.14 and the remark after it, so does A. If $\dim C_0 = 6$, then $D_3 = C_0$, $\dim S_1 \circ D_4 = 4$, and $y \circ R$, $z \circ R$ are independent modulo $S_1 \circ D_4$; the pairs (Q, R) with these open properties form an irreducible variety, which determines D_3, and $\tau(Q)$, E are free. The parabola rays (a) of the proof of Theorem E.2 exist for every R as soon as $Q(1, s_1, s_2)$ has a critical point, in particular when $Q(0, Y, Z)$ has four distinct roots. This open condition holds for $Q = Y^4 + Z^4 + XY^3 + XZ^3$, and script `l19_136621.sage` checks a member with this Q and $\dim C_0 = 6$; so it holds on a dense subset of the family.

In all cases Theorem 2.11(1) and Proposition 2.2(1) conclude. □

Appendix F. One quadric in the tangent cone

F.1. We turn to the tangent cones on a single quadric x^2. By the remark after Corollary 4.8, for socle degree at most six only the binary case remains, where D_3 contains all cubic forms in Y, Z. Write $\bar{T}_k = \mathbb{C}[Y,Z]_k$ and $\bar{S}_k = \mathbb{C}[y,z]_k$ for the forms and operators in the two variables of the plane $x = 0$.

Proof of Proposition 4.11(1). We find an element of $(I : \mathfrak{m}^2) \cap \mathfrak{m}^3$ whose cubic part does not vanish on the plane $x = 0$, and then apply Theorem 4.5. By Theorem 2.9(1) we may assume $I_2^* = \mathbb{C}x^2$. All first derivatives of a form spanning D_6 are proportional, as $H_A(5) = 1$, so D_6 is spanned by the sixth power of a linear form. Since $x^2 S_4 \subseteq I_6^*$, this linear form does not involve X, and we choose y, z so that $D_6 = \mathbb{C}Y^6$. Let $\Phi \in M$ have top form Y^6 and let Q_5 be its part of degree five. The elements of M of degree at most five are, up to adding multiples of $y \circ \Phi$, of degree at most four, so $M = \mathbb{C}\Phi + \mathbb{C}\,(y \circ \Phi) + (M \cap T_{\leq 4})$. An element $u \in \mathfrak{m}^3$ pairs to zero with $T_{\leq 2}$, so by the remark after Lemma 4.2 it lies in $(I : \mathfrak{m}^2)$ if and only if $\langle qu, \Phi \rangle = 0$ for all $q \in \mathfrak{m}^2$, that is, if and only if $u \circ \Phi$ has degree at most one. Writing $u = u_3 + u_4 + \cdots$ with $u_j \in S_j$, this means

$$u_3 \circ Y^6 = 0 \qquad \text{and} \qquad u_3 \circ Q_5 + u_4 \circ Y^6 = 0. \tag{15}$$

Since $u_4 \circ Y^6$ is 360 times the coefficient of y^4 in u_4 times Y^2, the second condition can be met by a choice of u_4 as soon as $u_3 \circ Q_5 \in \mathbb{C}Y^2$.

Let $f \in I$ have leading form x^2. The part of degree three of $f \circ \Phi = 0$ gives $x^2 \circ Q_5 \in \mathbb{C}Y^3$, so $Q_5 = \kappa X^2 Y^3 + X A_4 + A_5$ with $\kappa \in \mathbb{C}$, $A_4 \in \mathbb{C}[Y, Z]_4$ and $A_5 \in \mathbb{C}[Y, Z]_5$. We take $u_3 = xq_1 + zq_2$ with $q_1, q_2 \in \mathbb{C}[y, z]_2$; then $u_3 \circ Y^6 = 0$. A direct computation shows that $u_3 \circ Q_5$ has no term X^2, and that its coefficients of XY, XZ, YZ, and twice its coefficient of Z^2, are

$$12\kappa a + \langle yzq_2, A_4 \rangle, \qquad \langle z^2 q_2, A_4 \rangle,$$
$$\langle yzq_1, A_4 \rangle + \langle yz^2 q_2, A_5 \rangle, \qquad \langle z^2 q_1, A_4 \rangle + \langle z^3 q_2, A_5 \rangle,$$

where a is the coefficient of y^2 in q_1 (script `puresextic.sage`). With $\varphi(q) = (\langle yzq, A_4 \rangle, \langle z^2 q, A_4 \rangle)$, the condition $u_3 \circ Q_5 \in \mathbb{C}Y^2$ reads

$$\varphi(q_2) + (12\kappa a, 0) = 0, \qquad \varphi(q_1) + \psi(q_2) = 0,$$

with ψ linear. These are four linear equations in the six-dimensional space of pairs (q_1, q_2); we claim that they have a solution with $q_2 \neq 0$. If $\kappa = 0$, the solutions with $q_2 = 0$ form $\ker \varphi \times 0$, of dimension $3 - \operatorname{rk} \varphi$, while all solutions form a space of dimension at least $6 - \operatorname{rk} \varphi - 2$. If $\kappa \neq 0$, the solutions with $q_2 = 0$ are the pairs $(q_1, 0)$ with $q_1 \in \langle yz, z^2 \rangle$ and $\varphi(q_1) = 0$. If these are fewer than two dimensions, we are done, as all solutions form a space of dimension at least two. Otherwise $\varphi(yz) = \varphi(z^2) = 0$, so $\langle y^2 z^2, A_4 \rangle = \langle yz^3, A_4 \rangle = \langle z^4, A_4 \rangle = 0$, and the second coordinate of φ vanishes identically. Then the equations have rank at most three and a solution space of dimension at least three, larger than two.

So $(I : \mathfrak{m}^2) \cap \mathfrak{m}^3$ contains an element u whose cubic part restricts to $zq_2 \neq 0$ on the plane $x = 0$, and Corollary 4.8 applies, as $\mathfrak{m}^7 \subseteq I$. □

For the remaining parts we return to the method of Section 4: an element $u \in (I : \mathfrak{m}^2) \cap \mathfrak{m}^3$ whose cubic part does not vanish on the plane $x = 0$. By the remark after Lemma 4.2, u lies in $(I : \mathfrak{m}^2)$ if and only if $u \circ \Psi \in T_{\leq 1}$ for every $\Psi \in M$; this is automatic when $\deg \Psi \leq \deg u + 1$, so only the lifts of

the top forms of high degree impose conditions. In each part we may assume $I_2^* = \mathbb{C}x^2$, by Theorem 2.9(1).

Proof of Proposition 4.11(2) *when* $H_A(6) = 2$. By Theorem 2.9(1) we may assume $I_2^* = \mathbb{C}x^2$. By Lemma 2.16 with $k = 4$ and $n = 2$, $D_4 = \nu_4(\Gamma)$, $D_5 = \nu_5(\Gamma)$ and $D_6 \subseteq \nu_6(\Gamma)$ for a subscheme $\Gamma \subset \mathbb{P}^2$ of length two; as $\dim D_6 = 2$, $D_6 = \nu_6(\Gamma)$. Let $f \in I$ have leading form x^2. For $\Phi \in M$ with top form $F \in D_6$, the part of degree four of $f \circ \Phi = 0$ is $x^2 \circ F$, so $x^2 \circ D_6 = 0$. As $x^2 \circ P^6 = 30\, x(p)^2 P^4$, and $P^6 \in \nu_6(\Gamma)$ for every point $p = [P]$ of the support of Γ, the scheme Γ is supported on the line $x = 0$; it need not be contained in it. So in suitable coordinates y, z, with x fixed, Γ is one of: two points, $I_\Gamma = (x, yz)$; a double point tangent to the line $x = 0$, $I_\Gamma = (x, z^2)$; a double point not tangent to it, $I_\Gamma = (z, x^2)$.

Let F_1, F_2 be a basis of D_6, $\Phi_i = F_i + Q_i +$ (terms of degree at most four) the elements of M with top forms F_i, where $Q_i \in T_5$, and Ψ_1, Ψ_2 elements of M whose top forms span D_5. Then M is spanned by Φ_1, Φ_2, Ψ_1, Ψ_2 and $M \cap T_{\leq 4}$. Put $u_3 = z^2 y$; in all three cases $u_3 \in (I_\Gamma)_3$, so u_3 annihilates $\nu_k(\Gamma)$ for $k \geq 3$. For $u_4 \in S_4$ and $u = u_3 + u_4$, the elements $u \circ (M \cap T_{\leq 4})$ and $u \circ \Psi_j$ lie in $T_{\leq 1}$, and $u \circ \Phi_i = (u_3 \circ Q_i + u_4 \circ F_i) +$ (terms of degree at most one). By the remark after Lemma 4.2, $u \in (I : \mathfrak{m}^2)$ if and only if $\langle u, \mathfrak{m}^2 \circ M \rangle = 0$, that is, $u \circ M \subseteq T_{\leq 1}$; this holds if

$$u_3 \circ Q_i + u_4 \circ F_i = 0 \qquad (i = 1, 2). \tag{16}$$

The tails Q_i are constrained by the closedness of M. In the case of two points, take $F_1 = Y^6$ and $F_2 = Z^6$. Since $x \circ F_1 = z \circ F_1 = 0$, the elements $x \circ \Phi_1$ and $z \circ \Phi_1$ of M have degree at most four, so $x \circ Q_1$ and $z \circ Q_1$ lie in $D_4 = \langle Y^4, Z^4 \rangle$; as x and z map distinct monomials to distinct monomials, $Q_1 \in \langle XY^4, Y^5, Y^4 Z, Z^5 \rangle$, and likewise $Q_2 \in \langle XZ^4, Z^5, YZ^4, Y^5 \rangle$. Hence $u_3 \circ Q_1 = 0$ and $u_3 \circ Q_2 = 12\, g\, Z^2$, where g is the coefficient of YZ^4 in Q_2, and $u_4 = -\frac{g}{30} z^4$ satisfies (16), because $z^4 \circ Z^6 = 360 Z^2$ and $z^4 \circ Y^6 = 0$. The two double points are treated in the same way, with $F_1 = Y^6$ and $F_2 = Y^5 Z$, respectively $F_2 = XY^5$: the closedness conditions now include $z \circ Q_2 - \frac{1}{6} y \circ Q_1 \in D_4$, respectively $x \circ Q_2 - \frac{1}{6} y \circ Q_1 \in D_4$, and (16) is solved by $u_4 = -\frac{q}{60} y^4 - \frac{q'}{10} z y^3$, where q and q' are the coefficients of $Y^3 Z^2$ in Q_1 and Q_2, respectively by $u_4 = 0$ (script `l19_sexticplane.sage`, which solves the closedness conditions on (Q_1, Q_2) in all three cases and checks (16) on the whole solution space). In all cases $u = z^2 y + u_4$ lies in $(I : \mathfrak{m}^2) \cap \mathfrak{m}^3$, and its cubic part restricts to $z^2 y$ on the plane $x = 0$, which is not a cube. Corollary 4.6 concludes. □

Proof of Proposition 4.11(2) *when* $H_A(6) = 1$. By Lemma 2.16 with $k = 4$, $n = 2$, $D_4 = \nu_4(\Gamma)$, $D_5 = \nu_5(\Gamma)$ and $D_6 = \mathbb{C}F$ with $F \in \nu_6(\Gamma)$ for a subscheme $\Gamma \subset \mathbb{P}^2$ of length two. Since $x^2 \circ D_4 = 0$ and $x^2 \circ P^4 = 12\, x(p)^2 P^2$, the scheme Γ is supported on the line $x = 0$. So either $x \in I_\Gamma$, and $I_\Gamma = (x, h)$ with $h \in \bar{S}_2$, or Γ is a double point on this line with a tangent direction off

it, and $I_\Gamma = (z, x^2)$ in suitable coordinates y, z. Let $\Phi = F + R + \dots$ be the lift of F, with $R \in T_5$.

$I_\Gamma = (x, h)$. Then $D_k \subseteq \bar{T}_k$ for $k = 4, 5, 6$. We look for $u = p + u_4$ with $p \in h\bar{S}_1$ and $u_4 \in \bar{S}_4$. Then $pS_2 \subseteq (I_\Gamma)_5$ and $pS_3 \subseteq (I_\Gamma)_6$, so p annihilates D_5 and F. The only condition for $u \in (I : \mathfrak{m}^2)$ is therefore that the quadratic part $p \circ R + u_4 \circ F$ of $u \circ \Phi$ vanishes, that is, $p \circ R \in \bar{S}_4 \circ F$, as $u_4 \circ F$ runs over $\bar{S}_4 \circ F$ when u_4 runs over $\bar{S}_4$. The derivative $x \circ \Phi \in M$ has degree at most four, as $x \circ F = 0$, so $x \circ R \in D_4 \subseteq \bar{T}_4$. Hence $R = R^b + XG$ with $R^b \in \bar{T}_5$ and $G = x \circ R \in \nu_4(\Gamma)$, and for $p = hl$ with $l \in \bar{S}_1$,

$$p \circ R = (hl) \circ R^b + X\, l \circ (h \circ G) = (hl) \circ R^b \in \bar{T}_2,$$

because $h \circ \nu_4(\Gamma) = 0$. If F has two essential variables, then $\bar{S}_4 \circ F$ has dimension at least two: it has the dimension of $\bar{S}_2 \circ F$, by the symmetry of the pairing $(a, b) \mapsto \langle ab, F\rangle$ on $\bar{S}_4 \times \bar{S}_2$, and if all second derivatives of F were proportional, its first derivatives would be proportional to a fifth power, hence to each other. So the linear map $h\bar{S}_1 \to \bar{T}_2/(\bar{S}_4 \circ F)$, $hl \mapsto (hl) \circ R^b$, from a space of dimension two to a space of dimension at most one, has a nonzero kernel. If $F = L^6$, then $L^5 \in D_5 = \nu_5(\Gamma)$, so $[L] \in \Gamma$; choose coordinates with $L = Y$, so that $h = zl'$ for some $l' \in \bar{S}_1$, and $\bar{S}_4 \circ F = \mathbb{C}Y^2$. Now $z \circ \Phi$ has degree at most four, so $z \circ R = z \circ R^b \in \nu_4(\Gamma)$, and for every $l \in \bar{S}_1$ and $q \in z\bar{S}_1$ we get $\langle q, (hl) \circ R^b\rangle = \langle ql'l, z \circ R^b\rangle = 0$, as $ql'l \in h\bar{S}_2 \subseteq (I_\Gamma)_4$. So $(hl) \circ R^b \in (z\bar{S}_1)^\perp \cap \bar{T}_2 = \mathbb{C}Y^2$ for every l. In both cases some $u = hl + u_4$ with $l \neq 0$ lies in $(I : \mathfrak{m}^2)$, and its cubic part hl does not vanish on the plane $x = 0$.

$I_\Gamma = (z, x^2)$. Then $\nu_k(\Gamma) = \langle Y^k, XY^{k-1}\rangle$ and $F = aY^6 + bXY^5$. As $z \circ F = 0$, the element $z \circ \Phi$ has degree at most four, so $z \circ R \in D_4 = \langle Y^4, XY^3\rangle$, and $R \in \mathbb{C}[X, Y]_5 \oplus Z\langle Y^4, XY^3\rangle$. The cubic $u = z^2 y$ annihilates F, R, D_5 and D_4, as these have degree at most one in Z; so $u \in (I : \mathfrak{m}^2)$, and its cubic part yz^2 does not vanish on the plane $x = 0$ (script `l19_sextic2pts.sage` checks the shape of R).

As $\mathfrak{m}^7 \subseteq I$, Corollary 4.8 applies in all cases. □

The binary case $D_3 = \bar{T}_3$ is not needed: the placement of Γ on the line $x = 0$ follows from $x^2 \circ D_4 = 0$.

Proof of Proposition 4.11(3). Since $H_A(6) = 1$, the space D_7 is spanned by a seventh power L^7, and $x^2 \circ L^7 = 0$ forces $L \in \bar{T}_1$; choose y, z with $L = Y$. Then $D_6 = \mathbb{C}Y^6$, $D_5 = \mathbb{C}Y^5$, and for the lift $\Phi = Y^7 + Q_6 + Q_5 + Q_4 + Q_3$ of Y^7 we have $M = \mathbb{C}\Phi \oplus \mathbb{C}\,(y \circ \Phi) \oplus \mathbb{C}\,(y^2 \circ \Phi) \oplus (M \cap T_{\leq 4})$. So an element $u \in \mathfrak{m}^3$ lies in $(I : \mathfrak{m}^2)$ if and only if $u \circ \Phi \in T_{\leq 1}$.

The shape of Φ. The derivatives $x \circ \Phi$ and $z \circ \Phi$ have degree at most five, so $x \circ Q_6$ and $z \circ Q_6$ lie in $D_5 = \mathbb{C}Y^5$. Hence $Q_6 = (AX + BZ)Y^5 + \gamma Y^6$, and replacing Φ by $\Phi - \frac{\gamma}{7}\, y \circ \Phi$ we may assume $\gamma = 0$. Let $f = x^2 + f_3 + f_4 + \dots \in I$. The part of degree four of $f \circ \Phi = 0$ is $f_3 \circ Y^7$, so f_3 has no term y^3, and the transverse cubic γ_f is divisible by z. Write $f_3 = xr + zs$ with $r \in S_2$, $s \in \bar{S}_2$.

Then $f_3 \circ Q_6 = A\, r \circ Y^5 + B\, s \circ Y^5 \in \mathbb{C}Y^3$, and the part of degree three of $f \circ \Phi = 0$ gives $x^2 \circ Q_5 \in \mathbb{C}Y^3$, so

$$Q_5 = \kappa X^2 Y^3 + X A_4 + A_5, \qquad A_4 \in \bar{T}_4,\ A_5 \in \bar{T}_5.$$

The same holds for $\hat{Q}_5 = Q_5 - \frac{5}{21}(AX + BZ)^2 Y^3$.

Two elements of $(I : \mathfrak{m}^2)$. First, $V = zy^3 - \frac{B}{42}y^5$ lies in $(I : \mathfrak{m}^2)$: the quadratic part of $V \circ \Phi$ is $zy^3 \circ Q_6 - \frac{B}{42}y^5 \circ Y^7 = 60BY^2 - 60BY^2 = 0$, and its parts of higher degree vanish. Second, for $q_1, q_2 \in \bar{S}_2$ let $u_3 = xq_1 + zq_2$, so that $u_3 \circ Y^7 = 0$ and $u_3 \circ Q_6 = 20CY^3$ with $C = A\, q_1[y^2] + B\, q_2[y^2]$, where $q[y^2]$ is the coefficient of y^2; let $u_4 = -\frac{C}{42}y^4$. Then $(u_3 + u_4) \circ \Phi$ has degree at most two, and its quadratic part is $u_3 \circ \hat{Q}_5$ plus a multiple of Y^2 (script `l19_septic_h25.sage`, identities with all coefficients as indeterminates). Since $y^5 \circ Y^7 = 2520\, Y^2$, there is $c \in \mathbb{C}$ with $U = u_3 + u_4 + cy^5 \in (I : \mathfrak{m}^2)$ as soon as $u_3 \circ \hat{Q}_5 \in \mathbb{C}Y^2$. This is the linear system of the proof of part (1), with $\hat{Q}_5$ in place of Q_5; that proof shows that it has a solution with $q_2 \neq 0$. Fix one.

Jets. Apply Theorem 4.5 to $u = U + \sigma V$, $\sigma \in \mathbb{C}$, an element of $(I : \mathfrak{m}^2) \cap \mathfrak{m}^3$ whose cubic part restricts to $zq_2 \neq 0$ on the plane $x = 0$; the cubic part of G_t is $\gamma_f + tzq_2$. If zq_2 is not a cube, Corollary 4.6 applies. Otherwise $q_2 = c_0 z^2$ with $c_0 \neq 0$. If γ_f is not a cube, Theorem 2.9(2) applies; as z divides γ_f, it remains that $\gamma_f = az^3$, and the cubic part of G_t is $(a + tc_0)z^3$, nonzero for all but one t. We exchange the roles of y and z in Theorem 4.5. The coordinate change of the theorem, $x \mapsto x + \xi(y, z)$ with ξ of order at least two, changes $u(0, y, z)$ only in degrees at least four, and polynomially in t; so the coefficient of y^4 in G_t is a polynomial in t. If it is not identically zero, case (2) holds for all but finitely many t. If it is, consider the coefficient of zy^3: as V involves only y and z, it is unchanged by the coordinate change and $\partial_x V = 0$, so σV adds $t\sigma$ to this coefficient. For all but one σ it is a nonzero polynomial in t, and case (3) holds. □

The linear system of part (1) gives q_2 without using the dimensions of D_3 and D_4.

Proof of Proposition 4.11(4). As $H_A(6) = H_A(7) = 1$, $D_7 = \mathbb{C}Y^7$ for a linear form $Y \in \bar{T}_1$, as in the proof of part (3). By Lemma 2.16 with $k = 4$, $n = 2$, $D_4 = \nu_4(\Gamma)$ and $D_5 = \nu_5(\Gamma)$ for a subscheme Γ of length two; $Y^5 \in D_5$ gives $[Y] \in \Gamma$, and as in part (2), the support of Γ lies on the line $x = 0$. In suitable coordinates Γ is (a) $V(x, yz)$, (b) $V(x, z^2)$ or (c) $V(z, x^2)$, and $D_5 = \langle Y^5, G\rangle$ with $G = Z^5$, $Y^4 Z$, XY^4, respectively. Let $\Phi = Y^7 + Q_6 + Q_5 + \dots$ be the lift of Y^7, with Q_6 normalized modulo Y^6 as in part (3), and Ψ a lift of G. Then $M = \mathbb{C}\Phi + \mathbb{C}\,(y \circ \Phi) + \mathbb{C}\,(y^2 \circ \Phi) + \mathbb{C}\Psi + (M \cap T_{\leq 4})$, and $u = u_3 + u_4 + u_5 + \dots \in \mathfrak{m}^3$ lies in $(I : \mathfrak{m}^2)$ if and only if

(17)

$$u_3 \circ Y^7 = 0,\ \ u_3 \circ G = 0,\ \ u_3 \circ Q_6 + u_4 \circ Y^7 = 0,\ \ u_3 \circ Q_5 + u_4 \circ Q_6 + u_5 \circ Y^7 = 0.$$

The tails. As $x\circ Q_6$ and $z\circ Q_6$ lie in D_5, $Q_6 = AXY^5 + BZY^5 + C\,H$ with $H = Z^6$, Z^2Y^4, X^2Y^4 in the three cases. In cases (a) and (b), $x\circ Q_6 = AY^5$, so the element $x\circ\Phi - \frac{A}{42}y^2\circ\Phi$ has degree at most four, and $x\circ Q_5 - \frac{A}{42}y^2\circ Q_6$ is an element $\alpha Y^4+\beta Z^4$, respectively $\alpha Y^4+\beta Y^3Z$, of D_4; this fixes the part of Q_5 divisible by X up to the two constants α,β. In case (c), and in cases (a), (b) when $C=0$, we have $z\circ Q_6 = BY^5$, and likewise $z\circ Q_5 - \frac{B}{42}y^2\circ Q_6 \in D_4$, which gives linear relations among the coefficients of the binary part $\sum_i q_iY^{5-i}Z^i$ of Q_5. If $C\neq 0$ in cases (a), (b), then $z\circ\Phi - \frac{B}{42}y^2\circ\Phi$ has top form a multiple of G, so Ψ may be taken among the derivatives of Φ, and the condition $u_3\circ G = 0$ follows from the others. Only these conditions on Q_6 and Q_5 are used below; they hold for every algebra.

In each case we give an explicit solution $u = u_3+u_4+u_5$ of (17). The coefficients of u_4 and u_5 are polynomials in the tail coefficients divided by a nonzero constant times C, and $C\neq 0$ wherever C occurs in a denominator; the identities are checked in the polynomial ring after the denominators are cleared, so they hold at every parameter value of each branch (script `quadrictails4_cert.sage`; script `l19_13542211.sage` checks the same solutions over the field of rational functions).

Case (c). The z-condition forces the part of Q_5 of degree at least two in Z to be $\frac{5B^2}{21}Z^2Y^3$. Then $u = z^2y - \frac{B^2}{1764}y^5$ satisfies (17), and its cubic part yz^2 is not a cube: Corollary 4.6 applies.

Case (a). If $C\neq 0$, a solution is

$$\begin{aligned} u_3 &= 12q_3\,y^2z - (12q_2 - \tfrac{20}{7}B^2)\,yz^2,\\ u_4 &= -\tfrac{2}{7}Bq_3\,y^4 + \tfrac{42q_2q_4-10B^2q_4-21q_3^2}{105\,C}\,z^4,\\ u_5 &= (\tfrac{1}{35}q_2^2 - \tfrac{2}{35}q_1q_3 - \tfrac{1}{147}B^2q_2)y^5; \end{aligned}$$

if both coefficients of u_3 vanish, then $u = yz^2 - \frac{q_4}{30C}z^4 - \frac{B^2}{1764}y^5$ is a solution. If $C=0$, then $u = yz^2 - \frac{B^2}{1764}y^5$ is a solution. In all cases the cubic part restricted to $x=0$ is yz times a nonzero linear form, not a cube, and Corollary 4.6 applies.

Case (b). If $C=0$, then $u = yz^2 - \frac{q_2}{420}y^5$ is a solution. If $C\neq 0$, the cubic $u_3 = 60q_5\,yz^2 - (12q_4 - \frac{8}{35}C^2)\,z^3$ extends to a solution with $u_4 = -\frac{4}{7}Cq_5\,y^4 + \kappa\,y^3z$ and $u_5 = c\,y^5$, where $35C\kappa$ and $7350Cc$ are polynomials in $B, C, q_2, \dots, q_5$; if $12q_4 = \frac{8}{35}C^2$, the cubic $u_3 = yz^2$ extends to a solution with $u_4 = -\frac{C}{105}y^4 + \frac{2BC-21q_3}{84C}y^3z$ and $u_5 = c\,y^5$, $17640Cc = -10B^2C - 42Cq_2 + 105Bq_3$. The restriction of u_3 to $x=0$ is not a cube when $q_5\neq 0$ or $12q_4 = \frac{8}{35}C^2$, and in these cases Corollary 4.6 applies. There remains $C\neq 0$, $q_5 = 0$, $12q_4 \neq \frac{8}{35}C^2$. Then $u_3 = z^3 + xy^2$ extends to the solution with $u_4 = -\frac{A}{42}y^4 - \frac{\beta+4q_4}{8C}y^3z$ and $u_5 = c\,y^5$, $1680Cc = -8C\alpha + 5B\beta - 4Cq_3 + 20Bq_4$. As in part (3), γ_f has no term y^3, so if γ_f is a cube, $\gamma_f = az^3$, and Theorem 4.5, with y and z exchanged, applies to u: the cubic part of G_t is $(a+t)z^3$, and its coefficient of y^4 is $c_0 + c_1t - \frac{t^2}{4}$, because $\partial_x u(0,y,z)$ has quadratic part y^2; so

case (2) holds for all but finitely many t. If γ_f is not a cube, Theorem 2.9(2) applies. □

Suppose now that $I_2^* = \mathbb{C}x^2$ and $I_3^* = xS_2$, as in the binary case with $H_A(3) = 4$; that is, $D_3 = \bar{T}_3$ and $D_k \subseteq \bar{T}_k$ for $k \geq 3$. Then every two elements of I with leading form x^2 have the same transverse cubic γ_f. Since D_2 is spanned by all quadratic monomials except X^2, every element of M with top form $v \in \bar{T}_3$ is $C(v) = v + \chi(v)X^2$ modulo $T_{\leq 1}$, for a linear form χ on $\bar{T}_3$; we write $\chi(v) = \langle r, v\rangle$ with $r \in \bar{S}_3$. Pairing an element $f = x^2 + f_3 + \dots$ of I with $C(v)$ gives $2\chi(v) + \langle f_3, v\rangle = 0$, so $\gamma_f = -2r$.

Theorem F.2. *Every local algebra with Hilbert function* $(1, 3, 5, 4, 3, 2)$ *is smoothable.*

Proof. We may assume $I_2^* = \mathbb{C}x^2$ and, by the remark after Corollary 4.8, the binary case: $I_3^* = xS_2$, $D_3 = \bar{T}_3$, $W = D_4 \subseteq \bar{T}_4$ of dimension three and $V = D_5 \subseteq \bar{T}_5$ of dimension two, with $S_1 \circ V \subseteq W$. Let $P \subseteq \bar{S}_4$ and $L \subseteq \bar{S}_5$ be their orthogonal complements, of dimensions two and four; the condition says $\bar{S}_1 P \subseteq L$. If $\gamma_f \neq 0$, Lemma F.3 below applies; so let $\gamma_f = 0$, that is, $\chi = 0$.

The tails. We show that the algebras with given (P, L) and $\chi = 0$ form an affine space of dimension 20. As in Section 5.18, write the lifts of $w \in W$ and $v \in V$ with tails in the monomials divisible by X (and, in degree four, in a complement of W in $\bar{T}_4$):

$$Q(w) = w + X\alpha_2(w) + X^2\alpha_1(w) + \alpha_0(w)X^3 + c(w)X^2,$$
$$\Phi(v) = v + \eta(v) + X\,T(v) + X^2U(v) + \dots,$$

with $\alpha_k(w) \in \bar{T}_k$, $\eta(v)$ in the complement of W, $T(v) \in \bar{T}_3$, and so on. Differentiating and reducing as in Section 5.18: $x \circ Q(w)$ forces $\alpha_0 = 0$; $y \circ Q(w)$ and $z \circ Q(w)$ determine $\alpha_1(w) = \chi(y \circ w)Y + \chi(z \circ w)Z$; $x \circ \Phi(v)$ forces the quartic tail of $\Phi(v)$ to be $\eta(v) + XT(v)$ and determines the coefficient of X^3 in its cubic tail. The cubic part of $y \circ \Phi(v) - Q(y \circ v)$ must lie in $\bar{T}_3$; its X-part gives $y \circ T(v) = \alpha_2(y \circ v)$ and its X^2-part gives $\alpha_1(y \circ v) = 0$, that is, $\chi(y^2 \circ v) = \chi(yz \circ v) = 0$; similarly for z. Finally the quadratic parts of $y \circ \Phi(v)$ and $z \circ \Phi(v)$ determine the terms X^2Y and X^2Z of the cubic tail of $\Phi(v)$. So the conditions are

$$\chi(\bar{S}_2 \circ V) = 0, \qquad \ell \circ T(v) = \alpha_2(\ell \circ v) \quad (v \in V,\ \ell \in \{y, z\}),$$

together with formulas for the determined coefficients, and the free data are c (3), η (4), the coefficients of $X\bar{T}_2$ in the cubic tails of Φ (6), and of X^2 in their quadratic tails (2). Here $\chi = 0$, so the first condition is empty. The second is a system of 12 linear equations in the $9 + 8$ coefficients of α_2 and T, and it has rank 12 for every (P, L): a functional vanishing on its image is given by quadrics $q_{v,\ell} \in \bar{S}_2$ with $yq_{v,y} + zq_{v,z} = 0$, so $q_{v,y} = zs_v$, $q_{v,z} = -ys_v$ with $s_v = a_v y + b_v z$, and with $\sum_{v,\ell}(\ell \circ v) \otimes q_{v,\ell} = 0$; collecting the coefficients of y^2, yz, z^2 shows that $v_a = \sum a_v v$ and $v_b = \sum b_v v$ satisfy

$z \circ v_a = 0$, $y \circ v_b = 0$ and $y \circ v_a = z \circ v_b$, which forces $v_a = v_b = 0$. So the solutions form a space of dimension $17 - 12 = 5$, and the total dimension is $5 + 3 + 4 + 6 + 2 = 20$ (script `1135432.sage`).

Families. The pairs (P, L) form an irreducible variety $\mathcal{G}$ of dimension six: they are cut out of $\mathrm{Gr}(2, \bar{S}_4) \times \mathrm{Gr}(4, \bar{S}_5)$, of dimension 14, by the 8 conditions $\bar{S}_1 P \subseteq L$, and the pairs where $P = h\bar{S}_1$ for a cubic h form a family of dimension five, the others the open subset of $\mathrm{Gr}(2, \bar{S}_4)$ where $\bar{S}_1 P$ has dimension four; Lemma 2.17 applies. With the tails, the algebras with $\gamma_f = 0$ form an irreducible family over $\mathcal{G}$, a vector bundle of rank 20, in which the pencils P without common factor are dense. For them, two general quartics of P are coprime, and Theorem 3.6 applies with $k_1 = k_2 = 4$, as $16 < 18$. Proposition 2.2(1) concludes. □

Lemma F.3. *Let $A = S/I$ be a local algebra with $I_2^* = \mathbb{C}x^2$, $I_3^* = xS_2$ and $\mathfrak{m}^6 \subseteq I$. Then $\gamma_f \circ D_5 = 0$. In particular, if $\gamma_f \neq 0$, then A is smoothable.*

Proof. Let $F \in D_5$, with lift $\Phi = F + Q + \ldots$, $Q \in T_4$. As $x \circ F = 0$, the element $x \circ \Phi$ has degree at most three, so $x \circ Q \in D_3 = \bar{T}_3$, and Q has no monomial divisible by X^2. For $q \in \bar{S}_2$, the element $q \circ \Phi - C(q \circ F)$ of M has degree at most two, so its quadratic part lies in D_2, which has no term X^2. The quadratic part of $q \circ \Phi$ is $q \circ Q$, without X^2; so $\chi(q \circ F) = 0$, that is, $\langle rq, F\rangle = 0$ for all $q \in \bar{S}_2$, and $r \circ F = 0$. If $\gamma_f \neq 0$, the cubic r annihilates D_5, so it lies in $(I : \mathfrak{m}^2)$, as $\mathfrak{m}^6 \subseteq I$, and Corollary 4.8 applies. □

Lemma F.4. *Let $A = S/I$ be a local algebra with $I_2^* = \mathbb{C}x^2$, and let $f \in I$ have leading form x^2 and transverse cubic $\gamma_f = \alpha y^3$, $\alpha \neq 0$, for linear coordinates y, z on the plane $x = 0$. If $(I : \mathfrak{m}^2) \cap \mathfrak{m}^3$ contains an element u with cubic part xq, where $q \in S_2$ and $q(0, 0, 1) \neq 0$, then A is smoothable.*

Proof. We apply Theorem 4.5. In the formal coordinates of the theorem, $u(0, y, z)$ has order at least four, and $\partial_x u(0, y, z)$ is $q(0, y, z)$ plus terms of order at least three. So G_t has cubic part αy^3 and coefficient of z^4 equal to $c_0 + c_1 t - \frac{t^2}{4} q(0, 0, 1)^2$ for some $c_0, c_1 \in \mathbb{C}$, a nonzero polynomial in t, and case (2) holds for all but finitely many t. □

This is the mechanism of Theorem 4.12, with u only in $(I : \mathfrak{m}^2)$. For example, if $\mathfrak{m}^6 \subseteq I$, $x \circ D_5 = 0$ and some f has a transverse cubic $\gamma_f \neq 0$, then A is smoothable: if γ_f is not a cube, by Theorem 2.9(2), and if $\gamma_f = \alpha y^3$, by the lemma with $u = xz^2$, which lies in $(I : \mathfrak{m}^2)$ because the quadratic part of $u \circ \Phi$ is $z^2 \circ (x \circ F) = 0$ for every lift Φ of $F \in D_5$.

Lemma F.5. *Let $A = S/I$ be a local algebra with $I_2^* = \mathbb{C}x^2$, $I_3^* = xS_2$, socle degree six and $D_6 = \mathbb{C}F$. Write the lifts of $w \in D_4$ in the normal form of the proof of Theorem* F.2*, $Q(w) = w + X\alpha_2(w) + \ldots$ with $\alpha_2(w) \in \bar{T}_2$, and the lift of F as $\Phi = F + R +$ (terms of degree at most four) with $R \in T_5$. Then $G = x \circ R$ lies in D_4, $R = R^b + XG$ with $R^b \in \bar{T}_5$, and $q \circ G = \alpha_2(q \circ F)$ for all $q \in \bar{S}_2$.*

Proof. As $x \circ F = 0$, the element $x \circ \Phi$ has degree at most four, so its quartic part $G = x \circ R$ lies in $D_4 \subseteq \bar{T}_4$, and $R = R^b + XG$. Let $\Psi(v) = v + \eta(v) + XT(v) + \dots$ be the lifts of $v \in D_5$ in the same normal form, with $\eta(v) \in \bar{T}_4$ and $T(v) \in \bar{T}_3$. For $l \in \{y, z\}$, the element $l \circ \Phi - \Psi(l \circ F)$ of M has degree at most four, so its quartic part $l \circ R^b - \eta(l \circ F) + X\big(l \circ G - T(l \circ F)\big)$ lies in $D_4 \subseteq \bar{T}_4$, and $l \circ G = T(l \circ F)$. The closedness of $\Psi(v)$ gives $l' \circ T(v) = \alpha_2(l' \circ v)$ for $l' \in \{y, z\}$, as in the proof of Theorem F.2. Hence $l'l \circ G = \alpha_2(l'l \circ F)$. □

Theorem F.6. *Every local algebra with Hilbert function* $(1, 3, 5, 4, 4, 2)$ *is smoothable.*

Proof. If the transverse cubic is nonzero, the top forms give a turning family; otherwise the algebras form one irreducible family, and one member is linked to a scheme of length 13. We may assume $I_2^* = \mathbb{C}x^2$ and, by the remark after Corollary 4.8, the binary case; as $H_A(3) = 4$, $I_3^* = xS_2$. So $D_4 = g^\perp \cap \bar{T}_4$ for a binary quartic g, and $L = D_5^\perp \cap \bar{S}_5$ is a four-dimensional space containing $\bar{S}_1 g$. The pairs (g, L) form a $\mathrm{Gr}(2, 4)$-bundle over $\mathbb{P}(\bar{S}_4)$, smooth and irreducible of dimension eight. If $\gamma_f \neq 0$, Lemma F.3 applies.

Let $\gamma_f = 0$, that is, $\chi = 0$. We use the normal form of the lifts of the proof of Theorem F.2, with $W = D_4$ of dimension four instead of three: as there, the conditions are $\chi(\bar{S}_2 \circ D_5) = 0$, automatic here, and the 12 linear equations $l \circ T(v) = \alpha_2(l \circ v)$ in the $4 \cdot 3 + 2 \cdot 4 = 20$ coefficients of α_2 and T, together with formulas for the determined coefficients. The dual argument given there shows that these equations have rank 12 for every (g, L); it uses only $D_5 \subseteq \bar{T}_5$. The free data are c (4), η (2), the coefficients of $X\bar{T}_2$ in the cubic tails of the lifts of D_5 (6) and of X^2 in their quadratic tails (2), and the 8 solutions of the equations: in total 22. So the algebras with $\gamma_f = 0$ form a vector bundle of rank 22 over the irreducible variety of pairs (g, L) (script `l19_135442.sage` confirms the rank on nine types of (g, L)). One member is the algebra with ideal

$$I_0 = (x^2 + y^4 + z^4,\ xy^2,\ xyz,\ xz^2,\ y^4 - z^4,\ z^5,\ y^2z^3),$$

whose Hilbert function is $(1, 3, 5, 4, 4, 2)$, with $I_3^* = xS_2$ and $\gamma_f = 0$. Its elements $x^2 + y^4 + z^4$, xy^2, $y^4 - z^4$ are weighted homogeneous of degree four for the weights $(2, 1, 1)$ and have only the origin as common zero, so they define a complete intersection of length $4 \cdot 4 \cdot 4/2 = 32 < 38$, with residual scheme of length 13 (script `l19_135442.sage`). Proposition 2.24 concludes. □

Theorem F.7. *Every local algebra with Hilbert function* $(1, 3, 5, 5, 3, 2)$ *is smoothable.*

Proof. We may assume $I_2^* = \mathbb{C}x^2$ and the binary case: $I_3^* = x^2S_1 + xP_2$ for a pencil $P_2 \subset \bar{S}_2$, so $D_3 = \bar{T}_3 \oplus \mathbb{C}XQ_2$ with $Q_2 \in \bar{T}_2$ spanning $P_2^\perp$. As $I_3^* \subseteq xS_2$, the transverse cubic γ_f does not depend on f, and $D_4 \subseteq D_3' = \bar{T}_4 \oplus (\mathbb{C}XP^3$ if $Q_2 = P^2$ is a square). Let $V = D_5$.

Case I: $x \circ D_4 = 0$. Then $D_4 \subseteq \bar{T}_4$, and $V \subseteq \bar{T}_5$. If $\gamma_f \neq 0$, the example after Lemma F.4 applies. Let $\gamma_f = 0$; then every element of M with top form $b \in \bar{T}_3$ is b modulo $T_{\leq 1}$. As in the proof of Theorem F.2, write the lifts of XQ_2, of $w \in W = D_4$ and of $v \in V$ with tails in the monomials divisible by X and in fixed complements: $\Theta = XQ_2 + \mu X^2$,

$$Q(w) = w + X\alpha_2(w) + X^2\alpha_1(w) + \alpha_0(w)X^3 + c(w)X^2, \qquad \alpha_2(w) \in \bar{T}_2',$$

where $\bar{T}_2'$ is a complement of $\mathbb{C}Q_2$ in $\bar{T}_2$, and $\Psi(v) = v + \eta_0(v) + \sum_{i=1}^4 X^i\eta_i(v) + X\rho_1(v) + X^2\rho_2(v) + \rho_3(v)X^3 + \rho_0(v)X^2$ with $\eta_0(v)$ in a complement of W in $\bar{T}_4$, $\eta_i(v) \in \bar{T}_{4-i}$, $\rho_1(v) \in \bar{T}_2'$, $\rho_2(v) \in \bar{T}_1$. The derivatives give, in turn: $\alpha_0 = 0$ and $\alpha_1 = 0$ (from $x \circ Q(w)$ and $l \circ Q(w) - C(l \circ w)$); $\eta_3 = \eta_4 = 0$ and $\eta_2(v) = \nu(v)Q_2$ (from $x \circ \Psi(v)$); and, for $l \in \{y, z\}$, the cubic part of $l \circ \Psi(v) - Q(l \circ v)$ is $l \circ \eta_0(v) + X\big(l \circ \eta_1(v) - \alpha_2(l \circ v)\big) + \nu(v)X^2(l \circ Q_2)$, which lies in D_3 if and only if $\nu = 0$ and

$$l \circ \eta_1(v) - \alpha_2(l \circ v) = s_l(v)\, Q_2 \tag{18}$$

for scalars $s_l(v)$; then the coefficient of X^2 of the quadratic part gives $l \circ \rho_2(v) - c(l \circ v) - \mu s_l(v) = 0$, and the quadratic part of $x \circ \Psi(v)$ gives $\rho_3 = 0$. There are no other conditions. For fixed graded data and μ these are $12 + 4$ homogeneous linear equations in 35 unknowns, and they have rank 16. Indeed, $\rho_2(v)$ occurs only in the last four equations, which are therefore independent of the others; and a functional vanishing on the image of (18) is given by quadrics $q_{v,l} \in \bar{S}_2$ with $yq_{v,y} + zq_{v,z} = 0$, $\langle q_{v,l}, Q_2\rangle = 0$ and $\sum_{v,l}(l \circ v) \otimes q_{v,l} = 0$, and the argument of the proof of Theorem F.2 shows that all $q_{v,l}$ vanish. So the algebras of Case I with $\gamma_f = 0$ form a vector bundle of rank 19 over $\mathbb{P}(\bar{T}_2) \times \mathcal{G} \times \mathbb{A}^1_\mu$, where $\mathcal{G}$ is the irreducible variety of pairs (P, L) of the proof of Theorem F.2, with $P = D_4^\perp \cap \bar{S}_4$ and $L = V^\perp \cap \bar{S}_5$. On the dense open subset where the pencil P has no common factor, Theorem 3.6 applies with $k_1 = k_2 = 4$, as $16 < 19$, and Proposition 2.2(1) concludes.

Case II: $x \circ D_4 \neq 0$. Then $Q_2 = Y^2$ in suitable coordinates, and $D_4 = W_b \oplus \mathbb{C}(XY^3 + G_0)$ with $W_b = D_4 \cap \bar{T}_4$ of dimension two and $G_0 \in \bar{T}_4$. Writing $F \in V$ as $\sum_i X^i F_i$ with F_i binary and following the X-parts of $x \circ F$, $y \circ F$ and $z \circ F$ in D_4, one finds $F = F_0 + cXY^4$, with $Y^4 \in W_b$ if $c \neq 0$. As $\mathfrak{m}^6 \subseteq I$, every cubic u_3 with $u_3 \circ V = 0$ lies in $(I : \mathfrak{m}^2)$, and if moreover its restriction to $x = 0$ is nonzero, Corollary 4.8 applies. If $x \circ V = 0$, then $S_1 \circ V \subseteq W_b$, and Lemma 2.16 gives $W_b = \nu_4(\Gamma)$ and $V = \nu_5(\Gamma)$ with $I_\Gamma = (x, h)$; then $u_3 = hl$, $l \in \bar{S}_1 \setminus 0$, works. Otherwise $V = \langle F, F'\rangle$ with $F = F_0 + XY^4$ and F' binary; then $y \circ F_0 - 4G_0$ and $z \circ F_0$ lie in W_b. For $u_3 = p + xq'$ with $p \in \bar{S}_3$ and $q' \in S_2$, we have $u_3 \circ F' = p \circ F'$ and $u_3 \circ F = p \circ F_0 + 12\, q'[y^2]\, Y^2 + X(p \circ Y^4)$, so $u_3 \circ V = 0$ if and only if $p \circ F' = 0$, $p \circ Y^4 = 0$ and $p \circ F_0 \in \mathbb{C}Y^2$ (then choose $q'[y^2]$). If $F' \in \mathbb{C}Y^5$, the first condition follows from the second, and the others are three linear conditions on $p \in \bar{S}_3$, so some $p \neq 0$ satisfies them. Otherwise the space $\{H \in \bar{T}_5 : \bar{S}_1 \circ H \subseteq W_b\}$ contains Y^5 and F', so by Lemma 2.16 $W_b = \nu_4(\Gamma)$

with $I_\Gamma = (x, h)$, $h([Y]) = 0$, and $F' \in \nu_5(\Gamma)$. Take $p = hl$: then $p \circ F' = 0$, $p \circ Y^4 = 0$, and for $q = zm$, $m \in \bar{S}_1$, $\langle q, p \circ F_0\rangle = \langle mhl, z \circ F_0\rangle = 0$, as $mhl \in h\bar{S}_2$ and $z \circ F_0 \in \nu_4(\Gamma)$; so $p \circ F_0 \in (z\bar{S}_1)^\perp \cap \bar{T}_2 = \mathbb{C}Y^2$. □

Script `119_135532.sage` computes, as a check, the dimension $20 = 19 + 1$ of the fibres of Case I with $\gamma_f = 0$ over thirty types of graded data, and `119_135532_mu.sage` the rank in the variable μ; `119_135532_case2.sage` computes the cubics annihilating D_5 in examples of Case II.

Theorem F.8. *Every local algebra with Hilbert function* $(1, 3, 5, 4, 3, 2, 1)$ *is smoothable.*

Proof. We may assume $I_2^* = \mathbb{C}x^2$ and $I_3^* = xS_2$, so that $D_k \subseteq \bar{T}_k$ for $k \geq 3$. Let $D_6 = \mathbb{C}F$, $V = D_5$, $W = D_4$, $P = W^\perp \cap \bar{S}_4$, a pencil, and $\Phi = F + R^b + XG + \dots$ the lift of F, as in Lemma F.5.

$\gamma_f \neq 0$. Since $xz^2 \in I_3^*$, the ideal I contains an element u with leading form xz^2, and $u \in (I : \mathfrak{m}^2)$. If γ_f is not a cube, Theorem 2.9(2) applies, and if $\gamma_f = \alpha y^3$, Lemma F.4 applies with $q = z^2$. From now on $\gamma_f = 0$.

P without common factor. Theorem 3.6 applies with $k_1 = k_2 = 4$.

A criterion. Let $p \in \bar{S}_3$ and $q' \in S_2$. For $u_4 \in S_4$, the element $u = p + xq' + u_4$ lies in $(I : \mathfrak{m}^2)$ if and only if $u \circ M \subseteq T_{\leq 1}$, that is, if and only if the quadratic parts of $u \circ \Psi$, for the lifts Ψ of V, and the cubic and quadratic parts of $u \circ \Phi$ vanish. The quadratic part of $u \circ \Psi$ is $p \circ v$, where $v \in V$ is the top form of Ψ; the cubic part of $u \circ \Phi$ is $p \circ F$, and its quadratic part is $X(p \circ G) + p \circ R^b + q' \circ G + u_4 \circ F$. As $V \supseteq \bar{S}_1 \circ F$, the condition $p \circ V = 0$ implies $p \circ F = 0$. So $u \in (I : \mathfrak{m}^2)$ if and only if (1) $p \circ V = 0$, (2) $p \circ G = 0$, and

$$u_4 \circ F = -(p \circ R^b + q' \circ G).$$

As $u_4 \circ F$ runs through $\bar{S}_4 \circ F$ when u_4 runs through S_4, there exists $u_4 \in S_4$ with $p + xq' + u_4 \in (I : \mathfrak{m}^2)$ if and only if (1), (2) and (3) $p \circ R^b + q' \circ G \in \bar{S}_4 \circ F$ hold. Pairing the elements $xw + e_4 + \dots$ of I, $w \in S_2$, with Φ gives $w \circ G \in \bar{S}_4 \circ F$, so q' plays no role in (3). If $p \neq 0$ satisfies (1)–(3), then $(I : \mathfrak{m}^2)$ contains such an element u, and Corollary 4.8 applies.

(A) $\dim \bar{S}_2 \circ F = 3$. Then $V = \bar{S}_1 \circ F$, $W = \bar{S}_2 \circ F$ and $P = \mathrm{Ann}(F)_4$. The annihilator of F in $\mathbb{C}[y, z]$ is a complete intersection of degrees $a + b = 8$ with $a \geq 3$. If $a = b = 4$, the pencil P has no common factor. If $a = 3$, let p span $\mathrm{Ann}(F)_3$; then (1) holds, (2) holds as $G \in W = \bar{S}_2 \circ F$, and (3) is empty, as $\bar{S}_4 \circ F$ has the dimension of $\bar{S}_2 \circ F$, so it is $\bar{T}_2$.

(B) $\dim \bar{S}_2 \circ F = 2$, *$F$ not a sixth power.* Then $F \in \nu_6(\Gamma)$ for $\Gamma = V(x, g)$ of length two, $V = \nu_5(\Gamma)$ and $W = \nu_4(\Gamma) \oplus \mathbb{C}\omega$. By Lemma F.5, $g \circ G = \alpha_2(g \circ F) = 0$, so $G \in \nu_4(\Gamma)$. Take $p = gl$ with $l \in \bar{S}_1$: (1) and (2) hold, and as $\bar{S}_4 \circ F = \nu_2(\Gamma) = g^\perp \cap \bar{T}_2$, (3) is the single linear condition $\langle g^2 l, R^b\rangle = 0$ on l, which has a solution $l \neq 0$.

(C) $F = Y^6$. Then $Y^5 \in V$, $Y^4 \in W$, and the quartics of P vanish at $[Y]$. The element $z \circ \Phi$ has degree at most four, so $z \circ R^b + Xz \circ G \in W \subseteq \bar{T}_4$; hence $z \circ G = 0$, $G \in \mathbb{C}Y^4$, and $z \circ R^b \in W$. If $P = h\bar{S}_1$ for a cubic h, then

$h([Y]) = 0$, and $p = h$ satisfies (1), as $V^\perp \supseteq \bar{S}_1 P = h\bar{S}_2$, (2), as $h \circ Y^4 = 0$, and (3): for $q = zm$, $m \in \bar{S}_1$, $\langle q, h \circ R^b \rangle = \langle mh, z \circ R^b \rangle = 0$, as $mh \in P$, so $h \circ R^b \in \mathbb{C}Y^2 = \bar{S}_4 \circ Y^6$.

Otherwise $\dim \bar{S}_1 P = 4$, so $V^\perp \cap \bar{S}_5 = \bar{S}_1 P$ and V are determined by P, and the graded data are the pairs $([Y], P)$ with $[Y]$ on the line $x = 0$ and P a pencil of quartics vanishing at $[Y]$, other than the pencils $h\bar{S}_1$: an open subset of a $\mathrm{Gr}(2,4)$-bundle over $\mathbb{P}^1$, irreducible of dimension five. In the normal form of Theorem F.2 with $\chi = 0$, the lifts of W and V form a vector bundle of rank 20: there are 36 unknowns (α_2: 9, c: 3, η: 4, T: 8, and the coefficients K_1, K_2, k of $X\bar{T}_2$, $X^2\bar{T}_1$ and X^2 in the lower tails of the lifts of V: $6 + 4 + 2$), the equations $l \circ T(v) = \alpha_2(l \circ v)$ have rank 12 by the dual argument, and the equations relating K_2 and c have rank four. Going through the derivatives of the lift of Y^6 degree by degree, its tails are determined by these data and by six free coefficients, and the data satisfy three further equations, one of them $c(Y^4) = \frac{3}{100}e^2$, where $T(Y^5) = eY^3$. Over every base point, the solutions form the graph of a polynomial map on an affine space of dimension $17 + 6 = 23$; these affine spaces form an affine bundle over the base (Lemma 5.16), and the algebras of this stratum form an irreducible family. Script `l19_1354321_param.sage` eliminates all tails, in its own order, with $[Y] = [0 : 1 : 0]$ fixed (every point of the line $x = 0$ is equivalent to it under GL_2 acting on y, z) and the eight affine coordinates of $(W/\mathbb{C}Y^4, V/\mathbb{C}Y^5)$ as indeterminates, on the thirty standard charts of $\mathrm{Gr}(2,4) \times \mathbb{P}^4$, which cover the base: in each chart all pivots are nonzero constants, except for at most three linear equations whose rank is shown to be constant on the base by a saturation; script `l19_1354321_fibres.sage` computes the fibres at three points. Its member with ideal

$$I_1 = (x^2,\ xy^2,\ xyz,\ xz^2,\ y^3z + z^4,\ y^2z^2,\ y^7),$$

which has Hilbert function $(1, 3, 5, 4, 3, 2, 1)$, $D_6 = \mathbb{C}Y^6$ and $P = z\langle y^3 + z^3, y^2z\rangle$, lies in the complete intersection $V(x^2+y^7+z^6,\ xy^2+y^2z^2,\ y^3z+z^4)$, whose local length at the origin is $35 < 38$, as $\dim_{\mathbb{C}} S/(C + \mathfrak{m}^N) = 35$ for $N = 12, 13$, where C is its ideal; the residual scheme has length 16 (script `l19_1354321_anchor.sage`). Proposition 2.24 concludes. □

Appendix G. Very compressed algebras

G.1. In this appendix we consider the local algebras S/I with $\mathfrak{m}^4 \subseteq I \subseteq \mathfrak{m}^3$. Their Hilbert function is $(1, 3, 6, c)$ with $0 \le c \le 10$, and $I = \mathcal{C} + \mathfrak{m}^4$ for the subspace $\mathcal{C} = I \cap S_3$ of codimension c in S_3: the cubic part of an element of I lies in I, as its other terms lie in $\mathfrak{m}^4$. Conversely, every $\mathcal{C} \in \mathrm{Gr}(10 - c, S_3)$ gives such an algebra, with inverse system $T_{\le 2} \oplus \mathcal{C}^\perp$, and these form a family over the Grassmannian, by Lemma 2.3(2) applied to local frames of the tautological bundle with fibers $\mathcal{C}^\perp \subseteq T_3$. These are exactly the very compressed algebras of lengths ten to twenty in the sense of [7] (for $c = 0$, $I = \mathfrak{m}^3$), and Theorem G.4 below is a special case of [7, Proposition 1.2]. We give an explicit form of the proof of [7, Section 5].

For a finite set $\Xi \subset \mathbb{A}^3$ let $\mathcal{U}(\Xi) \subseteq S_{\leq 3}$ be the space of polynomials of degree at most three vanishing on Ξ, and $\mathcal{C}(\Xi) \subseteq S_3$ the space of their cubic parts.

Lemma G.2. *Let $0 \leq c \leq 10$ and let $\Xi \subset \mathbb{A}^3$ be a set of $10+c$ points on which no nonzero polynomial of degree at most two vanishes and which imposes independent conditions on polynomials of degree at most three. Then $\mathcal{C}(\Xi)$ has codimension c in S_3, and $S/(\mathcal{C}(\Xi)+\mathfrak{m}^4)$ is smoothable.*

Proof. The space $\mathcal{U}(\Xi)$ has dimension $20-(10+c) = 10-c$ and contains no polynomial of degree at most two, so taking cubic parts is injective on it. Similarly, the polynomials of degree at most four vanishing on Ξ form a space of dimension at least $35-(10+c)$, and modulo $\mathcal{U}(\Xi)$ their quartic parts fill a space of dimension at least 15, that is, all of S_4.

For $t \neq 0$ let $t\Xi = \{tp : p \in \Xi\}$. The subschemes $t\Xi$ form a flat family over $\mathbb{A}^1 \setminus \{0\}$ inside $\mathbb{P}^3$, which extends to a flat family over $\mathbb{A}^1$ [13, Proposition III.9.8]; its fiber Z_0 over 0 has length $10+c$ and is supported at the origin, the limit of all points tp. If f vanishes on Ξ and has degree e, then $t^e f(x/t, y/t, z/t)$ vanishes on $t\Xi$ and, at $t=0$, equals the top-degree part of f. So the ideal of Z_0 contains $\mathcal{C}(\Xi)$ and S_4. As $\mathcal{C}(\Xi)+\mathfrak{m}^4$ has colength $10+c$, it is the ideal of Z_0, which is a limit of reduced schemes. □

So the scheme defined by $\mathcal{C}(\Xi)+\mathfrak{m}^4$ is the limit of the configuration Ξ shrinking to the origin. This is the construction of [7, Section 5]. It remains to see that these limits are dense, that is, that the map $\Xi \mapsto \mathcal{C}(\Xi)$ from configurations to the Grassmannian is dominant. In characteristic zero it suffices to find one configuration at which its differential is surjective: then, by the inverse function theorem, the image contains an open subset of $\mathrm{Gr}(10-c, S_3)$ in the Euclidean topology, which is Zariski dense. In [7, Lemma 5.2] the surjectivity at some point is verified by a computer calculation. The following lemma makes the differential explicit.

Lemma G.3. *Let $\Xi = (p_1, \ldots, p_n)$, $n = 10+c$, be as in Lemma* G.2*, and let*

$$\mathcal{R}(\Xi) = \Big\{ w \in \mathbb{C}^n : \sum_{i=1}^{n} w_i h(p_i) = 0 \text{ for all } h \in S_{\leq 2} \Big\},$$

a space of dimension c. The differential of $\Xi \mapsto \mathcal{C}(\Xi)$ at Ξ is surjective if and only if the linear map

$$\mathcal{U}(\Xi) \otimes \mathcal{R}(\Xi) \to (\mathbb{C}^3)^n, \qquad f \otimes w \mapsto \big(w_i \nabla f(p_i)\big)_{i=1,\ldots,n},$$

is injective.

Proof. Move the points to $p_i + \epsilon v_i$. A polynomial $f \in \mathcal{U}(\Xi)$ deforms to $f + \epsilon g$ with $(f+\epsilon g)(p_i+\epsilon v_i) = O(\epsilon^2)$, that is, $g(p_i) = -\nabla f(p_i) \cdot v_i$; such $g \in S_{\leq 3}$ exists and is unique modulo $\mathcal{U}(\Xi)$. The differential sends (v_i) to the homomorphism $\mathcal{C}(\Xi) \to S_3/\mathcal{C}(\Xi)$ mapping the cubic part of f to the cubic part of g, in the tangent space $\mathrm{Hom}(\mathcal{C}(\Xi), S_3/\mathcal{C}(\Xi))$ of the Grassmannian. The map $S_{\leq 3} \to S_3/\mathcal{C}(\Xi)$ taking cubic parts vanishes on $S_{\leq 2}$ and $\mathcal{U}(\Xi)$, so

it factors through the evaluation at Ξ, and $(S_3/\mathcal{C}(\Xi))^\vee$ is identified with $\mathcal{R}(\Xi)$: the vector w corresponds to the functional $h \mapsto \sum_i w_i h(p_i)$. Pairing the image of (v_i) with $f \otimes w$ therefore gives $\sum_i w_i g(p_i) = -\sum_i w_i \nabla f(p_i) \cdot v_i$. So the dual of the differential is, up to sign, the map in the lemma. □

Theorem G.4 ([7, Proposition 1.2])**.** *Every local algebra S/I with $\mathfrak{m}^4 \subseteq I \subseteq \mathfrak{m}^3$ is smoothable.*

Proof. Let $p_1, \dots, p_{20}$ be the ten lattice points (i, j, k) with $i, j, k \geq 0$ and $i + j + k \leq 2$, in the order

$$(0,0,0),\ (1,0,0),\ (0,1,0),\ (0,0,1),\ (1,1,0),$$
$$(1,0,1),\ (0,1,1),\ (2,0,0),\ (0,2,0),\ (0,0,2),$$

followed by the ten points

$$(1,1,2),\ (1,2,1),\ (2,0,1),\ (0,1,2),\ (0,2,1),$$
$$(2,1,0),\ (0,0,3),\ (0,3,0),\ (1,0,2),\ (3,0,0).$$

For each $0 \leq c \leq 10$, the first $10 + c$ of these points satisfy the hypotheses of Lemma G.2. For $c = 0$ and $c = 10$ the Grassmannian is a point, and we are done. For $1 \leq c \leq 9$ the map of Lemma G.3 has rank $(10-c)c$, the dimension of $\mathrm{Gr}(10 - c, S_3)$. This is an exact computation over $\mathbb{Q}$ with matrices of size at most 25×57 (script `vc_dominance.sage`). Hence $\Xi \mapsto \mathcal{C}(\Xi)$ is dominant, the smoothable algebras are dense in the family over $\mathrm{Gr}(10 - c, S_3)$, and all of them are smoothable by Proposition 2.2(1). □

Remark G.5. Theorem G.4 has been formalized in the proof assistant Lean 4 (theorem `Hilb3.smoothable_veryCompressed_three` [24]) by the same method: for $3 \leq c \leq 10$ by exact certificates for the Jacobian of the chart map, at an explicit configuration of fourteen points, different from the points above, and at configurations obtained from it by removing or adding points, and for $c \leq 2$ through the formalized theorem for lengths at most twelve. In this paper, Theorem G.4 is used for $5 \leq c \leq 9$; for $5 \leq c \leq 7$, Remark 3.10 gives a second proof.

Appendix H. Computations

H.1. All computations are exact, over $\mathbb{Q}$ or over polynomial rings over $\mathbb{Q}$ and their fields of fractions, in SageMath 10.7. The scripts named below, the libraries and data files they load, the files `README.md` and `MANIFEST.md`, the outputs of a reference run, a script `VERIFY.sh` that reruns all computations listed in `runs.txt` and checks their outputs, and a list of checksums form the archive `nineteen_points_scripts.tar.gz`, which accompanies this paper on the author's web page. Its SHA-256 checksum is

`52cb92bf19e13b311633eecad42f990061431b03b5a1a272b35bc6b47d0bbe1c`

The file `MANIFEST.md` records, for each script, the result it serves, what it computes, the kind of certificate, the exact invocations, the expected output, the random seeds and the running time. Most scripts run in seconds and

the longest in a few minutes; `VERIFY.sh` reruns all 89 cited runs in about twenty-two minutes on one processor, and faster with parallel processes. Some witnesses were discovered by random search. The witnesses used in proofs are fixed in the accompanying files (as explicit polynomials, or by a fixed random seed, so that every run reproduces them) and verified by exact calculations; additional randomized tests are diagnostic only. Local lengths of complete intersections are certified by the criterion after Proposition 2.24, that is, by $\dim S/(J+\mathfrak{m}^N) = \dim S/(J+\mathfrak{m}^{N+1})$, or by a local monomial order. Saturations are Gröbner basis computations over $\mathbb{Q}$ in the polynomial ring of the parameters and the unknowns, chart by chart, as explained after Lemma 5.19. In the table, the kinds of certificate are: *identity* (a polynomial identity with all tails as indeterminates), *rank* (an exact rank, on each orbit of a group acting transitively on the stratum, or by elimination with constant pivots), *saturation*, *witness* (an explicit algebra or complete intersection), *check* (a computation confirming a statement proved by hand), and *count* (an enumeration).

result	script	certificate	kind
Tables 1, 2	`enumerate, table_counts`	the $21, 29, 39, 52, 70, 91, 119, 155$ Hilbert functions of lengths 12 to 19; the first row that applies	count
Lemma 4.4	`ade_weights`	the Milnor algebras of y^3+z^4, y^3+yz^3, y^3+z^5	check
Prop. 4.11	`puresextic`	the four coefficients of $u_3 \circ Q_5$	identity
Example 3.9	`linkage15`	the three cubics lie in I; local length 27; the residual ideal and its Hilbert function $(1,3,5,3)$	witness
Example 3.13	`excess, excess_graded`	local lengths $27+e(\mathfrak{B})$ in four examples; 35 for the explicit triple in the graded algebra	check, witness
Section 3.11, cone lemma	`cone_lemma`	the length $\sum_p n_p r_p$ in seven examples, with points of $\mathfrak{B}$ of each type	check
Corollaries 3.14, 3.15	`excess_planeschemes`	ν_3 of the subschemes of length three containing a double point; the bounds for all relevant Hilbert functions; the Schubert cells meeting $\langle X^2Z, XY^2\rangle$	check
Lemma D.3, Cor. D.4	`fat_free_tail, fat_graded`	members of the lines M_λ over fat points, including $F = X^5Z$ with $R = Z^4$, and graded worst cases; local lengths at most 35 (37 in Theorem D.7(b))	check

result	script	certificate	kind
Lemma A.3	`colon_socle`	$\dim \mathcal{K}_m(\Gamma, P) = n + \sigma + 1$ on many types with $n \leq 5$	check
Theorem 5.10	`tails136221, rays15`	ranks 24, 19 and tail dimensions 11, 16 on the four representatives; the parabola identities	rank
Theorem 5.21	`uniform_136c211`	the ray identities; the family M_t closed for all parameters and $\dim z^{-1}(D_{3,t}) = 5 + c$, as proved in the text	identity, check
Lemma 5.14	`l13632111_ray`	the operators x^4b, x^4c and the jet, all tails as indeterminates	identity
Theorem G.4	`vc_dominance`	hypotheses of Lemma G.2; rank $(10 - c)c$ for $1 \leq c \leq 9$	rank
Lemmas A.6, A.9	`pencils13632, kernels13642`	the five Jordan cases; solution dimensions 10 and 16; the explicit pencil in $\mathcal{N}$	identity, check
Prop. B.2, Thm. B.3	`l17_socle4, l18_socle4, l19_13672, l19_13654`	the examples for $c = 5, 6, 7$ (cubics without common zero); examples of case (c1)	witness, check
Theorem E.2	`l136321`	the Gorenstein example; the quartic on the conic	witness
Theorem C.3	`anchor136322`	the anchor M_0, the complete intersection K in its ideal, local lengths 9, 11, 13 and 33	witness
Theorem C.4	`l1363221_rays, l1363221_sat`	the identities over two points; saturation in $8 + 8$ charts	identity, saturation
Theorem C.5	`l136422_strata, l136422_doublepoint, l136422_rays, l136422_anchor`	ranks and dimensions of all strata; the identities of rays (b); local length 33	rank, witness
Theorem D.5	`l136331`	fiber dimensions 12, 16, 17 on the seven orbits	rank
Theorem D.6	`l1363311_dims, l1363311_ray`	dimensions; the ray identity	identity
Theorem E.3	`l136421`	the Gorenstein example; $r(D_4) = 4$ and $h = yz$ for $Q = Y^4 + Z^4 + XY^3 + XZ^3$	witness
Theorem E.5	`l136521`	a member with three reduced base points and local length 30; the examples with $\dim C_0 = 5$	witness
Theorem E.7, Ex. E.6	`l136431_nonpure, l136431_pure, l136431_excess`	the Gorenstein member; three members with local length 32 (pure case); length 31 in the example	witness, check
Theorem F.2	`l135432`	the dimension 20 of the tails with $\gamma_f = 0$, for four types of P	check

result	script	certificate	kind
Lemma 4.2, Cor. 4.9	`activation, socle5`	examples	check
Proposition 3.18, Theorem B.5	`l19_13663`	two members in global complete intersections of length 36, residual 17	check
Theorem C.10	`l19_136522`	facts (i)–(iii) on both forms of Γ; the member M_0, local length 32, residual 13	rank, witness
Theorem D.7	`l19_1364311_incidence`	on eleven types of Γ: $\dim \mathcal{K}_5 = 6$, the rank-one fibres, the rank 18 on the tails, closedness, and the ray after the coordinate change	check
Theorem 3.8, $c = 8$	`l19_13681`	monomial anchor; CI (x^3, y^3, z^4) of length 36, residual 17	witness
Theorem B.4	`l19_13645`	local length 36 of the explicit triple; residual 17	witness
Prop. C.6	`l19_13632211, l19_1363222, l1363221_rays`	the jet identities in cases (a) and (b) (the latter for $\dim D_6 = 2$ and $D_6 = \mathbb{C}X^6$), all tails as indeterminates	identity
Theorem C.7	`l19_1363222`	saturation in the chart $G = X^2Z + \dots$	saturation
Theorem C.8	`l19_13632211, l19_13632211_comp`	factor certificate for $G = XY^2$; the component over $t = 0$ for $G = X^2Z + \dots$ and its straight ray	saturation, rank
Theorem C.9	`l19_1364221_b3x, l19_1364221_nf, l19_1364221_comp`	constant pivots and generic straight rays over two points; normal forms and saturation in the thirteen remaining Schubert cells; components over $t = 0$ with rays or $D_3 \not\subseteq N$	rank, saturation
Theorem E.9	`l19_136441`	local lengths 31–33 for members with finite base	check
Theorem E.10	`l19_136432, l19_136432_ray`	tail dimensions; anchor of local length 36, residual 17; the explicit ray, all data symbolic	witness, identity
Lemma E.8, Theorem E.11	`l19_136531, l19_136531_strata`	four concise anchors, global length 36, residual 17; $\dim \mathcal{K}$ on representatives	witness, check
Lemma A.12, Theorems B.3, E.11	`l19_fivecubics_line`	the example of case (d) of the lemma; rays in case (d) of Theorem B.3 for $k = 3, 4$; $\dim \mathcal{K} = 6 + j$ for $F = X^5$; a member of step (iv) with a ray	check

result	script	certificate	kind
Lemma A.12, Theorems B.3, E.11	`l19_fivecubics_line2`	the criterion between cases (c) and (d); (11) for quartics with multiple roots; the anchor M_0 of step (iv), local length 36, residual 17	check, witness
Theorem E.12	`l19_136621`	anchor of local length 30, residual 11; a member with $Q = Y^4 + Z^4 + XY^3 + XZ^3$ and $\dim C_0 = 6$, with its parabola ray	witness
Prop. 4.11(2)	`l19_sexticplane`, `l19_sextic2pts`, `l1354221`	closedness conditions and the quartics u_4 in the three placements ($H(6) = 2$); the shape of R for $I_\Gamma = (z, x^2)$; the kernel for $I_\Gamma = (x, h)$	identity, check
Prop. 4.11(3)	`l19_septic_h25`	the identities (0)–(4)	identity
Prop. 4.11(4)	`l19_13542211`, `quadrictails4_cert`	the solutions u in all strata, with denominators cleared, and the nonvanishing of the denominators on each branch	identity
Theorem F.6	`l19_135442`	tail dimension 22 on nine types; weighted homogeneous CI of length 32, residual 13	check, witness
Theorem F.7	`l19_135532`, `l19_135532_mu`, `l19_135532_case2`	fibre dimensions 20 on thirty types; fibre dimension 19 for every μ; examples of Case II	check
Theorem F.8	`l19_1354321_param`, `l19_1354321_fibres`, `l19_1354321_anchor`	fibre dimension 23 over the whole base of (C), on thirty open charts; anchor of local length 35, residual 16	rank, saturation, witness

Dimension counts. Table 4 lists the expected-dimension counts (Lemmas 2.17 and 2.18) with the conventions stated after Lemma 2.18: Grassmannians for spaces of top forms, $[F]$ for a projectivized form, "raw" for a tail in T_k and "mod D_k" for a class in T_k/D_k; tails not listed are not part of the count, and each proof treats them over its strata. The last column gives $\dim \mathcal{Q}$, the rank of the bundle, and the lower bound of Lemma 2.17 for the dimension of every component.

Collision computations. In the proofs that use Lemma 5.19, the parameters other than t (for instance the coefficients of an extra cubic) are taken in finitely many locally closed strata, each an affine space, that cover their parameter space (affine charts, the strata where a given coefficient is the first nonzero one, or Schubert cells), and the saturation is checked in each stratum over $\mathbb{Q}$. This suffices: if a component $\mathcal{C}$ of the family lies over $t = 0$, and Σ is the stratum that contains the image of its generic point, then $\mathcal{C}$ meets the part of the family over Σ in a component of that part lying over $t = 0$. Moreover, the family over the t-line need not contain every algebra with the given graded data: it suffices that every algebra over the collision

TABLE 4. Expected-dimension counts and their conventions.

result	data	tails	$\dim \mathcal{Q} - \operatorname{rank} =$ bound
Prop. B.2, $c = 5, 6, 7$	$(D_4, D_3) \in \operatorname{Gr}(2, T_4) \times \operatorname{Gr}(c, T_3)$	none	$51 - 30 = 21$, $50 - 24 = 26$, $47 - 18 = 29$
Thm. 3.8, $c = 8$	$(D_3, [F]) \in \operatorname{Gr}(8, T_3) \times \mathbb{P}(T_4)$	none	$30 - 6 = 24$
Thm. B.5	$(D_3, D_4) \in \operatorname{Gr}(6, T_3) \times \operatorname{Gr}(3, T_4)$	none	$60 - 36 = 24$
Thm. C.5	$\Gamma \in \operatorname{Hilb}_2(\mathbb{P}^2)$, $K \in \operatorname{Gr}(2, 8)$	$R \bmod D_4$, $E \bmod D_3$	$54 - 24 = 30$
Thm. C.10	$\Gamma \in \operatorname{Hilb}_2(\mathbb{P}^2)$, $D_3/\nu_3(\Gamma) \in \operatorname{Gr}(3, 8)$	$R_i \bmod D_4$, $E_i \bmod D_3$	$55 - 20 = 35$
Thm. D.5	$\Gamma \in \operatorname{Hilb}_3(\mathbb{P}^2)$, F framed	$\tau \bmod D_3$, $R \bmod D_4$	$42 - 21 = 21$
Thm. D.7	$\Gamma \ni [X]$, $U \in \mathbb{P}^2$	R_5 raw	$14 - 2 = 12$
Thm. E.3, $r(D_4) = 2$	$(\Gamma, [X])$, $K \in \operatorname{Gr}(2, 8)$	$R \bmod D_4$	$29 - 12 = 17$
Thm. E.7	$\Gamma \in \operatorname{Hilb}_4(\mathbb{P}^2)$, $[u]$, $[F]$	none	$14 - 3 = 11$
Thm. E.10, $r = 3$	(D_4, D_5) for a fixed line	none	$14 - 8 = 6$
Thm. E.11 (i)	$[X]$, $D_4 \ni X^4$, $D_3 \ni X^3$	R raw, τ, $E \bmod D_3$	$76 - 40 = 36$
Thm. E.11 (ii)	$[z]$, $[F]$, $D_3 \supseteq S_2 \circ F$, $[u']$	R raw, $\tau(u')$, $E \bmod D_3$	$54 - 20 = 34$
Thm. E.11 (iii)	$[z]$, Γ, $[F]$, $D_3 \supseteq \nu_3(\Gamma)$, $[u']$	R raw, $\tau(u')$, $E \bmod D_3$	$57 - 20 = 37$
Thm. F.2	$(P, L) \in \operatorname{Gr}(2, \bar{S}_4) \times \operatorname{Gr}(4, \bar{S}_5)$	none	$14 - 8 = 6$

be isomorphic to its member at $t = 0$ for some value of the parameters. So the parameters may be normalized by coordinate changes that are valid only at $t = 0$, and a component lying over $t = 0$ may be treated directly (Theorems C.8 and C.9). The library `stages.sage` generates the equations and performs the eliminations; the saturations are tested by the scripts named in the proofs.

Some scripts take arguments, and some of their runs end with a line `FAIL` by design. The runs used in the text are `l1363221_sat` with `pure` and with `nonpure`; `l19_1363222` with `coll 5` and with `two`; `l19_13632211` with `colls 7`, `colls 5` and `two`; `l19_13632211_comp` with `5`; `l19_1364221_nf` with `p 600` and with `n 600`, followed by the thirteen cells i, k of the proof of Theorem C.9; and `l19_1364221_comp` with `p` i k `mp` for the nine cells (i, k) of that proof whose ideal is not t-saturated. The run `l19_1364221_nf p 600` prints `t-saturated: False` for the nine cells that are then passed to the script `l19_1364221_comp`. Apart from these lines, two kinds of output are expected to report a failure: `l19_13632211 colls 5` reports that the

ideal of the chart $G = X^2Z + \dots$ is not t-saturated, and `l19_1364221_comp p i 5 mp` reports `FAIL` for $i = 0, 1, 2$, because one component in each of these cells has no straight ray. These are exactly the components over $t = 0$ that the proofs treat directly: the component C of Theorem C.8, whose ray is found by `l19_13632211_comp 5`, and the three components of Theorem C.9 on which the script prints `rank N = 16, rank(N + D_3) = 17`. The file `MANIFEST.md` lists the expected output of every run.

The library `invsys.sage` computes inverse systems, Hilbert functions and annihilators; `stages.sage` generates the equations of Section 5.18. The script `l136431_incidence.sage` lists the strata of the incidence of Theorem E.7 for all types of Γ; it confirms the dimension count of the proof, which does not depend on it.

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