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Boundary colouring patterns at separated cuts in flower snarks

One cut copy and two copies joined crosswise

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Abstract

We cut one linking edge at each of b prescribed interfaces of a flower snark, with at least two claws between successive cuts in cyclic order. We characterize the assignments of colours to the resulting semiedges that extend to a proper 3-edge-colouring. A seven-state transfer matrix gives exact counts up to a simultaneous permutation of the colours, both for one copy and for two copies joined crosswise at matching cuts. At five cuts each copy admits 1,175 of the 2,461 parity-admissible patterns, and 750 patterns are compatible across the join.

1. Cuts and boundary patterns

Fix an odd integer $s \geq 5$. Arrange s copies of the claw $K_{1,3}$ in cyclic order and label the leaves of each claw 0, 1 and 2. Join equally labelled leaves of successive claws, interchanging labels 1 and 2 only at the seam from the last claw to the first. This is the flower snark studied here. A proper 3-edge-colouring assigns colours from $\{0, 1, 2\}$ so that the three edges incident with every vertex have distinct colours.

At each of b distinct interfaces cut one linking edge. Each cyclic interval between successive cuts, including the interval through the seam, must traverse at least two claws. Thus $1 \leq b \leq \lfloor s/2 \rfloor$. Write (a, d) for the semiedge colours in order from the preceding claw to the following claw. A **boundary word** is the resulting ordered list of pairs. Fix the starting interface, cyclic direction and leaf labels. A **boundary pattern** is an orbit of these words under one simultaneous permutation of the three colours.

> **Theorem 1.** Let $S(b)$ count the patterns that extend to a colouring of one cut copy. For two copies of the same length cut at the same interfaces, match (a, d) on the first copy to (d, a) on the second. Let $G(b)$ count the patterns that extend on both sides. For every permitted choice of the cut leaves and gaps,

$$S(b) = \frac{6^b - 3 \cdot 3^b + 3}{6} \tag{1}$$

$$G(b) = \frac{\left(3 + \sqrt{6}\right)^b + \left(3 - \sqrt{6}\right)^b - 2^{b+1} - (-1)^b - 3^b}{6} \tag{2}$$

2. Which boundary words extend?

Write u and v for the triples of incoming and outgoing colours at a claw, both indexed by the leaf labels. Let $n_c(u)$ be the multiplicity of colour c in u. The following local relation is the colouring relation of Isaacs's building block [1, 3], with the incoming and outgoing semiedges of a leaf given the same index.

> **Lemma 1.** A pair of triples extends across a claw if and only if $u_j \neq v_j$ at every leaf and $n_c(u) + n_c(v) = 2$ for every colour c.

Proof. Each spoke must receive the colour unused on the two linking edges at its leaf. Exactly $3 - n_c(u) - n_c(v)$ spokes then receive colour c. The three spoke colours are distinct precisely when this number is one for each colour. □

When all three colours occur, their cyclic order is preserved, and either of the other two rotations is possible after one claw. With two colours, the singleton colour persists and moves to either other leaf; the repeated and absent colours exchange roles. A walk on three positions that changes position at every step can reach any prescribed position in two or more steps. This gives the interval reduction used below; see also [2, 3].

Let t count the claws traversed from the seam. Immediately before and after a cut, normalize the colour counts as follows, where e_c is the unit vector for colour c.

$$m_c = (-1)^t (n_c - 1), m \in \{0\} \cup \{e_i - e_j : i \neq j\} \tag{3}$$

Crossing an ordinary claw replaces n_c by $2 - n_c$, leaving m unchanged. A cut (a, d) adds $\Delta = (-1)^t(e_d - e_a)$. The odd circuit length requires the final state to be the negative of the initial one. Thus the word fixes the initial state uniquely as $-\sum\Delta/2$.

> **Lemma 2.** A word with named colours extends if and only if the initial state and every updated state belong to (3), colour a is present immediately before each cut (a, d), and at least one pair has $a \neq d$.

Proof. If every pair has equal colours, we can rejoin the cut edges. The uncut flower cannot close: an all-distinct triple keeps its cyclic order until the seam reverses it, while a two-colour triple changes its repeated colour an odd number of times. The other conditions follow from the count update.

Conversely, an unequal pair forces a two-colour state somewhere on the circuit. In a two-colour segment place the required outgoing colour at the cut leaf. Lemma 1 lets us meet the next cut at the required singleton position. At a return to three colours the outgoing colour was repeated; either of the other two leaves can carry the singleton, so either cyclic order can be chosen. Within a three-colour segment a rotation suffices to put the required colour at its cut leaf, including at the cut that leaves this segment. For a segment through the seam reflect its endpoint labels before joining. If the entire circuit has two-colour states, only singleton positions need to be matched. □

3. Counting the patterns

Arrange the six nonzero states of (3) around a hexagon. Each has two outer neighbours obtained by moving one unit of colour count. For example, (1, −1, 0) is adjacent to (1, 0, −1) and (0, −1, 1). Let A be the adjacency matrix of this six-cycle, $\mathbf{1}$ the length-six column of ones, and R the permutation matrix for $m \mapsto -m$. This permutation fixes the centre and moves each outer state three places around the hexagon.

Weight a transition by the number of named colour pairs that realize it. At the centre a loop has weight three, one for each available colour. At an outer state only two colours are present, so its loop has weight two. Each admissible unequal pair gives one transition between the centre and an outer state or between adjacent outer states. The parity of the cut time changes the pair labelling a directed transition but preserves these weights.

For the two joined copies the reversed pair gives the negative increment. Uniqueness of the initial state puts the two copies in states m and $-m$ throughout. At outer states their only common colour is the singleton, so the common loop weight falls to one. For unequal pairs, reversing the pair realizes the corresponding transition on the other copy. Lemma 2 extends the word independently on each copy; the semiedge colours then agree across the join.

$$W_\ell = \begin{pmatrix} 3 & 1^T \\ 1 & A + \ell I_6 \end{pmatrix}, \ell \in \begin{Bmatrix} 1, 2 \end{Bmatrix} \tag{4}$$

Use $\ell = 2$ for one copy and $\ell = 1$ for the pair. The number of named words satisfying the count conditions is $\mathrm{tr}(W_\ell{}^b R)$. Each word determines one initial state and hence one state sequence. Conversely, the weights count exactly the choices of the colour pairs along the walk. The 3^b words consisting solely of equal pairs stay at the centre and must be removed. Every remaining word uses at least two named colours, so its orbit under colour permutation has size six.

On functions constant on the six outer states, the matrix reduces to B_ℓ below and R acts as the identity. On the five nonconstant outer Fourier modes, indexed by $j = 1,\ldots,5$, the respective eigenvalues of W_ℓ and R are λ_j and ρ_j.

$$B_\ell = \begin{pmatrix} 3 & 6 \\ 1 & \ell + 2 \end{pmatrix}, \lambda_j = \ell + 2\cos\left(\frac{2\pi j}{6}\right), \rho_j = (-1)^j \tag{5}$$

Proof of Theorem 1. The eigenvalues of B_2 are 6 and 1. Including the five other modes gives trace $6^b - 2\cdot 3^b + 3$. The eigenvalues of B_1 are $3 \pm \sqrt{6}$, and the remaining modes contribute $-2\cdot 2^b - (-1)^b$. Subtracting 3^b from each trace and dividing by six proves (1) and (2). □

Corollary. For any two separated-cut configurations with the same b, each consisting of two joined copies, reversing the colour pair at exactly the positions where the cut-time parities differ gives a bijection between their compatible pattern sets. It preserves Δ, and an equal pair of colour a is possible at an outer state of both copies exactly when $m_a = 0$.

4. Checks and related results

Parity alone permits $U(b) = (9^b + 15)/24$ boundary patterns [4, Proposition 3.1]. At five cuts each copy realizes fewer than half of these, even though 750 patterns extend on both sides.

Table 1. Boundary patterns up to colour permutation.

Cuts b	1	2	3	4	5	6
Parity-admissible U	1	4	31	274	2,461	22,144
One copy S	0	2	23	176	1,175	7,412
Two copies G	0	2	20	128	750	4,222

Verification. The verifier `verify_spaced_count_formulas.js` propagates all 27 possible triples through each cut and tests for the reflected initial triple at the seam. Direct enumeration on one configuration for each $b = 1,\ldots,6$ reproduced Table 1. The gap condition matters: a five-claw flower with gaps (1, 1, 3) and leaf labels (0, 1, 0) at the three cuts admits 21 patterns, whereas (1) gives 23. For arbitrary b, Lemma 2 tests an ordered word in $O(b)$ local operations using seven states and a bit recording whether an unequal pair has occurred.

Known sequences. The number of named words, $6S(b)$, equals OEIS A059517, recorded in 2001 [5]. It counts placements of b distinguishable intervals, including point intervals, on three ordered points with every point used as an endpoint. With three, two or one available points, each interval has six, three or one choices, respectively. Inclusion–exclusion gives $6^b - 3\cdot3^b + 3$. Likewise, $G(b)$ = A158869($b - 1$) − A087432(b) [6, 7].

Equation (1) gives the interval count a boundary-colouring interpretation for flower snarks. Equation (2) follows by lowering the outer loop weight from two to one and imposing the odd-circuit closure. The sources examined did not yield the two-copy result under these cut conditions. The size of an exact representation after successively eliminating three or more copies remains open here.

References

[1] R. Isaacs. Infinite families of nontrivial trivalent graphs which are not Tait colorable. *Amer. Math. Monthly* 82 (1975), 221–239, Theorem 4.1.1. Source

[2] M. Chladný and M. Škoviera. Factorisation of Snarks. *Electron. J. Combin.* 17 (2010), R32, Lemma 4.1 and Example 5.5. Source

[3] J. Mazák, J. Rajník and M. Škoviera. Morphology of small snarks. *Electron. J. Combin.* 29 (2022), P4.30, §5.5. Source

[4] M. A. Fiol and J. Vilaltella. Some Results on the Structure of Multipoles in the Study of Snarks. *Electron. J. Combin.* 22 (2015), P1.45. Source

[5] H. Bottomley. OEIS A059517. Interval endpoint arrangements (2001). Source

[6] OEIS A158869. Brick tilings of a $2 \times 3 \times 2n$ cuboid. Source

[7] P. Barry. OEIS A087432 (2003). Source

Contributions and AI use. 北山　昌武 proposed the research direction. OpenAI Codex developed the seven-state criterion, derived the formulas and proofs, searched the literature, wrote and ran the verification code, and drafted the note and its English translation.