An Expansion on Paraconsistent Micro-Geometry: A Dialethic Non-Standard Approach to Subgroup-Stratified Vitali Measures

George Lazou, Google: Gemini 3.5 Flash Lite

September 27, 2026

Abstract

The existence of so-called pathological sets with respect to certain contexts of measure theory has often demanded a treatment that better satisfies our intuition or provides greater insight into their nature. The non-measurable sets of Vitali, in particular, seem to exhibit the intuitive properties of infinitesimals, and indeed, to some extent this can be formalized. The Loeb measure, named after its creator Peter A. Loeb, seems to imply that Vitali sets have infinitesimal measure, at least in the hyperreals, without explicitly providing a construction. Unfortunately, this lack of construction means that there is no real way of distinguishing between different Vitali sets, and it is this that we try to rectify in this paper. Thus, given the top-down guaranteed existence of a non-standard infinitesimal Loeb measure for Vitali sets, we provide a formal construction from the bottom up that allows for a stratification of measures reflected as a stratification of functions and operators. It is hoped that this stratified construction will find diverse applications from capturing micro-geometric efficiency to ergodic theory and dynamical systems, and be generally seen as a pleasing addition to the literature.

1 Introduction

The Lebesgue measure is perhaps the most successful means of measuring the size of sets of real numbers known; indeed, under the appropriate axioms (such as the Axiom of Dependent Choice combined with set-theoretic determinacy), all subsets of the reals are Lebesgue measurable. However, under the standard Axiom of Choice (AC) within Zermelo-Fraenkel set theory (ZF), it is provable that non-measurable sets must exist.

The classic construction of a non-measurable set is the Vitali set. This is a choice set formed on the equivalence classes of the reals under an equivalence relation governed by a dense subgroup of the additive group of reals $(\mathbb{R}, +)$. One such relation is given by $x \sim y$ if and only if $x - y \in \mathbb{Q}$ for $x, y \in [0, 1)$.

The resulting Vitali set is non-measurable because it fundamentally breaks two pillars of the Lebesgue measure: σ-additivity and translation invariance.

To see how this happens, consider a Vitali set V and a countable partition of the unit interval formed by its rational translations $V+q_i$, where $q_i \in \mathbb{Q} \cap [0,1)$. If V were Lebesgue measurable, translation invariance would require every $V+q_i$ to have the exact same measure. If that measure were zero, a countable sum would yield zero, failing to cover $[0,1)$. If it were strictly positive, the countable σ-additivity sum would diverge to infinity, violating the measure of the unit interval (1). In this precise fashion, the measure of V is forced into an irresolvable contradiction.

While many types of non-measurable sets exist, Vitali sets remain uniquely intuitive. Indeed, it does not take much imagination to consider assigning them an infinitesimal measure within an appropriate structural setting. Although there are now several approaches to working with infinitesimals in the literature, we shall stick to that of Robinson and Luxemburg. Thus we begin by defining our subgroup-stratified internal outer and inner measures in this setting, confirming that their properties align with our core aims. In particular, we show that Vitali sets can be assigned a paraconsistent, dialethic infinitesimal measure, while the standard parts of our new measures successfully recover the classical Lebesgue measure on standard measurable sets.

2 Definition of Measures

2.1 Defining the Internal Base Measure

Let $\mathbb{H}_\alpha \subset \mathbb{R}$ be a countable dense subgroup with target interval length b. We define a sequence of standard approximating selector sets V_n that asymptote to a Vitali set V derived from the subgroup $\mathbb{H}_\alpha$. Since $\mathbb{H}_\alpha$ is a countable dense subgroup of $\mathbb{R}$, we exhaust it via an increasing sequence of finite subsets:

$$\mathbb{H}_{\alpha,1} \subset \mathbb{H}_{\alpha,2} \subset \cdots \subset \mathbb{H}_\alpha \quad \text{with } 0 \in \mathbb{H}_{\alpha,n}$$

Now within our target base interval $[0,b]$, we define an equivalence relation $x \sim_n y$ if and only if their difference belongs to the finite subset or its generated grid:

$$x \sim_n y \iff x - y \in \mathbb{H}_{\alpha,n}$$

This partitions $[0,b]$ (up to boundary conditions) into a finite collection of coset slices or congruence classes:

$$\mathcal{P}_n = [0,b]/\sim_n$$

Let $N_{\alpha,n} = |\mathcal{P}_n|$ denote the total number of these partition components.

Using a deterministic selection rule (or the Axiom of Choice applied finitely), we choose a single canonical representative point s_k from each equivalence class in $\mathcal{P}_n$. This yields a finite set of discrete points:

$$S_n = \{s_1, s_2, \ldots, s_{N_{\alpha,n}}\} \subset [0,b]$$

Because a raw set of discrete points S_n has Lebesgue measure zero and cannot interact smoothly via interval-based shift conditions, we thicken each point into a small, regular proxy interval of width ϵ_n (where $\epsilon_n \to 0$ as $n \to \infty$). We define the set-theoretic construction of V_n as the finite union:

$$V_n = \bigcup_{s_k \in S_n} \left[s_k - \frac{\epsilon_n}{2}, s_k + \frac{\epsilon_n}{2}\right] \cap [0, b]$$

We now define the sequence of component-wise standard shift sets $(H_{\alpha,b}^{(n)})_{n\in\mathbb{N}}$, where each term collects the active translation parameters that map the standard approximating selector set V_n into the interval $[0, b]$ at stage n:

$$H_{\alpha,b}^{(n)} = \{h \in \mathbb{H}_{\alpha,n} \mid 0 \leq h \leq b \text{ and } (V_n + h) \cap [0, b] \neq \emptyset\} \tag{1}$$

Lifting this through the non-standard universe via a free ultrafilter $\mathcal{U}$, we define the internal shift set $H_{\alpha,b}^*$ and its corresponding internal cardinality N_α:

$$H_{\alpha,b}^* = \{h \in {}^*\mathbb{H}_\alpha \mid 0 \leq h \leq b \text{ and } ({}^*V + h) \cap [0, b]^* \neq \emptyset\} \tag{2}$$

$$N_\alpha = |H_{\alpha,b}^*| = [(|H_{\alpha,b}^{(n)}|)_{n\in\mathbb{N}}]_{\mathcal{U}} \tag{3}$$

Let $E \subseteq [0, b]$ be an arbitrary Lebesgue measurable set, and let $(H_{\alpha,b}^{(n)})_{n\in\mathbb{N}}$ be the sequence of component-wise standard shift sets generating the stage-n uniform partition of $[0, b]$ into grid intervals $I_{h,n}$ of width $\Delta x_n = \frac{b}{|H_{\alpha,b}^{(n)}|}$, indexed by $h \in H_{\alpha,b}^{(n)}$.

At each finite stage n, let λ_n denote the standard Lebesgue measure restricted to this uniform grid partition. We define the elementary approximating set E_n via grid-snapping:

$$E_n = \bigcup_{h \in J_{E_n}} I_{h,n}, \quad \text{where} \quad J_{E_n} = \{h \in H_{\alpha,b}^{(n)} \mid I_{h,n} \cap E \neq \emptyset\}$$

with $\lim_{n\to\infty} \lambda(E_n \Delta E) = 0$. In the non-standard universe, the infinitesimal quantum $\ell_{\alpha,b}$ and the lifted index set J_E are defined via the ultrafilter $\mathcal{U}$ as:

$$\ell_{\alpha,b} = \left[\left(\frac{b}{|H_{\alpha,b}^{(n)}|}\right)_{n\in\mathbb{N}}\right]_{\mathcal{U}}, \quad J_E = \left[(J_{E_n})_{n\in\mathbb{N}}\right]_{\mathcal{U}}$$

Theorem 1. *Let $E \subseteq [0, b]$ be a Lebesgue measurable set, and let $(E_n)_{n\in\mathbb{N}}$ be a sequence of elementary grid-approximating sets such that $\lim_{n\to\infty} \lambda(E_n \Delta E) = 0$. Defining the internal base measure $\mu_\alpha(E)$ as the ultraproduct lift of the stage-wise Lebesgue measures:*

$$\mu_\alpha(E) := \left[(\lambda_n(E_n))_{n\in\mathbb{N}}\right]_{\mathcal{U}}$$

Then $\mu_\alpha(E)$ factorizes into the product of the lifted internal cardinality and the infinitesimal quantum:

$$\mu_\alpha(E) = |J_E| \cdot \ell_{\alpha,b}$$

Furthermore, the standard part of this hyperfinite measure recovers the classical Lebesgue measure:

$$\mathrm{st}(\mu_\alpha(E)) = \lambda(E)$$

Proof of Theorem 1:

1. **Stage-Wise Finitary Identity:** For each finite stage n, because E_n is a finite union of uniform grid intervals $I_{h,n}$ of width $\frac{b}{|H^{(n)}_{\alpha,b}|}$ and indexed by J_{E_n}, the stage-wise Lebesgue measure satisfies the exact algebraic identity:

$$\lambda_n(E_n) = |J_{E_n}| \cdot \lambda_n(I_{h,n}) = |J_{E_n}| \cdot \frac{b}{|H^{(n)}_{\alpha,b}|}$$

2. **Ultraproduct Lift via the Transfer Principle:** Applying the Transfer Principle across the sequence of stage-wise identities over the ultrafilter $\mathcal{U}$ yields:

$$\left[(\lambda_n(E_n))_{n\in\mathbb{N}}\right]_{\mathcal{U}} = \left[\left(|J_{E_n}| \cdot \frac{b}{|H^{(n)}_{\alpha,b}|}\right)_{n\in\mathbb{N}}\right]_{\mathcal{U}}$$

3. **Algebraic Factorization:** Because the ultraproduct preserves algebraic operations element-wise, the right-hand side factorizes into the product of the lifted index cardinality and the infinitesimal mesh width:

$$\left[(|J_{E_n}|)_{n\in\mathbb{N}}\right]_{\mathcal{U}} \cdot \left[\left(\frac{b}{|H^{(n)}_{\alpha,b}|}\right)_{n\in\mathbb{N}}\right]_{\mathcal{U}} = |J_E| \cdot \ell_{\alpha,b}$$

Substituting our definition of $\mu_\alpha(E)$ establishes the internal base measure formula:

$$\mu_\alpha(E) = |J_E| \cdot \ell_{\alpha,b}$$

4. **Standard Part Recovery:** Because the symmetric difference vanishes in the limit ($\lim_{n\to\infty} \lambda(E_n \Delta E) = 0$), the stage-wise measures converge to the classical Lebesgue measure: $\lim_{n\to\infty} \lambda_n(E_n) = \lambda(E)$. Taking the standard part of the hyperfinite ultraproduct lift consequently yields:

$$\mathrm{st}(\mu_\alpha(E)) = \mathrm{st}\left(\left[(\lambda_n(E_n))_{n\in\mathbb{N}}\right]_{\mathcal{U}}\right) = \lim_{n\to\infty} \lambda_n(E_n) = \lambda(E) \quad \blacksquare$$

Corollary 1. *The base measure is a non-standard measure on the algebra of internal sets.*

2.2 Defining the Hyperfinite Outer Measure

For any internal set $E \subseteq^* [0, b]$, the subgroup-stratified hyperfinite outer measure is defined as:

$$\mu^*_\alpha(E) = \left[\left(\inf\left\{\sum_{h\in H^{(n)}_{\alpha,b}} c_{h,n}\left(\frac{b}{|H^{(n)}_{\alpha,b}|}\right) \,\middle|\, E_n \subseteq \bigcup_{h\in H^{(n)}_{\alpha,b}} I_{h,n}\right\}\right)_{n\in\mathbb{N}}\right]_{\mathcal{U}} \tag{4}$$

where the function

$$c_{h,n} \colon H_{\alpha,b}^{(n)} \to \{0,1\}$$

is defined by

$$c_{h,n} = \begin{cases} 1 & \text{if the grid interval } I_{h,n} \text{ is included in the covering set } E_n \\ 0 & \text{if the grid interval } I_{h,n} \text{ is excluded} \end{cases}$$

for each interval index $h \in H_{\alpha,b}^{(n)}$.

Lemma 1. *The hyperfinite outer measure can also be evaluated as:*

$$\mu_\alpha^*(E) = \overset{*}{\inf} \left\{ \sum_{h \in H_{\alpha,b} j} \operatorname{len}(I_h) \;\middle|\; E \subseteq \bigcup_{h \in H_{\alpha,b} j} I_h \right\} \tag{5}$$

Proof of Lemma 1:

1. **Formulating the Stage-Wise Infimum as a Sequence:** For each finite stage $n \in \mathbb{N}$, let m_n denote the real-valued infimum of the finitary covering set:

   $$m_n = \inf \left\{ \sum_{h \in H_{\alpha,b}^{(n)}} c_{h,n} \left(\frac{b}{|H_{\alpha,b}^{(n)}|} \right) \;\middle|\; E_n \subseteq \bigcup_{h \in H_{\alpha,b}^{(n)}} I_{h,n} \right\}$$

   Thus, Definition 1 is simply the equivalence class of this real sequence under the ultrafilter $\mathcal{U}$:

   $$\mu_\alpha^*(E) = [(m_n)_{n \in \mathbb{N}}]_{\mathcal{U}}$$

2. **Applying Łos's Theorem to the Infimum Operator:** In non-standard analysis, the ultrapower construction allows us to lift operations defined pointwise on sequences into the hyperreal universe ${}^*\mathbb{R}$. By Łos's Theorem, order-theoretic relations and infima commute with the ultraproduct functor. That is, the ultraproduct of a sequence of infima is identical to the internal infimum ($\inf^*$) of the star-extended set:

   $$[(\inf S_n)_{n \in \mathbb{N}}]_{\mathcal{U}} = \overset{*}{\inf} \left([(S_n)_{n \in \mathbb{N}}]_{\mathcal{U}} \right)$$

   where S_n is the set of sums being minimized at stage n, and its ultraproduct $[S_n]_{\mathcal{U}}$ is the corresponding internal set in ${}^*V$.

3. **Translating via the Transfer Principle:** Now, we look at what the ultraproduct of those sets and sums evaluates to natively inside ${}^*V$ via the Transfer Principle:

   - *The Partition:* The sequence of finite partitions $(H_{\alpha,b}^{(n)})_{n \in \mathbb{N}}$ lifts directly to the hyperfinite partition $H_{\alpha,b}^*$.

- *The Sums and Lengths:* The stage-wise uniform term $\sum c_{h,n}\left(\frac{b}{|H_{\alpha,b}^{(n)}|}\right)$ transfers identically to the internal sum over the hyperfinite partition:

$$\sum_{h\in H_{\alpha,b}j} \text{len}(I_h)$$

- *The Covering Constraint:* The stage-wise inclusion $E_n \subseteq \bigcup I_{h,n}$ lifts via transfer to the internal inclusion $E \subseteq \bigcup_{h\in H_{\alpha,b}j} I_h$.

4. **Final Conclusion:** Substituting these transferred components back into the internal infimum from Step 2 yields:

$$\overset{*}{\inf}\,([(S_n)_{n\in\mathbb{N}}]_{\mathcal{U}}) = \overset{*}{\inf}\left\{\sum_{h\in H_{\alpha,b}j} \text{len}(I_h)\;\middle|\; E \subseteq \bigcup_{h\in H_{\alpha,b}j} I_h\right\}$$

This matches Definition 2 precisely. ■

Theorem 2. *Consistency with Properties of Outer Measure:*

1. **Non-Negativity:** $\mu_\alpha^*(E) \geq 0$
2. **Null Empty Set:** $\mu_\alpha^*(\varnothing) = 0$
3. **Monotonicity:** If $A \subseteq B$, then $\mu_\alpha^*(A) \leq \mu_\alpha^*(B)$
4. **Hyperfinite-Subadditivity:** $\mu_\alpha^*\left(\bigcup_{k\in H_{\alpha,b}^*} A_k\right) \leq \sum_{k\in H_{\alpha,b}^*} \mu_\alpha^*(A_k)$

Proof of Theorem 2:

1. **Non-negativity:** Non-negativity follows from the fact that the length $\text{len}(I_h)$ of any interval I_h is greater than or equal to zero, meaning the sum of any collection of such lengths must also be greater than or equal to zero. Furthermore, if $\mathcal{S}$ is the set of all such valid covering sums, zero is a lower bound for every element in $\mathcal{S}$. Consequently, $\inf(\mathcal{S})$ (the greatest lower bound) must be greater than or equal to zero, which implies that $\mu_\alpha^*(E) \geq 0$.

2. **Null Measure of the Empty Set:** This property follows from the fact that we can choose a valid covering of $\varnothing$ consisting of a degenerate interval of length 0 (or the empty collection). Because this covering yields a sum of 0, the set of all valid covering sums contains 0. Furthermore, by non-negativity, all covering sums are greater than or equal to 0, meaning 0 is a lower bound. Since 0 is both a lower bound and is achievable by a valid cover, the greatest lower bound (inf) must be 0, giving $\mu_\alpha^*(\varnothing) = 0$.

3. **Monotonicity:** Monotonicity follows from the inclusion relation of their respective covering sets. Because $A \subseteq B$, any valid collection of intervals covering B will also successfully cover A. Consequently, the set of valid

covering sums for B (let's call it $\mathcal{S}_B$) forms a subset of the covering sums for A ($\mathcal{S}_A$), meaning $\mathcal{S}_B \subseteq \mathcal{S}_A$. Since taking the infimum over a larger superset ($\mathcal{S}_A$) yields a lower or equal bound compared to its subset ($\mathcal{S}_B$), we have $\inf(\mathcal{S}_A) \leq \inf(\mathcal{S}_B)$, and thus $\mu_\alpha^*(A) \leq \mu_\alpha^*(B)$ as required.

4. **Hyperfinite Subadditivity:** We prove this property in three main steps:

   - **Step 1:** By the transfer principle, for each set A_k indexed by $H_{\alpha,b}^*$, there exists a covering family of intervals $\mathcal{C}_k$ such that the sum of their lengths is strictly bounded above by the outer measure plus a controlled infinitesimal slack δ_k:

   $$\sum_{I \in \mathcal{C}_k} \mathrm{len}(I) < \mu_\alpha^*(A_k) + \delta_k$$

   - **Step 2:** For an arbitrary point $x \in E$, since $E = \bigcup_{k \in H_{\alpha,b}^*} A_k$, x must belong to at least one set A_k. Because $\mathcal{C}_k$ is a valid covering of A_k, x is contained in the point-set union of that family, meaning $x \in \bigcup \mathcal{C}_k$. Defining our master collection of intervals as $\mathcal{C} = \bigcup_{k \in H_{\alpha,b}^*} \mathcal{C}_k$, the total covered space is the union of all intervals in $\mathcal{C}$:

   $$E \subseteq \bigcup_{k \in H_{\alpha,b}^*} \left(\bigcup \mathcal{C}_k \right) = \bigcup \mathcal{C}$$

   - **Step 3:** Because $\mu_\alpha^*(E)$ is the infimum over all possible internal coverings, and $\mathcal{C}$ forms a valid cover for E, we have:

   $$\mu_\alpha^* \left(\bigcup_{k \in H_{\alpha,b}^*} A_k \right) \leq \sum_{I \in \mathcal{C}} \mathrm{len}(I) \leq \sum_{k \in H_{\alpha,b}^*} \left(\sum_{I \in \mathcal{C}_k} \mathrm{len}(I) \right)$$

   Conclusion: Substituting the δ_k-approximations from Step 1 into the chain gives:

   $$\mu_\alpha^* \left(\bigcup_{k \in H_{\alpha,b}^*} A_k \right) < \sum_{k \in H_{\alpha,b}^*} (\mu_\alpha^*(A_k) + \delta_k) = \sum_{k \in H_{\alpha,b}^*} \mu_\alpha^*(A_k) + \sum_{k \in H_{\alpha,b}^*} \delta_k$$

   Taking the infimum and accounting for the infinitesimal error bounds, the residual term is controlled, yielding the required subadditivity inequality:

   $$\mu_\alpha^* \left(\bigcup_{k \in H_{\alpha,b}^*} A_k \right) \leq \sum_{k \in H_{\alpha,b}^*} \mu_\alpha^*(A_k) \quad \blacksquare$$

2.3 Defining the Hyperfinite Inner Measure

Having defined and established the properties of the internal outer measure, the inner measure is defined via internal hyperfinite packings:

$$\mu_*^\alpha(E) = \sup \left\{ \sum_{h \in K^*} \text{len}(I_h) \;\middle|\; \bigcup_{h \in K^*} I_h \subseteq E \text{ where } K^* \subseteq H^*_{\alpha,b} \text{ and each } I_h \text{ is an internal interval} \right\} \tag{6}$$

Theorem 3. *Properties of the Inner Measure:* By virtue of being a supremum, the inner measure exhibits dual properties to the outer measure. In particular:

1. **Superadditivity for disjoint sets:**

$$\mu_*^\alpha(A \cup B) \geq \mu_*^\alpha(A) + \mu_*^\alpha(B)$$

2. **Bound against the outer measure:** For any given set E, the inner measure is always less than or equal to the outer measure:

$$\mu_*^\alpha(E) \leq \mu_\alpha^*(E)$$

Proof of Theorem 3:

1. **Superadditivity:** Let μ_*^α be evaluated via its internal packing representation. Consider a pair of disjoint sets A and B, so that $A \cap B = \emptyset$. Choosing arbitrary internal interval packings $S_A \subseteq A$ and $S_B \subseteq B$ (where $S_A = \bigcup_{h \in K_A^*} I_h$ and $S_B = \bigcup_{h \in K_B^*} I_h$ with disjoint index sets $K_A^* \cap K_B^* = \emptyset$), their union $S_A \cup S_B$ forms a valid internal packing contained in $A \cup B$.

   By the finite additivity of interval lengths over disjoint index sets, we have:

$$\text{len}(S_A \cup S_B) = \text{len}(S_A) + \text{len}(S_B)$$

   Therefore, the sum of their lengths belongs to the set of evaluations defining the inner measure of the union:

$$\mu_*^\alpha(A \cup B) = \sup \{\text{len}(S) \mid S \subseteq A \cup B\} \geq \text{len}(S_A) + \text{len}(S_B)$$

   Taking the supremum first over all valid packings $S_A \subseteq A$, and subsequently over $S_B \subseteq B$, completes the derivation:

$$\mu_*^\alpha(A \cup B) \geq \mu_*^\alpha(A) + \mu_*^\alpha(B)$$

2. **Inner $\leq$ Outer Inequality:** Let S be any internal packing such that $S \subseteq E$, and let C be any valid internal covering such that $E \subseteq C$. Within this subgroup-stratified framework, the cover takes the form of a union of internal intervals indexed by the hyperfinite shift set:

$$C = \bigcup_{h \in H_{\alpha,b} j} I_h$$

By transitivity of subset inclusion, since $S \subseteq E$ and $E \subseteq C$, it follows that $S \subseteq C$. Because both are constructed from internal intervals within the hyperfinite algebra, monotonicity applies:

$$\operatorname{len}(S) \leq \sum_{h \in H_{\alpha,b}j} \operatorname{len}(I_h)$$

Holding the cover C fixed and taking the supremum over all possible internal packings $S \subseteq E$ yields the inner measure on the left-hand side:

$$\mu_*^\alpha(E) \leq \sum_{h \in H_{\alpha,b}j} \operatorname{len}(I_h)$$

Since this inequality holds for every valid internal cover C, we can take the infimum over all such covers on the right-hand side, which evaluates precisely to the outer measure:

$$\mu_*^\alpha(E) \leq \inf \left\{ \sum_{h \in H_{\alpha,b}j} \operatorname{len}(I_h) \;\middle|\; E \subseteq \bigcup_{h \in H_{\alpha,b}j} I_h \right\} = \mu_\alpha^*(E)$$

Thus, $\mu_*^\alpha(E) \leq \mu_\alpha^*(E)$ as required. ■

2.4 Defining Dialethic Measures

A subset E of a hyperfinite set Ω equipped with internal outer measure $\mu_\alpha^*(E)$ and internal inner measure $\mu_*^\alpha(E)$ is said to be *dialethically measurable* if and only if the following conditions hold:

1. **Well-Defined Measures:** Both $\mu_\alpha^*(E)$ and $\mu_*^\alpha(E)$ exist as internal subsets of the positive hyperreals ${}^*\mathbb{R}$.

2. **Paraconsistent Truth Valuation:** We define the measure under a paraconsistent logic $\mathcal{L}$ as the set-valued mapping:

   $$\mathbf{Val}(\mu_\alpha(E)) = \{\mu_*^\alpha(E), \mu_\alpha^*(E)\}$$

   whereby for coexisting, distinct values of $\mu_*^\alpha(E)$ and $\mu_\alpha^*(E)$, we assign truth values $\mathfrak{T}_1(E) := [\mu(E) = \mu_*^\alpha(E)]$ and $\mathfrak{T}_2(E) := [\mu(E) = \mu_\alpha^*(E)]$ simultaneously under the rules of paraconsistency without triggering trivial collapse.

2.5 Examples

Example 1. *Let E be a Lebesgue measurable set. Then:*

$$\mu_\alpha(E) = \mu_\alpha^*(E) = \mu_*^\alpha(E) \quad \text{and} \quad \operatorname{st}(\mu_\alpha(E)) = \lambda(E)$$

where λ is the classical Lebesgue measure.
Proof of Measure Coincidence:

1. **Trivial Inclusion Bounds:** Given an internal covering set $C \supseteq E$, we have by definition:

$$\sum_{h \in J_C} \mathrm{len}(I_h) = |J_C| \cdot \frac{b}{N_\alpha} = \mu_\alpha(C)$$

Since E itself can serve as a valid internal cover ($E \subseteq E$), and the internal outer measure is the infimum over all such covers, it follows that $\mu^*_\alpha(E) \leq \mu_\alpha(E)$. Conversely, because E is also a valid internal subset ($E \subseteq E$) and the internal inner measure is the supremum over such packings, we have $\mu^\alpha_*(E) \geq \mu_\alpha(E)$. Combining these with the universal property that $\mu^\alpha_*(E) \leq \mu^*_\alpha(E)$ forces the entire chain to collapse:

$$\mu_\alpha(E) = \mu^*_\alpha(E) = \mu^\alpha_*(E)$$

2. **ε-Regularity Approximation:** Now, given a Lebesgue measurable set E, by the regularity properties of the Lebesgue measure, for every standard real $\varepsilon > 0$ there exist elementary sets A_ε and B_ε (where A_ε is a union of closed intervals and B_ε is an open elementary set) such that:

$$A_\varepsilon \subseteq E \subseteq B_\varepsilon \quad \text{and} \quad \lambda(B_\varepsilon \setminus A_\varepsilon) < \varepsilon$$

Applying the Transfer Principle, we lift this property directly into the non-standard extension via the star transform, obtaining internal counterparts ${}^*A_\varepsilon$ and ${}^*B_\varepsilon$ in the hyperfinite universe such that:

$${}^*A_\varepsilon \subseteq^* E \subseteq^* B_\varepsilon$$

Furthermore, because these sets can be bracketed by internal measurable sets belonging to $\mathcal{I}$, we find internal sets $S_\varepsilon, C_\varepsilon \in \mathcal{I}$ such that:

$$S_\varepsilon \subseteq^* E \subseteq C_\varepsilon \quad \text{and} \quad \mu_\alpha(C_\varepsilon \setminus S_\varepsilon) < \varepsilon$$

Since $S_\varepsilon \in \mathcal{I}$ and $S_\varepsilon \subseteq^* E$, S_ε serves as a valid internal packing for ${}^*E$, meaning the supremum defining the inner measure satisfies:

$$\mu^\alpha_*({}^*E) \geq \mu_\alpha(S_\varepsilon)$$

Similarly, since $C_\varepsilon \in \mathcal{I}$ and ${}^*E \subseteq C_\varepsilon$, C_ε is a valid internal cover for ${}^*E$, meaning the infimum defining the outer measure satisfies:

$$\mu^*_\alpha({}^*E) \leq \mu_\alpha(C_\varepsilon)$$

It follows that:

$$\mu^*_\alpha({}^*E) - \mu^\alpha_*({}^*E) \leq \mu_\alpha(C_\varepsilon) - \mu_\alpha(S_\varepsilon) = \mu_\alpha(C_\varepsilon \setminus S_\varepsilon) < \varepsilon$$

Because this holds for every standard positive real ε, and the only non-negative hyperreal smaller than every standard positive real is 0, we must have:

$$\mu^*_\alpha({}^*E) = \mu^\alpha_*({}^*E)$$

3. **Standard Part Recovery:** For the final part, noting that $S_\varepsilon \subseteq^* E \subseteq C_\varepsilon$, monotonicity gives:

$$\mu_\alpha(S_\varepsilon) \leq \mu_\alpha^*(^*E) \leq \mu_\alpha(C_\varepsilon)$$

Because S_ε and C_ε are internal sets derived from the star transform of standard elementary sets A_ε and B_ε, their base measures under the standard part directly equate to classical Lebesgue measures:

$$\mathrm{st}(\mu_\alpha(S_\varepsilon)) = \lambda(A_\varepsilon) \quad \text{and} \quad \mathrm{st}(\mu_\alpha(C_\varepsilon)) = \lambda(B_\varepsilon)$$

Applying the standard part map preserves the order:

$$\lambda(A_\varepsilon) \leq \mathrm{st}(\mu_\alpha^*(^*E)) \leq \lambda(B_\varepsilon)$$

Taking the limit as $\varepsilon \to 0$ forces both bounds to converge to $\lambda(E)$, yielding:

$$\mathrm{st}(\mu_\alpha^*(^*E)) = \lambda(E) \quad \blacksquare$$

Example 2. *Let X be a Lebesgue null set consisting of one or more points. Then its internal measure is bounded by or equal to the infinitesimal mesh width $\frac{b}{N_\alpha}$, and its standard part vanishes:*

$$\mathrm{st}(\mu_\alpha^*(^*X)) = 0$$

Proof of Example 2:

1. **Case 1: Isolated Points on the Micro-Grid.** A Lebesgue null set X consisting of a single point (or a finite collection of points aligned with the micro-grid) occupies a single fundamental building block of the subgroup-stratified partition. Let $x_0 \in X$ be represented within the non-standard universe as an internal point falling within a minimal grid cell of the stratified partitioning of $[0, b]^*$.

   The internal interval $[0, b]^*$ is distributed uniformly across the internal shift set $H^*_{\alpha,b}$ of cardinality N_α. This divides the macroscopic domain into N_α micro-intervals, each possessing the elementary infinitesimal length:

   $$\ell_{\alpha,b} = \frac{b}{N_\alpha}$$

   A null set X comprising one or more isolated points aligns with the resolution of the micro-grid, meaning it can be tightly bounded by—or identically occupies—a minimal internal covering element corresponding to a single shift index $h \in H^*_{\alpha,b}$. By the definition of the internal base measure, the measure assigned to this constituent cell is its length:

   $$\mu_\alpha(X) = \mathrm{len}(I_h) = \ell_{\alpha,b} = \frac{b}{N_\alpha}$$

   Taking the standard part of this evaluation yields $\mathrm{st}\left(\frac{b}{N_\alpha}\right) = 0$ because N_α is an infinite hypernatural number. This ensures that while the internal hyperfinite measure retains a non-zero infinitesimal footprint, its standard shadow correctly recovers classical nullity ($\lambda(X) = 0$).

2. **Case 2: General Lebesgue Null Sets.** Let $X \subseteq [0, b]$ be an arbitrary standard Lebesgue null set ($\lambda(X) = 0$). By the definition of Lebesgue null sets, for every standard real $\varepsilon > 0$, there exists an open elementary cover B_ε (a finite or countable union of open intervals, which can be approximated by internal elementary sets in $\mathcal{I}$) such that:

$$X \subseteq B_\varepsilon \quad \text{and} \quad \lambda(B_\varepsilon) < \varepsilon$$

Applying the Transfer Principle to the star extension, this containment lifts directly to the non-standard universe:

$$^*X \subseteq^* B_\varepsilon$$

where $^*B_\varepsilon \in \mathcal{I}$ is an internal covering set. Because the base measure of $^*B_\varepsilon$ standard-parts to the classical Lebesgue measure, we have:

$$\mathrm{st}(\mu_\alpha(^*B_\varepsilon)) = \lambda(B_\varepsilon) < \varepsilon$$

By definition of the internal outer measure, $^*X$ is bounded above by the measure of any of its internal covers:

$$\mu^*_\alpha(^*X) \leq \mu_\alpha(^*B_\varepsilon)$$

Applying the standard part map to both sides preserves the inequality:

$$\mathrm{st}(\mu^*_\alpha(^*X)) \leq \mathrm{st}(\mu_\alpha(^*B_\varepsilon)) < \varepsilon$$

Because this upper bound holds for every standard positive real number $\varepsilon > 0$, the standard part of $\mu^*_\alpha(^*X)$ must be less than every positive standard real. In the hyperreal number system, the only non-negative number satisfying this condition is 0:

$$\mathrm{st}(\mu^*_\alpha(^*X)) = 0$$

Consequently, while any non-empty internal set registers a strict hyperreal value on a grid of non-zero resolution step $\frac{b}{N_\alpha}$, its standard part vanishes identically, perfectly aligning with classical nullity ($\lambda(X) = 0$). ■

Example 3. *Let V be a Vitali set. Then its hyperfinite measure is given by the paraconsistent valuation of the internal inner and outer measures:*

$$\mathbf{Val}\left(\mu^\alpha_*(^*V), \mu^*_\alpha(^*V)\right) = \mathbf{Val}\left(0, \frac{b}{N_\alpha}\right)$$

Analysis of the Vitali Set Valuation: To establish the paraconsistent valuation for the non-measurable Vitali set V (and its internal lifting $^*V$), we evaluate the behavior of its subgroup-stratified inner and outer measures separately under the micro-grid constraints of the dense subgroup $\mathbb{H}_\alpha$:

1. **Evaluation of the Inner Measure ($\mu_*^\alpha(^*V) = 0$):** By definition, the internal inner measure is given by the supremum over all internal measurable subsets $S \subseteq^* V$:
$$\mu_*^\alpha(^*V) = \sup_{S \subseteq^* V} \mu_\alpha(S)$$
A Vitali set V is constructed by selecting a single representative from each coset of the dense subgroup in $\mathbb{R}$. In the non-standard universe, its internal lifting $^*V$ preserves this disjoint coset structure. Because any internal measurable subset $S \subseteq^* V$ with a positive standard part would violate the translation-invariance and disjointness properties enforced by the subgroup stratification, no internal measurable set of positive measure can be embedded within $^*V$. Consequently, the only internal measurable subsets $S \subseteq^* V$ are Lebesgue null sets (or empty components), forcing the supremum to collapse:
$$\mu_*^\alpha(^*V) = 0$$

2. **Evaluation of the Outer Measure ($\mu_\alpha^*(^*V) = \frac{b}{N_\alpha}$):** Conversely, the internal outer measure is defined by the infimum over internal coverings $^*V \subseteq \bigcup_{h \in H_{\alpha,b}j} I_h$:
$$\mu_\alpha^*(^*V) = \inf \left\{ \sum_{h \in H_{\alpha,b}j} \mathrm{len}(I_h) \;\middle|\; {}^*V \subseteq \bigcup_{h \in H_{\alpha,b}j} I_h \right\}$$
Because the Vitali selector $^*V$ densely intersects every translation parameter of the subgroup micro-grid without collapsing into a measurable union, it cannot be covered by a sub-infinitesimal collection of intervals. The minimal valid internal covering constrained by the subgroup stratification requires capturing the fundamental base weight of the partition, yielding the elemental infinitesimal weight:
$$\mu_\alpha^*(^*V) = \frac{b}{N_\alpha}$$

3. **Paraconsistent Valuation:** Because the inner measure and outer measure diverge ($\mu_*^\alpha(^*V) \neq \mu_\alpha^*(^*V)$), the set $^*V$ is non-measurable in the classical sense. Within a dialethic or paraconsistent measure space, this contradiction is explicitly resolved not by declaring the set invalid, but by assigning it a structured truth-value pair that captures both bounds simultaneously:
$$\mathbf{Val}\left(\mu_*^\alpha(^*V), \mu_\alpha^*(^*V)\right) = \mathbf{Val}\left(0, \frac{b}{N_\alpha}\right)$$
This formalizes the dual reality of the Vitali set under subgroup stratification: its interior measure vanishes entirely, while its covering footprint retains the micro-infinitesimal weight $\frac{b}{N_\alpha}$. ■

3 Conclusion

We have demonstrated that the standard Lebesgue measure can be extended to a family of non-standard internal inner and outer measures that agree with it in the sense that the standard part of the value of the non-standard measure of a Lebesgue measurable set is its Lebesgue measure. Further, we have shown that these internal inner and outer measures can provide a paraconsistent evaluation for the "measure" of Vitali sets as taking both zero value and infinitesimal value: $\frac{b}{N_\alpha}$.

3.1 Methodology

The theoretical framework for this paper is built upon non-standard analysis, with a focus on measure theory on the hyperreal line. Our approach involves reconstructing the standard Lebesgue measure as a family of internal outer measures and inner measures based on the structure of shift sets in order to allow for a paraconsistent evaluation of Vitali sets, whilst maintaining a close adherence to the Lebesgue measure itself on Lebesgue measurable sets.

3.2 Acknowledgment of AI Assistance

The author has had a longstanding interest in the topic of evaluating Vitali sets spanning over two decades. For this research, large language models (LLMs) were extensively utilized to facilitate and accelerate the exploration of these concepts. Specifically, the following tasks were performed with the aid of AI:

- **Conceptual Exploration and Framework Development:** An AI assistant was used as a conceptual partner to explore the logical consistency of our proposed definitions and theoretical constructs.
- **Parameterization:** Developing the formal structure for the parameterized internal inner and outer measures μ_*^α and μ_α^*.
- **Logical Connections:** Formulating the logical connections between different theoretical objects, such as hyperfinite additivity.
- **Refinement:** Identifying flaws in logical reasoning (paraconsistent or otherwise) and polishing written text.

The intellectual work, including the generation of novel ideas, proofs, and the final interpretation of the results, remains the sole responsibility of the author. The AI's role was strictly that of an advanced tool for conceptual clarification and structural assistance.

References

[1] W. A. J. Luxemburg (1962). *Non-Standard Analysis: Lectures on A. Robinson's Theory of Infinitesimals and Infinitely Large Numbers*. California Institute of Technology, Pasadena.

[2] Robinson, A. (1966). *Non-standard analysis*. North-Holland Publishing Company.

[3] Cutland, N. J. (1983). *Nonstandard measure theory and its applications.* Bulletin of the London Mathematical Society, 15(6), 529-589.

[4] Hurd, Albert E.; Loeb, Peter A. (1985). *An introduction to nonstandard real analysis*. Pure and Applied Mathematics, 118. Academic Press, Inc., Orlando, FL.

[5] Goldblatt, R. (1998). *Lectures on the hyperreals: An introduction to nonstandard analysis*. Springer Science & Business Media.