

Putnam-Style Examination, Session A, Problem A1

Variation, Concentration, and Middle-Layer Incidence in a Differential Poset

Solutions

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Notation and the one fact used about P

Identify a function v on partitions with the vector $\sum_{\lambda} v(\lambda) \lambda \in \mathbb{R}P$. For $w \in \mathbb{R}P$ and $j \geq 0$,

$$(D^j w)(\mu) = \sum_{|\lambda|=|\mu|+j} f^{\lambda/\mu} w(\lambda),$$

where $f^{\lambda/\mu}$ denotes the number of saturated chains $\mu = \rho^0 \triangleleft \rho^1 \triangleleft \cdots \triangleleft \rho^j = \lambda$ in the Young lattice, equivalently the number of standard Young tableaux of skew shape λ/μ . The only property needed below is

$$f^{\lambda/\mu} \geq 1 \iff \mu \subseteq \lambda, \quad f^{\lambda/\mu} = 0 \text{ otherwise,}$$

together with the fact that D lowers rank by exactly 1. The commutation relation $DU - UD = I$ is not required for any part of this problem.

Part A. Support and tail variation

(a)

Since v is supported on the rank level $|\lambda| = 2c$, the vector $D^j v$ is supported on rank $2c - j$. Expanding the binomial (the operators I and D commute),

$$T_F(v) = (I + D)^F v = \sum_{j=0}^F \binom{F}{j} D^j v.$$

Evaluate at μ with $|\mu| = c + F$. The j -th summand is supported on rank $2c - j$, so it can be nonzero at μ only when $2c - j = c + F$, i.e. $j = c - F$. Consequently, on the entire rank level $c + F$ — in particular on $O_{c,F}$ —

$$T_F(v)|_{O_{c,F}} = \binom{F}{c-F} D^{c-F} v|_{O_{c,F}}.$$

If $F > c$ then $c - F < 0$ and no summand survives. If $F < c/2$ then $c - F > F$, so the required index $j = c - F$ lies outside the range $0 \leq j \leq F$ of summation and again no summand survives. In both cases $T_F(v)$ vanishes identically on $O_{c,F}$.

If $c/2 \leq F \leq c$ then $0 \leq c - F \leq F$, the index $j = c - F$ does occur, $\binom{F}{c-F} \geq 1$, and the displayed identity is exactly the assertion of the problem. ■

(b)

For $F > c$, part (a) gives $T_F(v) \equiv 0$ on $O_{c,F}$, so no value is negative and $\Delta(v, F) = 0$.

For $F = c$ we have $O_{c,c} = \{\mu : |\mu| = 2c, \mu_1 = c\} = S_c$, and the coefficient is $\binom{c}{0} = 1$ with $D^0 = I$. Hence $T_c(v)|_{S_c} = v$ and

$$\Delta(v, c) = \#\{\lambda \in S_c : v(\lambda) < 0\}. \quad \blacksquare$$

(c)

Every $\mu \in O_{c,F}$ with $c/2 \leq F \leq c$ is contained in some $\lambda \in S_c$. Let $\mu \in O_{c,F}$, so $|\mu| = c + F$ and $\mu_1 = F \leq c$. Define

$$\lambda = (c, \mu_2, \mu_3, \dots),$$

i.e. replace the first row of μ by a row of length c . This is a partition, since $\mu_2 \leq \mu_1 = F \leq c$ and the remaining parts are already weakly decreasing. Its size is

$$|\lambda| = c + (|\mu| - \mu_1) = c + (c + F - F) = 2c,$$

and $\lambda_1 = c$, so $\lambda \in S_c$. Containment of Ferrers diagrams is checked row by row: $\lambda_1 = c \geq F = \mu_1$ and $\lambda_i = \mu_i$ for $i \geq 2$. Hence $\mu \subseteq \lambda$.

Positivity. Suppose $v > 0$ throughout S_c and $c/2 \leq F \leq c$. By (a),

$$T_F(v)(\mu) = \binom{F}{c-F} \sum_{\substack{\lambda \in S_c \\ \lambda \supseteq \mu}} f^{\lambda/\mu} v(\lambda),$$

where the sum ranges over a nonempty index set by the previous paragraph, each $f^{\lambda/\mu} \geq 1$, each $v(\lambda) > 0$, and $\binom{F}{c-F} \geq 1$. Therefore $T_F(v) > 0$ throughout $O_{c,F}$, and in particular $\Delta(v, F) = 0$.

Thus $\Delta(v, F)$ can vanish for a reason entirely different from the vanishing in (b): here $T_F(v)$ is nowhere zero on $O_{c,F}$, and it is the pointwise positivity of v , propagated by an operator with nonnegative matrix entries, that forces $\Delta = 0$. ■

Part B. Macroscopic anti-concentration

Answer

No. For no $\alpha \in [\frac{1}{2}, 1]$ does such a constant $\varepsilon(\alpha) > 0$ exist.

Proof

First note that $F = \lceil \alpha c \rceil$ does lie in the admissible range of Part A(a): we have $F \geq \alpha c \geq c/2$, and $F \leq c$ because c is an integer with $c \geq \alpha c$. Also $O_{c,F} \neq \emptyset$: any partition of $c + F$ whose largest part is exactly F belongs to it, e.g. $(F, 1, 1, \dots, 1)$ with c ones (valid since $F \geq 1$), so the claimed inequality is not vacuous for trivial reasons.

Take $v \equiv 1$ on S_c . Every lateral path in S_c contributes $\sum_i |1 - 1| = 0$, so $\text{Var}(v) = 0 \leq 1$ and v is admissible. By Part A(c), $T_F(v) > 0$ throughout $O_{c,F}$, whence

$$\Delta(v, F) = 0 < \varepsilon(\alpha) |O_{c,F}|$$

for every $\varepsilon(\alpha) > 0$ and every $c \geq 1$. This contradicts the proposed bound. ■

Remarks: why the anti-concentration heuristic fails, and how badly

(i) The failure is structural, not an artifact of taking v constant. The hypothesis class is invariant under the shift $v \mapsto v + t$ for $t \in \mathbb{R}$, because Var depends only on increments $v(\lambda^{(i)}) - v(\lambda^{(i-1)})$. The conclusion is not shift-invariant: for any v_0 and any $t > -\min_{S_c} v_0$, the function $v_0 + t$ is strictly positive, so $\Delta(v_0 + t, F) = 0$ by A(c). So one may start from any nonconstant v_0

with $\text{Var}(v_0) = 1$ and shift it; the shift preserves $\text{Var} = 1$ and annihilates Δ . A quantity controlled only by a variation seminorm can never lower-bound a quantity that counts sign changes.

(ii) **The natural repair also fails, at least at $\alpha = 1$.** Adding the normalization $\sum_{\lambda \in S_c} v(\lambda) = 0$ removes the shift freedom but not the conclusion. Write $N = |S_c|$, fix $\lambda_0 \in S_c$, and set

$$v(\lambda_0) = -\frac{N-1}{2N}, \quad v(\lambda) = \frac{1}{2N} \quad (\lambda \neq \lambda_0).$$

Then $\sum_{\lambda} v(\lambda) = 0$. A lateral path consists of *distinct* elements, so it visits λ_0 at most once and hence has at most two nonzero increments, each of absolute value $\frac{N-1}{2N} + \frac{1}{2N} = \frac{1}{2}$; therefore $\text{Var}(v) \leq 1$. By Part A(b), $\Delta(v, c) = 1$, which is $o(|S_c|)$ as $c \rightarrow \infty$ since $|S_c| = p(c) \rightarrow \infty$ (here $|S_c|$ is the number of partitions of $2c$ with largest part exactly c , which is $p(c)$, the number of partitions of c : delete the first row). So even normalized, no constant fraction is forced.

Part C. Balanced all-small incidence

(a) σ is well defined, $\{-1, 0, +1\}$ -valued, and conjugation-odd

B_k is closed under conjugation. Indeed $|\lambda'| = |\lambda| = 2k$, and conjugation exchanges the two defining conditions: $\lambda_1 = k \iff \ell(\lambda') = k$, and $\ell(\lambda) = k \iff \lambda'_1 = k$. So $\lambda \in B_k$ implies $\lambda' \in B_k$, and both indicators appearing in

$$\sigma(\lambda, \mu) = \mathbf{1}[\mu \subseteq \lambda] - \mathbf{1}[\mu \subseteq \lambda']$$

refer to legitimate elements of B_k ; σ is a difference of two $\{0, 1\}$ -valued quantities and hence takes values in $\{-1, 0, +1\}$.

Since $(\lambda')' = \lambda$,

$$\sigma(\lambda', \mu) = \mathbf{1}[\mu \subseteq \lambda'] - \mathbf{1}[\mu \subseteq (\lambda')'] = \mathbf{1}[\mu \subseteq \lambda'] - \mathbf{1}[\mu \subseteq \lambda] = -\sigma(\lambda, \mu).$$

In particular $\sigma(\lambda, \mu) = 0$ for every self-conjugate $\lambda \in B_k$ and every μ . ■

(b) The conjugation identity

Split the defining sum and reindex the second half by the involution $\lambda \mapsto \lambda'$ of B_k established in (a):

$$H(v)(\mu) = \sum_{\lambda \in B_k} \mathbf{1}[\mu \subseteq \lambda] v(\lambda) - \sum_{\lambda \in B_k} \mathbf{1}[\mu \subseteq \lambda'] v(\lambda).$$

In the second sum substitute $\lambda \mapsto \lambda'$ (a bijection of B_k onto itself), turning it into $\sum_{\lambda \in B_k} \mathbf{1}[\mu \subseteq \lambda] v(\lambda')$. Hence

$$H(v)(\mu) = \sum_{\lambda \in B_k} \mathbf{1}[\mu \subseteq \lambda] (v(\lambda) - v(\lambda')) = \sum_{\substack{\lambda \in B_k \\ \lambda \supseteq \mu}} (v(\lambda) - v(\lambda')). \quad \blacksquare$$

(c) Sign, nonemptiness, and nonvacuity

Sign. If v is conjugation-monotone, every summand in (b) is ≥ 0 , so $H(v)(\mu) \geq 0$ — for every μ of size k , in particular for every $\mu \in A_k$.

A degeneracy worth flagging. Taken with the quantifier as literally stated (“ $v(\lambda) \geq v(\lambda')$ for every $\lambda \in B_k$ ”), the hypothesis is self-conjugate: applying it at $\lambda' \in B_k$ gives $v(\lambda') \geq v(\lambda)$, so $v(\lambda) = v(\lambda')$ for all $\lambda \in B_k$ and $H(v) \equiv 0$. The inequality $H(v)(\mu) \geq 0$ holds, but only with equality.

The nondegenerate form of the hypothesis — and the one under which the exercise has content — is the one-sided version: $v(\lambda) \geq v(\lambda')$ for λ ranging over one set of representatives of the conjugate pairs in B_k . The proof of the sign is unchanged, and the sum in (b) then has genuinely positive terms available.

Nonemptiness of the index set. Let $|\mu| = k$ and put

$$\lambda^w = (k, \mu_1, \mu_2, \dots).$$

This is a partition, since $k \geq \mu_1$ (as $|\mu| = k$) and $\mu_1 \geq \mu_2 \geq \dots$. Its size is $k + |\mu| = 2k$ and $\lambda_1^w = k$, so $\lambda^w \in B_k$. It contains μ row by row: $\lambda_1^w = k \geq \mu_1$ and $\lambda_i^w = \mu_{i-1} \geq \mu_i$ for $i \geq 2$. Hence $\{\lambda \in B_k : \lambda \supseteq \mu\} \neq \emptyset$ for every μ of size k ; the restriction $\mu_1 \leq k/2$ is not needed for this. So the conclusion of (c) is not vacuous.

Where A_k does earn its keep. By (a), self-conjugate λ contribute nothing, so nonvacuity in the sharper sense requires a *non*-self-conjugate $\lambda \in B_k$ containing μ . Applying the construction above to μ' and conjugating gives

$$\lambda^t = ((k, \mu'_1, \mu'_2, \dots))',$$

which lies in B_k (it has exactly k rows) and contains μ , because containment is reversed twice under conjugation. Suppose both λ^w and λ^t were self-conjugate. Self-conjugacy of λ^w forces $\ell(\lambda^w) = \lambda_1^w = k$, i.e. $\ell(\mu) = k - 1$; self-conjugacy of λ^t forces $\lambda_1^t = \ell(\lambda^t) = k$, i.e. $1 + \mu_1 = k$. The second gives $\mu_1 = k - 1$, and $\mu \in A_k$ demands $\mu_1 \leq k/2$, so $k \leq 2$. The first gives $k = |\mu| \geq \mu_1 + \ell(\mu) - 1 = (k - 1) + (k - 1) - 1 = 2k - 3$, so $k \leq 3$. Combining with $|\mu| = k$ and $\mu_1 = k - 1$: for $k = 1$, $\mu_1 = 0$ contradicts $|\mu| = 1$; for $k = 2$, $\mu = (1, 1)$ has $\mu_1 = 1 \leq 1$ but $\ell(\mu) = 2 \neq k - 1 = 1$. So the two conditions are never simultaneously satisfiable for $\mu \in A_k$.

Hence for every $\mu \in A_k$ the index set in (b) contains a partition $\lambda \neq \lambda'$. Under the one-sided hypothesis the corresponding term $v(\lambda) - v(\lambda')$ can be strictly positive, so H is not forced to vanish identically on A_k and the inequality $H(v)(\mu) \geq 0$ has content. ■

Sanity check

Take $k = 4$ and $\mu = (1, 1, 1, 1) \in A_4$ (indeed $\mu_1 = 1 \leq 2$). The partition $\lambda = (4, 2, 2)$ lies in B_4 (size 8, first part 4) and does *not* contain μ , having only three rows; but $\lambda' = (3, 3, 1, 1)$ does contain μ . So $\sigma(\lambda, \mu) = 0 - 1 = -1$, confirming that the incidence operator genuinely realizes both signs, and that (b) is not a disguised statement about a nonnegative kernel.