A q-rious counterexample

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Abstract. We disprove Warnaar and Zudilin's q-rious positivity conjecture. We verify Landau's condition for a family of q-factorial ratios using equation (1.7) of de la Bretèche and Tenenbaum [2]. For the resulting polynomials P, we prove $|P(z)| > P(|z|)$ at a complex point, which forces a negative coefficient. We formalise the counterexample theorem in Lean, taking the cited estimate as a hypothesis.

AI assistance & Acknowledgements

ChatGPT-6 Astra found the counterexample after about two hours of reasoning while trying to deduce the q-rious positivity conjecture from the results in [5]. The human authors carefully studied and verified the proof, and gave ChatGPT-6.1 Sol general guidance on rewriting it for readability. ChatGPT-6.1 Sol also provided the Lean formalisation. Gemini 4 Argon independently audited the manuscript and the Lean formalisation.

We plan to post a final version on arXiv soon.

1. Introduction

For $n \geq 1$, write $[n]_q = 1 + q + \cdots + q^{n-1}$ and $[n]_q! = \prod_{j=1}^{n} [j]_q$.

For tuples $\mathbf{a} = (a_i)_{i=1}^{u}$ and $\mathbf{b} = (b_j)_{j=1}^{v}$ of positive integers, set

$$P_{\mathbf{a},\mathbf{b}}(q) = \frac{\prod_{i=1}^{u} [a_i]_q!}{\prod_{j=1}^{v} [b_j]_q!}, \qquad \Delta_{\mathbf{a},\mathbf{b}}(x) = \sum_{i=1}^{u} \lfloor a_i x \rfloor - \sum_{j=1}^{v} \lfloor b_j x \rfloor.$$

Landau's condition is

$$\Delta_{\mathbf{a},\mathbf{b}}(x) \geq 0 \qquad (x \geq 0). \tag{1}$$

If (1) holds, factoring the q-factorials into cyclotomic polynomials gives $P_{\mathbf{a},\mathbf{b}} \in \mathbb{Z}[q]$. Warnaar and Zudilin conjectured that all its coefficients are then nonnegative [1, Conjecture 1].

For $m \geq 1$, let $B_m(q) = [2m]_q!/([m]_q!)^2$. Write μ for the Möbius function, with $\mu(1) = 1$, $\mu(m) = 0$ when a prime square divides m, and $\mu(m) = (-1)^a$ when m is a product of a distinct primes. For an integer $k \geq 1$, let

$$N = 2^k, \qquad H = 4kN \operatorname{lcm}(1, \ldots, N).$$

Define

$$P_k(q) = B_1(q)^H \prod_{m=1}^{N} B_m(q)^{-H\mu(m)/m} \prod_{m=1}^{4N} B_m(q)^{H/(4kN)}. \tag{2}$$

Let e_m be the sum of the exponents of B_m in (2). It is an integer because $m \mid H$ for $m \leq N$ and $4kN \mid H$. If $e_m \geq 0$, put e_m copies of $2m$ in the numerator tuple and $2e_m$ copies of m in the denominator tuple. If $e_m < 0$, put $-e_m$ copies of $2m$ in the denominator and $-2e_m$ copies of m in the numerator. Call the two tuples $\mathbf{a}_k$ and $\mathbf{b}_k$. Their ratio is P_k, and their entries have the same sum because $2m = m + m$.

Theorem 1.1. *For every sufficiently large integer k, the tuples $\mathbf{a}_k, \mathbf{b}_k$ satisfy (1), and P_k is a polynomial in $\mathbb{Z}[q]$ with a negative coefficient.*

2. Landau's condition

Proposition 2.1. *For all sufficiently large k, $\Delta_{\mathbf{a}_k,\mathbf{b}_k}(x) \geq 0$ for every real x.*

Write $\{x\} = x - \lfloor x \rfloor$. The floor difference for B_m is $\lfloor 2mx \rfloor - 2\lfloor mx \rfloor = b(mx)$, where

$$b(x) = \lfloor 2x \rfloor - 2\lfloor x \rfloor = \lfloor 2\{x\} \rfloor = \begin{cases} 0, & 0 \leq \{x\} < 1/2, \\ 1, & 1/2 \leq \{x\} < 1. \end{cases}$$

Write $\mathbf{1}_E$ for 1 if E holds and 0 otherwise. In (2), $e_m/H = \mathbf{1}_{m=1} - \mathbf{1}_{m\leq N}\mu(m)/m + 1/(4kN)$ for $1 \leq m \leq 4N$. Thus

$$\frac{\Delta_{\mathbf{a}_k,\mathbf{b}_k}(x)}{H} = \underbrace{b(x) - \sum_{m=1}^{N} \frac{\mu(m)}{m} b(mx)}_{A_N(x)} + \underbrace{\frac{1}{4kN} \sum_{m=1}^{4N} b(mx)}_{W_k(x)}. \tag{3}$$

The right side has period 1 because $b(mx + m) = b(mx)$ for integer m.

Let $\psi(t) = \{t\} - 1/2$. Set $B(t) = 0$ for integer t and $B(t) = \psi(t)$ otherwise. By de la Bretèche and Tenenbaum [2, p. 4, Eq. (1.7)] (which, in turn, is due to Davenport) with $A = 2$, there is a constant $C_2 \geq 0$ such that

$$\left| \frac{\sin(2\pi x)}{\pi} + \sum_{1\leq m\leq y} \frac{\mu(m)}{m} B(mx) \right| \leq \frac{C_2}{(\log y)^2} \qquad (y \geq 2,\ x \in \mathbb{R}). \tag{4}$$

Lemma 2.2 (The bound for ψ)**.** *Put $S_N(x) = \sum_{m=1}^{N} \mu(m)\psi(mx)/m$ and $M(n) = \sum_{m=1}^{n} \mu(m)/m$, with $M(0) = 0$. For $C = C_2/(\log 2)^2$ we have*

$$\left| S_N(x) + \frac{\sin(2\pi x)}{\pi} \right| \leq \frac{C}{k^2} \qquad (x \in \mathbb{R}). \tag{5}$$

$$|M(N)| \leq \frac{2C}{k^2}. \tag{6}$$

Proof. Apply (4) with $y = N$ at $x + h$ for $h > 0$. For fixed x, N, take

$$0 < h < \min_{1\leq m\leq N} \frac{\lfloor mx \rfloor + 1 - mx}{m}.$$

Then $m(x + h)$ lies strictly between $\lfloor mx \rfloor$ and $\lfloor mx \rfloor + 1$, so $B(m(x + h)) = \psi(mx) + mh$. Let $h \downarrow 0$ in the finite sum and use $\log N = k \log 2$ to get (5). At $x = 0$, $S_N(0) = -M(N)/2$, giving (6).

Lemma 2.3 (A bound for $A_N(x)$)**.** *With $V(x) = (2\sin(2\pi x) - \sin(4\pi x))/\pi$,*

$$|A_N(x) - b(x) - V(x)| \leq \frac{4C}{k^2}. \tag{7}$$

Proof. Using $b(x) = 1/2 + 2\psi(x) - \psi(2x)$ in (3) gives

$$A_N(x) = b(x) - \tfrac{1}{2}M(N) - 2S_N(x) + S_N(2x).$$

Subtract $b(x) + V(x)$. By (5) at x and $2x$, and by (6),

$$\begin{aligned} |A_N(x) - b(x) - V(x)| &\leq \tfrac{1}{2}|M(N)| + 2\left| S_N(x) + \frac{\sin(2\pi x)}{\pi} \right| + \left| S_N(2x) + \frac{\sin(4\pi x)}{\pi} \right| \\ &\leq (1 + 2 + 1)C/k^2. \end{aligned}$$

Lemma 2.4 (A lower bound for $W_k(x)$)**.** *If $1/(2N) \leq x < 1/2$, then $W_k(x) \geq 1/(6k)$.*

Proof. Fix $y \in \mathbb{R}$ and $1/4 \le t \le 3/4$. Put $u = \{y\}$. If $u \ge 1/2$, then $b(y) = 1$. If $1/2 \le u + t < 1$, then $b(y + t) = 1$. For $u < 1/2$, the other two cases give

$$\begin{aligned} u + t < 1/2 &\implies 1/2 \le 2t \le u + 2t \le 2(u + t) < 1, \\ u + t \ge 1 &\implies t > 1/2, \qquad 3/2 < 1 + t \le u + 2t < 2. \end{aligned}$$

The upper bound in the last line uses $u < 1/2$ and $2t \le 3/2$. In both cases $\{u + 2t\} \in [1/2, 1)$, so $b(y + 2t) = 1$.

Put $s = \lceil 1/(4x) \rceil$ and $t = sx$. Since $0 < 1/(4x) \le N/2$, we have $1 \le s \le N$. Also $1/4 \le t < 1/4 + x < 3/4$. At $y = mx$, the sum of the three values of b is at least 1. Sum this inequality for $1 \le m \le 4N - 2s$, so that $m + 2s \le 4N$. This gives

$$\begin{aligned} 2N \le 4N - 2s &\le \sum_{m=1}^{4N-2s} [b(mx) + b((m + s)x) + b((m + 2s)x)] \\ &= \sum_{j=1}^{4N-2s} b(jx) + \sum_{j=1+s}^{4N-s} b(jx) + \sum_{j=1+2s}^{4N} b(jx) \\ &\le 3 \sum_{j=1}^{4N} b(jx). \end{aligned}$$

For the last inequality, extend each sum to $1 \le j \le 4N$ using $b(jx) \ge 0$. Divide by $12kN$ to get $W_k(x) \ge 1/(6k)$.

Proof of Proposition 2.1. By periodicity, take $0 \le x < 1$.

- If $0 \le x < 1/(2N)$, then $0 \le mx \le Nx < 1/2$ for $m \le N$. Hence $A_N(x) = 0$ and $W_k(x) \ge 0$.
- If $1/(2N) \le x < 1/2$, then $b(x) = 0$ and $\pi V(x) = 2 \sin(2\pi x)(1 - \cos(2\pi x)) \ge 0$. By (7) and Lemma 2.4, $A_N(x) + W_k(x) \ge 1/(6k) - 4C/k^2 \ge 0$ when $k \ge 24C$.
- If $1/2 \le x < 1$, then $b(x) = 1$ and $V(x) \ge -3/\pi$. Thus $A_N(x) + W_k(x) \ge 1 - 3/\pi - 4C/k^2 \ge 0$ when $k^2 \ge 4C/(1 - 3/\pi)$, since $\pi > 3$.

3. A test for negativity

Proposition 3.1. *If (1) holds, then $P_{\mathbf{a},\mathbf{b}} \in \mathbb{Z}[q]$. If also $|P_{\mathbf{a},\mathbf{b}}(z)| > P_{\mathbf{a},\mathbf{b}}(|z|)$ for some $z \in \mathbb{C}$, then $P_{\mathbf{a},\mathbf{b}}$ has a negative coefficient.*

Proof. The cyclotomic factorization of $q^j - 1$ gives $[j]_q = \prod_{d|j,\ d\ge 2} \Phi_d(q)$ after division by $q - 1$. Counting the multiples of d up to n gives

$$[n]_q! = \prod_{d=2}^{n} \Phi_d(q)^{\lfloor n/d \rfloor}, \qquad P_{\mathbf{a},\mathbf{b}}(q) = \prod_{d\ge 2} \Phi_d(q)^{\Delta_{\mathbf{a},\mathbf{b}}(1/d)}.$$

In the product for $P_{\mathbf{a},\mathbf{b}}$, the exponents are zero for d larger than every a_i and b_j. All exponents are nonnegative by (1). If every coefficient p_j of a polynomial P is nonnegative, then $|P(z)| \le \sum_j p_j |z|^j = P(|z|)$ by the triangle inequality.

4. The logarithm of $|P_k(z)|/P_k(r)$

Fix $r = 101/102$ and $z = (99 + 20i)/102$. Since $99^2 + 20^2 = 101^2$, we have $|z| = r < 1$. For $j \ge 1$, $|z^j| = r^j < 1$, so $1 - z^j \ne 0$. Also $1 - r^j > 0$. Thus $B_m(z) \ne 0$ and $B_m(r) > 0$ for every $m \ge 1$. Here log denotes the natural logarithm.

For integers $j, m \geq 1$, put

$$c_j = \log \frac{|1 - z^j|}{1 - r^j}, \qquad d_m = \log \frac{|B_m(z)|}{B_m(r)}.$$

Using $[j]_q = (1 - q^j)/(1 - q)$, the factors $1 - q$ cancel in B_m. Hence

$$d_m = \sum_{j=1}^{2m} c_j - 2 \sum_{j=1}^{m} c_j. \tag{8}$$

Since $B_1(q) = 1 + q$,

$$d_1 = \frac{1}{2} \log \frac{201^2 + 20^2}{203^2} = \frac{1}{2} \log \frac{40801}{41209}. \tag{9}$$

Taking logarithms in (2) gives

$$L_k = \frac{1}{H} \log \frac{|P_k(z)|}{P_k(r)} = d_1 - \sum_{m=1}^{N} \frac{\mu(m)}{m} d_m + \frac{1}{4kN} \sum_{m=1}^{4N} d_m. \tag{10}$$

Lemma 4.1 (Bounds for c_j and d_m)**.** *For $j, n, m \geq 1$,*

$$0 \leq c_j \leq \frac{2r^j}{1 - r^j} \leq 204 r^{j-1}, \qquad \sum_{j=1}^{n} c_j \leq 20808, \qquad |d_m| \leq 62424. \tag{11}$$

Proof. The triangle and reverse triangle inequalities give $1 - r^j \leq |1 - z^j| \leq 1 + r^j$. Divide by $1 - r^j$ and take logarithms to get

$$0 \leq c_j \leq \log \left(1 + \frac{2r^j}{1 - r^j} \right) \leq \frac{2r^j}{1 - r^j} \leq 204 r^j \leq 204 r^{j-1}.$$

Here $\log(1 + u) \leq u$ for $u \geq 0$ and $1 - r^j \geq 1 - r = 1/102$. Thus $\sum_{j=1}^{n} c_j \leq 204/(1 - r) = 20808$. By (8), $|d_m| \leq 3 \cdot 20808$.

Lemma 4.2 (The coefficient of c_j)**.** *Set*

$$h_{N,j} = 2M\big(\min(N, j - 1)\big) - M\big(\min(N, \lfloor (j - 1)/2 \rfloor)\big),$$
$$h_j = 2M(j - 1) - M(\lfloor (j - 1)/2 \rfloor).$$

Then

$$\sum_{m=1}^{N} \frac{\mu(m)}{m} d_m = -M(N) \sum_{j=1}^{2N} c_j + \sum_{j=1}^{2N} h_{N,j} c_j. \tag{12}$$

Moreover $h_{N,j} = h_j$ for $j \leq N$, and $|h_{N,j}|, |h_j| \leq 3j$.

Proof. In (8), the first sum contains c_j if $2m \geq j$, and the second contains c_j if $m \geq j$. For $1 \leq j \leq 2N$, the coefficient of c_j on the left of (12) is

$$\begin{aligned} \sum_{\substack{1 \leq m \leq N \\ 2m \geq j}} \frac{\mu(m)}{m} - 2 \sum_{\substack{1 \leq m \leq N \\ m \geq j}} \frac{\mu(m)}{m} &= M(N) - M\big(\min(N, \lfloor (j - 1)/2 \rfloor)\big) \\ &\quad - 2\big(M(N) - M(\min(N, j - 1))\big) \\ &= -M(N) + h_{N,j}. \end{aligned}$$

Empty sums equal zero. If $j \leq N$, the two minima are $j - 1$ and $\lfloor (j - 1)/2 \rfloor$, so $h_{N,j} = h_j$. Since $|M(n)| \leq \sum_{m=1}^{n} |\mu(m)|/m \leq n$, both $|h_{N,j}|$ and $|h_j|$ are at most $2(j - 1) + \lfloor (j - 1)/2 \rfloor \leq 3j$.

Proposition 4.3 (A bound after J terms)**.** *For $J \geq 1$, define*

$$\begin{aligned} F_J &= d_1 - \sum_{j=1}^{J} h_j c_j, \\ T_J &= 612 \sum_{j=J+1}^{\infty} j r^{j-1} = 612 r^J \left(\frac{J+1}{1-r} + \frac{r}{(1-r)^2} \right), \end{aligned} \tag{13}$$

and put $\varepsilon_k = 41616C/k^2 + 62424/k$. If $N \geq J$, then $|L_k - F_J| \leq T_J + \varepsilon_k$.

Proof. Put $j = J + 1 + \ell$ in the tail sum. Then

$$\sum_{j=J+1}^{\infty} j r^{j-1} = r^J \left((J+1) \sum_{\ell=0}^{\infty} r^\ell + \sum_{\ell=0}^{\infty} \ell r^\ell \right),$$

$$\sum_{\ell=0}^{\infty} \ell r^\ell = \sum_{a=1}^{\infty} \sum_{\ell=a}^{\infty} r^\ell = \frac{r}{(1-r)^2}.$$

The double sum counts r^ℓ exactly ℓ times. Using $\sum_{\ell \geq 0} r^\ell = 1/(1-r)$ gives (13). Substituting (12) into (10) and using $h_{N,j} = h_j$ for $j \leq J$ gives

$$L_k - F_J = -\sum_{j=J+1}^{2N} h_{N,j} c_j + M(N) \sum_{j=1}^{2N} c_j + \frac{1}{4kN} \sum_{m=1}^{4N} d_m.$$

Since $|h_{N,j} c_j| \leq 612 j r^{j-1}$, the first term has absolute value at most T_J. By (6) and (11), the absolute values of the second and third terms are at most $(2C/k^2) \cdot 20808$ and $62424/k$, respectively.

Corollary 4.4 (The limit of L_k)**.** *The series $\sum_{j \geq 1} h_j c_j$ converges absolutely, and*

$$\lim_{k \to \infty} L_k = d_1 - \sum_{j=1}^{\infty} h_j c_j. \tag{14}$$

Proof. Since $\sum_{j>J} |h_j c_j| \leq T_J$ and $T_J \to 0$, the series converges absolutely. For $2^k \geq J$, Proposition 4.3 bounds the absolute difference between L_k and the right side of (14) by $2T_J + \varepsilon_k$. Given $\eta > 0$, choose J with $2T_J < \eta/2$. For all sufficiently large k, $2^k \geq J$ and $\varepsilon_k < \eta/2$.

5. A numerical check

Proposition 5.1. *At $J = 2048$, $F_J - T_J > 9/100$.*

Proof. Write $a_j + i b_j = (99 + 20i)^j$. Then

$$X_j = \frac{|1 - z^j|^2}{(1 - r^j)^2} = \frac{(102^j - a_j)^2 + b_j^2}{(102^j - 101^j)^2}.$$

Since $2c_j = \log X_j$, we have

$$2F_J = \log(40801/41209) + \sum_{j=1}^{J} |h_j| \log Y_j, \qquad Y_j = \begin{cases} X_j, & h_j < 0, \\ X_j^{-1}, & h_j \geq 0. \end{cases}$$

For a positive fraction x, round x down to u at 40 significant digits. Python's `Decimal.ln` rounds $\log u$ to the nearest representable number at that precision [4]. Take the next smaller number for `ln_lower(x)`, or use 0 if $u = 1$. This gives $\texttt{ln_lower}(x) \leq \log u \leq \log x$. Use exact fractions for the other operations. Since every $|h_j| \geq 0$, the following calculation gives a lower bound for F_J and checks that it exceeds $T_J + 9/100$.

```
lower = ln_lower(40801/41209) / 2
for j = 1, ..., 2048:
    lower += abs(h_j) * ln_lower(Y_j) / 2
```

```
check lower - T_J > 9/100
```

The code is in `verify-lean.md`. Rounding the lower bound down and the upper bound up to six decimal places gives

$$F_{2048} \geq 0.326718, \qquad T_{2048} \leq 0.231640, \qquad F_{2048} - T_{2048} \geq 0.095078 > 9/100. \tag{15}$$

Proof of Theorem 1.1. For sufficiently large k, we have $2^k \geq 2048$ and $\varepsilon_k < 9/100$. By Propositions 4.3 and 5.1,

$$L_k \geq F_{2048} - T_{2048} - \varepsilon_k > 9/100 - \varepsilon_k > 0$$

Since $H > 0$, this gives $|P_k(z)| > P_k(r)$. By Proposition 2.1, (1) holds for sufficiently large k. Proposition 3.1 now gives $P_k \in \mathbb{Z}[q]$ and a negative coefficient.

6. Unimodality and the Lean proof

Corollary 6.1. *For every sufficiently large k, $(1+q)P_k(q)$ fails to be unimodal, giving a counterexample to [3, Conjecture 5].*

Proof. The identity $[n]_{1/q} = q^{1-n}[n]_q$ gives $q^D P_k(1/q) = P_k(q)$, where D is the numerator degree minus the denominator degree. Since $[n]_0 = 1$, we have $P_k(0) = 1$. Hence $D = \deg P_k$ and $p_j = p_{D-j}$. Set $p_j = 0$ outside $0 \leq j \leq D$. The coefficients $a_j = p_j + p_{j-1}$ of $(1+q)P_k$ are also symmetric. If they were unimodal, then $a_j \geq a_{j-1}$ for $1 \leq j \leq \lfloor D/2 \rfloor$, so

$$p_j - p_{j-2} = a_j - a_{j-1} \geq 0 \qquad (1 \leq j \leq \lfloor D/2 \rfloor).$$

Apply $p_j \geq p_{j-2}$ separately for even and odd j, starting with $p_0 = 1$ and $p_{-1} = 0$. This gives $p_j \geq 0$ for $0 \leq j \leq \lfloor D/2 \rfloor$. By symmetry, every coefficient is nonnegative, contrary to Theorem 1.1. This is the argument of [3, Lemma 6].

Lean proof. In `QRiousPositivity.lean`, `paper_theorem` proves Theorem 1.1 from (4), which is the $A = 2$ case of [2, Eq. (1.7)]. The theorem `gap_after_terms` proves the inequality in Proposition 4.3. Lean numbers the terms from 0, so c_j has index $j - 1$. For (15), Lean bounds the natural logarithms by sums of rational numbers and checks the arithmetic with `native_decide`, which uses `Lean.trustCompiler`. We checked the file with Lean and mathlib 4.24.0.

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