Octahedron Comparison in Geodesic CAT(0) Spaces

An exact computer-assisted proof candidate

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Status. This manuscript presents a complete computer-assisted exact proof candidate. Generative-AI systems made substantial contributions to mathematical idea and proof generation, verifier and code generation, exposition, and adversarial review. All referenced exact verifiers pass. A separate generative-AI research agent conducted the 28 July 2026 clean-room adversarial audit, independently verified the eleven rank-three phase-one families, and reported no material gap. Here "independent" means separated from the construction workflow; it does not mean independent human verification. This does not constitute journal peer review or proof-assistant verification. The status line is part of the mathematical claim and should not be removed before such review.

Abstract

Let O_3 be the one-skeleton of the octahedron. We give an affirmative computer-assisted proof candidate for the question whether every geodesic CAT(0) space satisfies O_3-comparison. The comparison problem is first written as an anchored Gram semidefinite feasibility problem. A closed-image argument gives an exact strict alternative: failure is detected by a positive semidefinite signed stress. Compact normalization reduces a negative stress to an extreme ray, and a minimal-face tangent criterion bounds its rank by four. Ranks one and four admit solver-free proofs. Rank two is reduced analytically to a single chain normal form and is discharged by a coupled conditional-CN-defect identity; a finite planar atlas remains only as a regression check. Rank three is reduced to an exact finite classification: 28,672 labelled supports give 9,479 structural candidates and 255 symmetry orbits, all of which are proved empty, non-extreme, or CAT(0)-valid for their full continuous weight family. The final retained count is zero. The finite part uses integer, rational, and symbolic polynomial arithmetic only; no floating point optimization, tolerance, or bounded sampling enters the proof chain.

1 Introduction

For a graph Γ, a metric space meets Γ-comparison if every configuration labelled by the vertices of Γ has a Hilbert-space model in which graph edges do not get longer and nonedges do not get shorter. The case $\Gamma = O_3$, the octahedral graph, was posed by Petrunin as an open problem; see [1]. Products of trees satisfy this comparison [2], while genuinely six-point CAT(0) inequalities are known to contain information not visible on five-point subsets [3].

Let

$$V = \{0, 1, 2, 3, 4, 5\}, \qquad M = \{01, 23, 45\}, \qquad E = \binom{V}{2} \setminus M.$$

Thus M is the matching of the three octahedral diagonals and E is the edge set of O_3.

Theorem 1.1 (Main theorem candidate)**.** *Let X be a geodesic* CAT(0) *space and let $x_0, \dots, x_5 \in X$. There are points $p_0, \dots, p_5$ in a Hilbert space such that*

$$\|p_i - p_j\| \le d(x_i, x_j) \quad (ij \in E), \qquad \|p_i - p_j\| \ge d(x_i, x_j) \quad (ij \in M).$$

Consequently every geodesic CAT(0) *space meets O_3-comparison.*

The proof has two deliberately separated parts. Sections 2–6 contain the convex reduction and the solver-free rank-one, rank-four, and rank-two mathematics. Section 7 states and proves the exact computer-assisted rank-three classification theorem. Section 9 records the audit boundary: which assertions are regenerated by the master verifier and which are proved by pinned dependency theorems.

2 The exact semidefinite alternative

Put $\delta_{ij} = d(x_i, x_j)^2$. Anchor $p_0 = 0$, let $G \in \mathbb{S}^5_+$ be the Gram matrix of $p_1, \dots, p_5$, and set $r_0 = 0$, $r_a = e_a$ for $1 \le a \le 5$. For

$$B_{ij} = (r_i - r_j)(r_i - r_j)^\mathsf{T},$$

the squared Hilbert distance is $\langle B_{ij}, G\rangle$. The desired comparison is therefore equivalent to

$$\langle B_e, G\rangle \le \delta_e \quad (e \in E), \qquad \langle B_m, G\rangle \ge \delta_m \quad (m \in M), \qquad G \succeq 0. \tag{1}$$

For $i, j \in V$, write $b_{ij} = e_i - e_j \in \mathbb{R}^6$. Define the signed stress cone

$$\mathcal{C} = \{\Omega \succeq 0 : \Omega\mathbf{1} = 0, \quad \Omega_{ij} \le 0 \ (ij \in E), \quad \Omega_{ij} \ge 0 \ (ij \in M)\}. \tag{2}$$

Every $\Omega \in \mathcal{C}$ has the unique multiplier representation

$$\Omega = \sum_{e \in E} \alpha_e \, b_e b_e^\mathsf{T} - \sum_{m \in M} \beta_m \, b_m b_m^\mathsf{T}, \qquad \alpha_e, \beta_m \ge 0, \tag{3}$$

because $\alpha_e = -\Omega_e$ and $\beta_m = \Omega_m$ in the fifteen off-diagonal coordinates. For a six-tuple x, put

$$\Phi_\Omega(x) = \sum_{i<j} \Omega_{ij} d(x_i, x_j)^2 = -\sum_{e \in E} \alpha_e \delta_e + \sum_{m \in M} \beta_m \delta_m. \tag{4}$$

Proposition 2.1 (Strict exact alternative)**.** *Exactly one of the following holds:*

1. *the Gram feasibility problem* (1) *has a solution;*
2. *there is $\Omega \in \mathcal{C}$ for which $\Phi_\Omega(x) > 0$.*

Proof. The only point not supplied by formal semidefinite separation is closedness of the relevant linear image of $\mathbb{S}^5_+$ plus nonnegative slack. The twelve upper-bound pairs E form the connected graph O_3. Hence a sequence of model configurations with bounded squared lengths on E has all pairwise model distances bounded, by fixed graph paths and the triangle inequality. After anchoring $p_0 = 0$, every Gram entry is then bounded by the polarization identity. Thus every convergent sequence in the projected image-plus-slack cone has a convergent Gram subsequence, proving closedness.

Strong separation now gives nonnegative multipliers satisfying (3) and

$$\sum_{e \in E} \alpha_e \delta_e - \sum_{m \in M} \beta_m \delta_m < 0.$$

This is precisely $\Phi_\Omega(x) > 0$ by (4). Conversely, taking the inner product of (3) with any feasible Gram matrix shows that such a strict inequality is incompatible with (1). $\square$

3 Extreme stresses and the rank reduction

Normalize a nonzero stress by

$$\sum_{e\in E}\alpha_e+\sum_{m\in M}\beta_m=1.$$

Uniqueness of the multiplier coordinates makes the normalized section a closed subset of a simplex, hence compact. If $\Phi_\Omega(x)>0$ anywhere, it attains a positive maximum on an exposed face, which contains an extreme point. Therefore a failed comparison has a positive extreme stress certificate.

Write a rank-r stress as $\Omega=UU^\top$, where $U^\top\mathbf{1}=0$, and let $Z(\Omega)=\{ij:\Omega_{ij}=0\}$.

Lemma 3.1 (Minimal-face extremality criterion)**.** *The ray $\mathbb{R}_+\Omega$ is extreme in $\mathcal{C}$ if and only if*

$$\Big\{H\in \mathrm{Sym}_r : (UHU^\top)_{ij}=0 \text{ for all } ij\in Z(\Omega)\Big\}=\mathbb{R}I_r. \tag{5}$$

Proof. Every PSD summand of Ω lies in its minimal PSD face and is therefore of the form $UHU^\top$. At a zero sign coordinate, two sign-compatible summands adding to zero must both vanish. This proves necessity. Conversely, a nonscalar solution of the homogeneous zero equations gives, for sufficiently small $\varepsilon>0$, the two-sided sign-preserving perturbations $U(I_r\pm\varepsilon H)U^\top$, contradicting extremality. □

Since $\dim \mathrm{Sym}_r=r(r+1)/2$, an extreme ray requires at least $r(r+1)/2-1$ independent zero equations. Therefore

$$|\operatorname{supp}\Omega|\le 15-\left(\frac{r(r+1)}{2}-1\right)=16-\frac{r(r+1)}{2}. \tag{6}$$

For $r=5$, the support contains at most one pair, but a zero-row-sum stress supported on one pair has rank at most one. Thus every extreme certificate relevant to Proposition 2.1 has rank at most four.

4 Basic CAT(0) defect calculus

Let $\gamma_{xy}(t)$, $0\le t\le 1$, be the constant-speed geodesic from x to y. Define the CN defect

$$\Delta_t(z;x,y)=(1-t)d(z,x)^2+td(z,y)^2-t(1-t)d(x,y)^2-d(z,\gamma_{xy}(t))^2. \tag{7}$$

The CAT(0) inequality says $\Delta_t\ge 0$.

Lemma 4.1 (Defect transfer)**.** *For fixed $(z;x,y)$, the function $t\mapsto\Delta_t(z;x,y)$ is concave and vanishes at 0 and 1. Consequently, if $0<r\le s<1$, then*

$$\Delta_r\le\frac{1-r}{1-s}\Delta_s,\qquad \Delta_s\le\frac{s}{r}\Delta_r. \tag{8}$$

Proof. For $0\le u<v\le 1$ and $0\le\theta\le 1$, the CAT(0) inequality on the subsegment from $\gamma_{xy}(u)$ to $\gamma_{xy}(v)$ gives

$$d(z,\gamma_{xy}((1-\theta)u+\theta v))^2\le(1-\theta)d(z,\gamma_{xy}(u))^2+\theta d(z,\gamma_{xy}(v))^2-\theta(1-\theta)(v-u)^2d(x,y)^2.$$

The quadratic comparison term in the definition of Δ satisfies the same interpolation formula with equality. Subtraction therefore shows that $t\mapsto\Delta_t(z;x,y)$ is concave, with $\Delta_0=\Delta_1=0$. Comparing its chord slopes on $[0,r]$, $[r,s]$, and $[s,1]$ gives the two inequalities. □

We record the exact identity used in rank two. Let $c, e \geq 0$ and $d > 0$ have total $m = c + d + e$, let $a, b \geq 0$ satisfy $a + b = m$, and put

$$s = \frac{d+e}{m}, \quad t = \frac{c}{m}, \quad r = \frac{e}{d+e}, \quad q = \frac{b}{m}.$$

Set

$$c_{45} = \gamma_{x_4 x_5}(r), \qquad z = \gamma_{c_{45} x_3}(t), \qquad b_{01} = \gamma_{x_0 x_1}(q).$$

Lemma 4.2 (Conditional $2+3$ identity)**.** *For $v = (a, b, 0, -c, -d, -e)$, one has the exact identity*

$$\begin{aligned}\frac{\Phi_{vv^\mathsf{T}}}{m^2} &= st\Delta_r(x_3; x_4, x_5) - s\frac{a}{m}\Delta_r(x_0; x_4, x_5) - \frac{a}{m}\Delta_t(x_0; c_{45}, x_3) \\ &\quad - s\frac{b}{m}\Delta_r(x_1; x_4, x_5) - \frac{b}{m}\Delta_t(x_1; c_{45}, x_3) \\ &\quad - \Delta_q(z; x_0, x_1) - d(z, b_{01})^2. \end{aligned} \tag{9}$$

The same identity holds after arbitrary relabelling.

Proof. Insert definition (7) into the right side. The coefficients of all distances involving c_{45}, z, b_{01} cancel. The coefficient of $d(x_i, x_j)^2$ is then $v_i v_j / m^2$ for each $i < j$. This is a formal identity; its expansion is also checked coefficient-by-coefficient by the exact rank-two verifier. □

5 Ranks one and four

5.1 Rank one

The rank-one sign conditions imply, after an O_3-relabeling and an overall factor sign choice,

$$\Omega = vv^\mathsf{T}, \qquad v = (a_1, a_2, -b_1, -b_2, 0, 0), \qquad a_i, b_j \geq 0, \qquad a_1 + a_2 = b_1 + b_2.$$

Indeed, two nonzero coordinates of the same sign have positive product and therefore must form one of the three matching pairs. Consequently each sign class has size at most two, with zero coordinates added as necessary. The two-measure variance inequality in a CAT(0) space gives

$$\sum_{i,j} a_i b_j d(x_i, y_j)^2 \geq a_1 a_2 d(x_1, x_2)^2 + b_1 b_2 d(y_1, y_2)^2.$$

The right side minus the left side is Φ_Ω, so every rank-one stress satisfies $\Phi_\Omega \leq 0$.

5.2 Rank four

Theorem 5.1 (Rank-four classification)**.** *Every rank-four extreme ray is a Hamiltonian-path stress*

$$\Omega_P = \sum_{i=1}^{5} c_i b_{v_{i-1}v_i} b_{v_{i-1}v_i}^\mathsf{T} - \frac{1}{R} b_{v_0 v_5} b_{v_0 v_5}^\mathsf{T}, \qquad R = \sum_{i=1}^{5} \frac{1}{c_i}, \qquad c_i > 0, \tag{10}$$

where $(v_0, \ldots, v_5)$ is a permutation of V, every consecutive pair lies in E, and $v_0 v_5 \in M$. Conversely every stress (10) *spans a rank-four extreme ray.*

Proof. An extreme stress has one active positive-support component. Rank four forces all six vertices to be active: with five active vertices there are too few nontrivial zero equations to cut Sym_4 to the scalar line. The support bound (6), connectedness, and the presence of a negative pair force exactly five positive edges and one negative pair. Hence the positive support is a spanning tree T.

For the active negative pair st, the one-column Schur complement gives $\lambda R^T_{\mathrm{eff}}(s,t) \le 1$; rank four forces equality. Edges off the unique s-t path split off as a nonzero positive Laplacian, which contradicts extremality. Thus the tree is the Hamiltonian s-t path and the stress has form (10).

Weighted Cauchy shows that the kernel of (10) is spanned by the constant vector and the voltage vector $\phi_{v_j} = \sum_{i \le j} c_i^{-1}$, so its rank is four. If $0 \preceq A \preceq \Omega_P$ has the same sign support, then $\ker \Omega_P \subseteq \ker A$. Applying $A\phi = 0$ at successive internal vertices forces every path conductance of A to be the same scalar multiple of c_i, and the endpoint equation forces the negative coefficient to have the same scalar. Hence A is proportional to Ω_P, proving the converse extremality. $\square$

For points in any metric space, triangle inequality and weighted Cauchy give

$$\frac{1}{R} d(v_0, v_5)^2 \le \sum_{i=1}^{5} c_i d(v_{i-1}, v_i)^2.$$

Thus $\Phi_{\Omega_P} \le 0$ without using the CAT(0) hypothesis.

6 All rank-two extreme stresses

Let $\Omega = UU^\mathsf{T}$ have rank two and row vectors $u_i \in \mathbb{R}^2$. Lemma 3.1 says that the active zero forms $u_i^\mathsf{T} H u_j$ must span the two-dimensional dual of $\mathrm{Sym}_2 / \mathbb{R} I_2$.

Theorem 6.1 (Exact planar normal form)**.** *Up to* $\mathrm{Aut}(O_3)$*, every rank-two extreme stress admits the active chain zeros*

$$\Omega_{12} = \Omega_{34} = 0.$$

Every such stress admits a Schur chart

$$\Omega = P + Q, \qquad P = \kappa p p^\mathsf{T}, \quad Q = \lambda r r^\mathsf{T}, \tag{11}$$

with

$$p = (a, b, 0, -c, -d, -e), \qquad\qquad r = (A, 0, -B, -C, D, E), \tag{12}$$

where

$$\begin{gathered} a, e, A, E \ge 0, \qquad b, c, d, B, C, D, \kappa, \lambda > 0, \\ a + b = c + d + e, \qquad A + D + E = B + C, \\ \lambda CD = \kappa cd, \qquad \kappa ce - \lambda CE \le 0, \qquad -\kappa ae + \lambda AE \le 0. \end{gathered} \tag{13}$$

No other rank-two extreme cell occurs. Independently, the accompanying finite atlas enumerates 194 labelled feasible extreme zero patterns in nine $\mathrm{Aut}(O_3)$*-orbits; these counts are regression data, not the source of the universal normal-form claim.*

Proof. Two independent active zero pairs are necessary. The 105 unordered choices split into seven orbits under the 48-element group $(C_2)^3 \rtimes S_3$. An orbit containing a matching zero is excluded by the planar quadrant signs: after placing the two nonzero orthogonal rows on the coordinate axes, every other row lies in the same closed opposite quadrant, so every active-zero form is proportional to the off-diagonal coordinate functional. A shared-vertex pair gives proportional tangent forms because both counterpart rows lie in the same one-dimensional orthogonal complement. The remaining crossing orbit is excluded by writing $u_2 = \alpha J u_0$, $u_3 = \beta J u_1$ and using

$$(u_0 \cdot u_3)(u_1 \cdot u_2) = -\alpha\beta \det(u_0, u_1)^2.$$

If $u_0 \cdot u_1 = 0$, this reduces to the matching-zero case. Otherwise the matching signs give $\alpha\beta > 0$, while the two edge signs make the left side nonnegative. Thus the determinant vanishes and the two active forms are again proportional. Only the chain orbit survives; relabel its zeros as $\Omega_{12} = \Omega_{34} = 0$.

The rows u_1, u_2 and u_3, u_4 are nonzero orthogonal frames. If any of $\Omega_{13}, \Omega_{14}, \Omega_{23}, \Omega_{24}$ were zero, the two frames would share an axis, making their active-zero forms proportional. Hence the signed cone makes these four entries respectively strict negative, negative, positive, negative.

Choose orthonormal coordinates with

$$u_1 = (b, 0), \quad u_2 = (0, -B), \qquad b, B > 0.$$

The four strict cross signs and the signs at $01, 02, 15, 25$ then give

$$\begin{aligned} &u_3 = (-c, -C), \quad u_4 = (-d, D), \quad c, d, C, D > 0, \\ &u_0 = (a, A), \qquad\quad u_5 = (-e, E), \quad a, e, A, E \geq 0. \end{aligned}$$

Since $\Omega\mathbf{1} = 0$, $\|U^\mathsf{T}\mathbf{1}\|^2 = \mathbf{1}^\mathsf{T}\Omega\mathbf{1} = 0$, so

$$a + b = c + d + e, \qquad A + D + E = B + C.$$

The active zero 34 gives $CD = cd$, while the edge signs at 35 and 05 give $CE - ce \geq 0$ and $ae - AE \geq 0$. Thus $\Omega = pp^\mathsf{T} + rr^\mathsf{T}$ has (12)–(13) with $\kappa = \lambda = 1$; positive rescaling of the coordinate columns gives the displayed general κ, λ. This derivation includes inactive rows and equality faces and uses no finite direction grid.

For regression only, the exact direction-cell enumerator places the weak phase classes in six formal slots and recovers 194 labelled patterns in nine orbits. The universal assertion above does not rely on interpreting those slots as a complete sampling of the continuous parameter space. □

Theorem 6.2 (Rank-two validity). *Every rank-two extreme stress $\Omega \in \mathcal{C}$ satisfies $\Phi_\Omega(x) \leq 0$ on every geodesic* CAT(0) *six-tuple.*

Proof. Apply Lemma 4.2 to P with common subpair $(4, 5)$ and third point 3, and to Q with the same subpair and third point 0. Put

$$r_P = \frac{e}{d+e}, \qquad r_Q = \frac{E}{D+E}.$$

After retaining only the four defects needed for coupling, the two exact identities give

$$\begin{aligned} &\Phi_P \leq \kappa c(d+e)\Delta_{r_P}(x_3; 4, 5) - \kappa a(d+e)\Delta_{r_P}(x_0; 4, 5) + N_P, \\ &\Phi_Q \leq \lambda A(D+E)\Delta_{r_Q}(x_0; 4, 5) - \lambda C(D+E)\Delta_{r_Q}(x_3; 4, 5) + N_Q, \end{aligned} \tag{14}$$

where $N_P, N_Q \leq 0$ are the remaining explicitly displayed negative defects and squared-distance term in (9).

Assume first $e > 0$. From the middle equality and first weak inequality in (13),

$$D(\lambda CE - \kappa ce) = \kappa c(dE - eD) \geq 0,$$

so $r_P \le r_Q$. Lemma 4.1 gives

$$\Delta_{r_P} \le \frac{d(D+E)}{D(d+e)}\Delta_{r_Q}.$$

The x_3-coefficients cancel exactly because

$$\kappa c(d+e)\frac{d(D+E)}{D(d+e)} = \lambda C(D+E).$$

In the reverse direction,

$$\Delta_{r_Q} \le \frac{E(d+e)}{e(D+E)}\Delta_{r_P},$$

and the available minus required x_0-coefficient is

$$\frac{d+e}{e}(\kappa ae - \lambda AE) \ge 0.$$

Thus all terms in $\Phi_P + \Phi_Q$ are nonpositive.

If $e = 0$, then $\Delta_{r_P} = \Delta_0 = 0$. The last inequality in (13) forces $A = 0$ or $E = 0$; in either case the only potentially positive Q-defect also vanishes. This includes the inactive and equality charts. Therefore $\Phi_\Omega \le 0$ in every atlas cell. □

7 The exact rank-three classification

For rank three, (6) gives $|\operatorname{supp}\Omega| \le 10$. An *exact support* is a pair (P, N), where $P \subseteq E$ and $\varnothing \ne N \subseteq M$ are precisely the strict positive multiplier sets in (3). Thus there are

$$2^{12}(2^3 - 1) = 28{,}672 \tag{15}$$

labelled exact supports before structural filtering.

Theorem 7.1 (Computer-assisted exact support classification)**.** *Among the supports in* (15)*, exact structural conditions leave 9,479 labelled candidates in 255* $\mathrm{Aut}(O_3)$*-orbits. Every one of these 255 orbits satisfies one of the following mutually sufficient conclusions:*

1. *the corresponding semialgebraic PSD rank-three support cell is empty;*
2. *the cell contains no extreme ray;*
3. *every stress in the full feasible continuous cell is* $\mathrm{CAT}(0)$*-valid.*

Consequently every rank-three extreme ray in $\mathcal{C}$ *is* $\mathrm{CAT}(0)$*-valid.*

Proof. The exact set flow is

$$255 \longrightarrow 157 \longrightarrow 155 \longrightarrow 51 \longrightarrow 17 \longrightarrow 15 \longrightarrow 0.$$

The stages and their certificate types are listed in Table 1. Each removal theorem is quantified over the full feasible weight family of its exact support; no unit witness or numerical sample removes a support.

For clarity, the two dedicated full-cell inputs in the $17 \to 15$ step are the following. Their complete symbolic identities and support contracts are included in the artifact map.

Stage	retained	exact reason for removed cells
Structural quotient	255	connectivity, active-size, rank/support conditions
First proof filters	157	one-demand, bridge, tree/path, cycle and C_4 lemmas
First family theorems	155	full `r3_0159`, `r3_0177` cells
Recursive classifier	51	exact pivot/Schur sign charts for 104 cells
Phase one	17	23 empty, 9 metric-valid, 2 non-extreme cells
Dedicated theorems	15	full `r3_0123`, `r3_0093` cells
Final partition	0	2 frame-valid and 13 non-extreme cells

Table 1: Exact rank-three support flow.

Lemma 7.2 (The `r3_0123` full cell)**.** *Every PSD rank-three zero-row-sum stress with strict negative off-diagonal support*

$$\{02, 03, 04, 12, 14, 15, 25, 35\}$$

and strict positive off-diagonal support $\{23, 45\}$ *is a positive sum of weighted quadrilateral atoms. Hence it is* CAT(0)*-valid.*

Certificate outline. Schur elimination at vertex 4 leaves a PSD rank-two residual $H = pp^{\mathsf{T}}/X + rr^{\mathsf{T}}/Y$. With $s = H_{15}$ and $L = bs + uh$, the three exhaustive charts

$$s \geq 0, \qquad s < 0,\ L \geq 0, \qquad s < 0,\ L \leq 0$$

give exact positive decompositions into $2 + 2$ vectors. The two latter matrix identities and their boundary cases are checked symbolically. □

Lemma 7.3 (The `r3_0093` full cell)**.** *Every PSD rank-three zero-row-sum stress with strict negative off-diagonal support*

$$\{02, 03, 04, 12, 13, 14, 25, 35\}$$

and strict positive off-diagonal support $\{01, 45\}$ *is a positive sum of four weighted quadrilateral atoms. Hence it is* CAT(0)*-valid.*

Certificate outline. The zero pairs make g_2, g_3, g_4 an orthogonal Gram basis. In $g_5^{\perp}$, the shared rays $g_2 \times g_5$ and $g_3 \times g_5$, together with either $g_0 \times g_5$ or $g_1 \times g_5$, form a positive planar frame. If

$$D_0 = \det(u_0, u_1), \quad L_0 = \det(u_0, q), \quad L_1 = \det(u_1, q),$$

the fourth factor is a quadrilateral atom whenever

$$D_0 L_0 > 0 \quad \text{or} \quad D_0 L_1 < 0,$$

with determinant-zero cases giving valid three-active-point degenerations. The determinant identity $D_0 q = -L_1 u_0 + L_0 u_1$ proves that these alternatives and their boundaries are exhaustive. □

All set operations, orbit representatives, rational factorizations, sign identities, tangent ranks, and per-family provenance are checked by exact integer/rational or symbolic-polynomial code. A clean-room enumerator independently regenerates (15), all 48 automorphisms, the 9,479 labelled structural candidates, and the 255 canonical orbits. The immutable machine-readable support-flow snapshot verifies that the removed and retained sets at each stage are disjoint and exhaustive.

It remains to explain the last fifteen, because this gives a short human- readable terminal certificate independent of the earlier chronological decompositions. They split as

$$15 = 2_{\text{frame valid}} + 2_{\text{triangle nonextreme}} + 11_{K_{2,2}\text{ nonextreme}}.$$

For `r3_0103` and `r3_0114`, the zero graph contains a signed Hamiltonian path. Four normal vectors give the positive-frame identity

$$I_3 = \sum_{ij \in \{AC,AD,BC,BD\}} \lambda_{ij} n_{ij} n_{ij}^{\mathsf{T}}, \qquad \lambda_{ij} > 0.$$

Multiplication by the Gram factor decomposes every stress in either full cell into four rank-one $2+2$ quadrilateral atoms, each valid by the CAT(0) quadrilateral inequality.

For `r3_0122` and `r3_0205`, three nonzero Gram rows form an orthogonal triangle. The zero equations first restrict $H \in \mathrm{Sym}_3$ to the three-dimensional diagonal subspace. There is at most one remaining zero equation with both endpoints outside the triangle, so the zero-preserving tangent space has dimension at least two. Lemma 3.1 rules out extremality. The nonzero hypothesis is guaranteed by strict incident support coefficients and is checked explicitly.

For each of the remaining eleven supports, the zero graph contains a $K_{2,2}$ between disjoint pairs P, Q, plus at most two further zero edges. If $U = \mathrm{span}\{u_p : p \in P\}$ and $W = \mathrm{span}\{u_q : q \in Q\}$, then $U \perp W$ in $\mathbb{R}^3$, so

$$\dim U \ \dim W \leq 2.$$

The four cross-zero equations therefore have joint rank at most two. With at most two extra zero equations, the zero-preserving subspace of Sym_3 has dimension at least $6-2-2=2$, again excluding an extreme ray. The eleven canonical support identifiers are

`r3_0011`, `r3_0013`, `r3_0015`, `r3_0023`, `r3_0027`, `r3_0080`,
`r3_0095`, `r3_0105`, `r3_0201`, `r3_0202`, `r3_0215`.

This exhausts the final fifteen and completes the exact set flow to zero. □

Remark 7.4 (Boundary rule)**.** Rank-three boundary cells are not inferred by closing a generic strict cell. When any multiplier vanishes, the stress belongs to a different exact support already present in the 28,672-support enumeration. The 255-orbit exhaustion therefore includes every coefficient boundary as its own support key.

Remark 7.5 (Relabelling rule)**.** The quotient uses the full 48-element group preserving the three opposite pairs. A theorem for a representative is transported by relabelling; no distance table is averaged over the group.

8 Proof of the main theorem

Assume Theorem 1.1 fails for a six-tuple in a geodesic CAT(0) space. Proposition 2.1 gives $\Omega \in \mathcal{C}$ with $\Phi_\Omega > 0$. Compact normalization gives such a stress on an extreme ray. Section 3 shows that its rank is one, two, three, or four. Rank one is excluded by the variance inequality; rank four by Theorem 5.1 and the metric path inequality; rank two by Theorem 6.2; and rank three by Theorem 7.1. In every case $\Phi_\Omega \leq 0$, a contradiction. Hence the Gram problem (1) is feasible, and a Gram factor gives the required Hilbert configuration.

9 Exact verification and reproducibility

The computational proof boundary is intentionally narrow and explicit.

Solver-free mathematical part

The following arguments do not depend on finite enumeration: the anchored Gram equivalence; connected-graph closedness; strict separation and compact extreme selection; Lemma 3.1; the rank bound; rank-one variance; the rank-four path classification; defect concavity and transfer; the conditional $2 + 3$ identity; the rank-two Schur defect coupling; and the terminal orthogonal-triangle and $K_{2,2}$ tangent theorems.

Exact finite part

The finite component generates the supports and 48 symmetries, applies the structural predicates, verifies the orbit quotient and exact set flow, checks rational Gram factorizations and symbolic identities, and validates the certificate provenance. It uses Python integers, exact rational arithmetic, and SymPy polynomial identities. It does not call a numerical SDP solver.

The active support-flow integration checker prints

```
PASS historical provenance pins (non-theorem)
PASS active theorem bindings
PASS immutable dependency hashes, manifests, and active premises
PASS clean-room support enumeration and rank-four path orbit
PASS clean-room 255 -> 157 proof-filter quotient
PASS immutable rank-three set flow 157 -> 155 -> 51 -> 17 -> 15 -> 0
PASS exact-support boundary and full Aut(O3) relabeling audit
VERDICT ACTIVE_SUPPORT_FLOW_INTEGRATION_PASS
```

The artifact map accompanying this draft records the entry points, expected output, and SHA-256 manifests. Historical provenance pins do not authenticate active premises. Acceptance requires every active theorem leaf, the semantic guard, and the root verifier to exit with status zero, followed by a passing manifest check.

10 Review status and residual epistemic risk

The v2 external adversarial audit found artifact-portability failures and two places where the encoded coverage was weaker than the prose: the continuous rank-two normal-form interface and the rank-three $155 \to 51$ reduction. Version 3 replaces the former by the analytic proof in Theorem 6.1 and adds an independent exact checker for the latter. It also incorporates the condition that the three rows in the orthogonal-triangle lemma are nonzero. Both applications satisfy and verify this condition.

Subsequent withheld-mutation audits found no mathematical counterexample, but did expose narrower artifact guards: deletion of the third `r3_0123` chart, replacement of the `r3_0093` chart disjunction by a conjunction, and sentinel-only or named-no-op verifier substitutions. The current package independently checks the two full-cell support contracts, exact feasible exclusive-chart witnesses, and normalized verifier syntax; it also locks the complete active and legacy-consumed leaf payload. Twenty-one named mutations are rejected, nine after coordinated manifest and active-pin regeneration. The trusted inventory is checked both before and after the leaf graph runs. This internal lock does not replace the externally communicated archive SHA-256.

The remaining risk is not a known counterexample but external validation of a long proof chain. A 28 July 2026 clean-room adversarial audit by a separate generative-AI research agent reported `NO_MATERIAL_GAP_FOUND` and `PHASE1_11_INDEPENDENTLY_VERIFIED`. It independently reconstructed the phase-one continuous-family mechanisms, actual support flow, and targeted exact theorem-chain identities. Its finite exact search found no positive obstruction among 72 stress rays on 55,611 Euclidean-grid tables and 90,465 metric-tree tables; this finite search is corroboration, not a proof of the universal theorem.

AI-assistance and audit disclosure

This project is a human-curated, substantially AI-assisted mathematical proof candidate. Generative-AI systems contributed to problem exploration, mathematical conjecture and proof development, exact and numerical code, LaTeX exposition, counterexample search, and adversarial criticism. The 28 July audit was assigned to a separate AI research agent that was not part of the construction workflow. Its "independence" records that workflow separation only; it is not qualified human peer review, institutional certification, provider endorsement, or proof-assistant verification. Agent verdicts and passing verifier output are evidence to be scrutinized, not a substitute for specialist mathematical review.

The remaining specialist review should now concentrate on:

1. a solver-free line-by-line review of connected-image-cone closedness and exact separation;
2. the minimal-face/extreme-stress and rigidity correspondence;
3. the CAT(0) variance, network, polygon, and defect-transfer lemmas;
4. an independent check that the formulation in Theorem 1.1 matches the current published statement of Petrunin's problem; and
5. a literature and priority review.

A The final fifteen rank-three supports

Support	terminal certificate	conclusion
`r3_0103`, `r3_0114`	four-normal positive frame	full family valid
`r3_0122`, `r3_0205`	nonzero orthogonal triangle	non-extreme
`r3_0011`, `r3_0013`, `r3_0015`	$K_{2,2}$+extras	non-extreme
`r3_0023`, `r3_0027`, `r3_0080`	$K_{2,2}$+extras	non-extreme
`r3_0095`, `r3_0105`	$K_{2,2}$+extras	non-extreme
`r3_0201`, `r3_0202`, `r3_0215`	$K_{2,2}$+extras	non-extreme

B Clean-room enumeration pseudocode

Let $P \subseteq E$ and $\varnothing \neq N \subseteq M$. The master enumerator performs the following exact procedure.

```
for positive_mask in range(2^12):
    P = selected O3 edges
    for negative_mask in range(1, 2^3):
        N = selected opposite pairs
        reject if |P|+|N| > 10
        reject unless every negative endpoint lies in one P-component
        reject unless exactly one component contains all active vertices
        reject unless active vertex and nontrivial-zero counts permit rank 3
        canonicalize (P,N) under all 48 Aut(O3) permutations
apply exact theorem filters to each canonical support
assert removed and retained sets are disjoint and exhaustive at every stage
assert final retained set is empty
```

The structural predicate is necessary only; every later removal carries a separate exact certificate of emptiness, non-extremality, or full-family validity.

References

[1] A. Petrunin, *Puzzles in geometry: Octahedron comparison*, open problem collection, `https://github.com/anton-petrunin/puzzles/blob/master/open.tex`.

[2] N. Lebedeva and A. Petrunin, *Trees meet octahedron comparison*, J. Topol. Anal. **17** (2025), no. 5, 1205–1209; `https://arxiv.org/abs/2212.06445`.

[3] T. Toyoda, *Inequalities on six points in a* CAT(0) *space*, arXiv:2411.13877, `https://arxiv.org/abs/2411.13877`.