Counting Crossings Among N Regular Polygons Inscribed in a Circle

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Abstract

Consider regular polygons $P_3, P_4, \ldots, P_N$ inscribed in a circle and aligned so that they share a common vertex. Counting the distinct proper crossings among their sides as N increases produces an integer sequence. We show that any two polygons P_m and P_k, with $3 \leq m < k$, have exactly

$$2\big(m - \gcd(m, k)\big)$$

proper side crossings. We further prove that no proper crossing is shared by sides from three or more polygons of distinct orders. It follows that for every integer $N \geq 3$, the total number of distinct proper crossing points is exactly

$$A(N) = 2 \sum_{3 \leq m < k \leq N} \big(m - \gcd(m, k)\big)$$

We also give an arithmetic reformulation of the sequence in terms of Pillai's arithmetical function. Finally, we consider a second polygon alignment rule in which the midpoint of one side of every polygon lies on a common fixed radius, yielding a second integer sequence of crossing numbers.

AI contribution: Eddie Lin Rui formulated the original problem. GPT-5.6 Sol (OpenAI) contributed substantially to the mathematical derivation, proof development, and preparation of this manuscript. The proof presented here has not been independently validated by a human expert.

1 Introduction

Consider drawing regular polygons with $3, 4, \ldots, N$ sides inscribed in the same circle, with every polygon sharing a common vertex. As more polygons are added, their sides form an increasingly dense pattern of intersections. How can we determine the number of distinct proper crossing points?

Let $A(N)$ denote this number. Intersections occurring at polygon vertices are excluded. The values of $A(N)$, beginning with $N = 3$, are

$$0, 4, 14, 26, 54, 86, 132, 188, 276, 352, 482, 614, \ldots$$

As N varies, $A(N)$ defines an integer sequence that arises naturally from the geometric construction.

The counting problem has two parts. First, we determine the number of proper crossings between a fixed pair of polygons. For $3 \le m < k$, we show that this number is

$$I(m, k) = 2\big(m - \gcd(m, k)\big).$$

Pairwise counting alone, however, does not necessarily give the correct number of distinct crossing points: if sides from three or more polygons could pass through the same interior point, summing the pairwise counts would overcount such intersections. The main challenge is therefore to determine whether such higher-order concurrences can occur. We prove that, under the common-vertex alignment, they cannot. The proof reduces a hypothetical triple concurrence to a rational chord-concurrence problem and uses Poonen and Rubinstein's classification of concurrent diagonals [1].

Consequently, every proper crossing point belongs to exactly one pair of polygon orders, and the pairwise counts may be summed directly. This gives

$$A(N) = 2 \sum_{3 \le m < k \le N} \big(m - \gcd(m, k)\big).$$

The same approach also yields a related integer sequence when the polygons are aligned so that the midpoint of one side of every polygon lies on a common fixed radius. After establishing the geometric results, we give an arithmetic reformulation of the common-vertex sequence in terms of Pillai's arithmetical function.

2 Setup and main results

We parameterize the common circumcircle by $\mathbb{R}/\mathbb{Z}$, the real numbers modulo integers, so that one unit corresponds to one full turn:

$$\theta \longmapsto (\cos 2\pi\theta, \sin 2\pi\theta).$$

For $n \ge 3$, let P_n be the regular n-gon with one of the following vertex sets:

$$V_n^{\mathrm{V}} = \left\{ \frac{j}{n} : 0 \le j < n \right\},$$
$$V_n^{\mathrm{M}} = \left\{ \frac{2j+1}{2n} : 0 \le j < n \right\}.$$

The first is **vertex alignment**: all polygons share the vertex 0.

The second is **side-midpoint alignment**: the midpoint directions of the sides of P_n are exactly

$$\frac{1}{n}\mathbb{Z}/\mathbb{Z},$$

so one side midpoint of every polygon lies on the fixed radius 0.

A **proper crossing** is an intersection lying in the relative interiors of two sides belonging to distinct polygons. Let $A_{\mathrm{V}}(N)$ and $A_{\mathrm{M}}(N)$ denote the numbers of distinct proper crossing points among

$$P_3, P_4, \ldots, P_N$$

under the two alignments.

Theorem 1.

$$\boxed{A_{\mathrm{V}}(N) = 2 \sum_{3 \le m < k \le N} \bigl(m - \gcd(m,k)\bigr).}$$

Theorem 2.

$$\boxed{A_{\mathrm{M}}(N) = 2 \sum_{3 \le m < k \le N} \left(m - \mathbf{1}_{\{\nu_2(m)=\nu_2(k)\}} \gcd(m,k)\right).}$$

Both theorems follow from the following fact.

Theorem 3 (Nonconcurrence)**.** *Under either alignment, no point lies in the relative interiors of sides belonging to three polygons of pairwise distinct orders.*

3 Pairwise crossings

Lemma 4. *If $3 \le m < k$, no side of P_m is collinear with a side of P_k.*

Proof. Two collinear chords of the same circle coincide. But a side of P_n has length

$$2 \sin \frac{\pi}{n},$$

which is strictly decreasing in $n \ge 3$. Hence sides of P_m and P_k cannot coincide. $\square$

Two distinct sides of a fixed P_n have disjoint relative interiors, since P_n is strictly convex. Let

$$C(m,k) = |V_m \cap V_k|$$

and let $I(m,k)$ be the number of proper crossings between P_m and P_k.

Proposition 5. *For $3 \le m < k$,*

$$\boxed{I(m,k) = 2\bigl(m - C(m,k)\bigr).}$$

Proof. Fix a side $e = ab$ of P_m and let α denote its open minor arc, of length $1/m \le 1/3$.

Crossing criterion. A chord of the circle meets e in the relative interiors of both if and only if exactly one of its endpoints lies in α. Indeed, two chords of a circle meet in their relative interiors precisely when the endpoints of one separate the endpoints of the other on the circle; since the endpoints of e are the endpoints of α, this says exactly that one endpoint of the chord lies in α and the other in the complementary open arc. By Lemma 4 the sides of P_k are not collinear with e, so each such crossing is a single point.

The vertex block. Since $1/m > 1/k$, the arc α contains a nonempty consecutive block B of vertices of P_k. Moreover $B \ne V_k$: an open arc containing all k vertices of P_k would have length exceeding $(k-1)/k \ge 2/3$, whereas α has length $1/m \le 1/3$. Hence B is a nonempty proper consecutive block, and exactly two sides of P_k have one endpoint in B and the other outside α, namely the two sides leaving B at its two ends. Every other side of P_k has both endpoints in α or both outside, and so does not cross e properly.

Counting. Each of the two leaving sides crosses e properly unless its exterior endpoint is a or b, in which case the two chords share that circle point and meet nowhere else. Thus

$$\#\{\text{proper crossings on } e\} = 2 - |\{a, b\} \cap V_k|.$$

Summing over the m sides of P_m, every common vertex is subtracted twice, giving

$$I(m, k) = 2m - 2C(m, k).$$

□

Proposition 6. *Let*

$$g = \gcd(m, k), \qquad m = ga, \qquad k = gb, \qquad \gcd(a, b) = 1.$$

Then

$$C_{\mathrm{V}}(m, k) = g,$$

while

$$C_{\mathrm{M}}(m, k) = \mathbf{1}_{\{\nu_2(m)=\nu_2(k)\}} \gcd(m, k).$$

Proof. For vertex alignment,

$$V_m^{\mathrm{V}} \cap V_k^{\mathrm{V}} = \frac{1}{g}\mathbb{Z}/\mathbb{Z},$$

so there are g common vertices.

For midpoint alignment,

$$V_n^{\mathrm{M}} = \frac{1}{2n} + \frac{1}{n}\mathbb{Z}/\mathbb{Z}.$$

The two cosets intersect exactly when

$$\frac{1}{2m} - \frac{1}{2k} = \frac{b-a}{2\,\mathrm{lcm}(m,k)}$$

is an integral multiple of $1/\mathrm{lcm}(m,k)$, equivalently when $a \equiv b \pmod 2$. Since $\gcd(a,b) = 1$, this means a, b are both odd, equivalently

$$\nu_2(m) = \nu_2(k).$$

When the intersection is nonempty it contains exactly g points. □

Hence

$$\boxed{I_{\mathrm{V}}(m,k) = 2\big(m - \gcd(m,k)\big)}$$

and

$$\boxed{I_{\mathrm{M}}(m,k) = 2\left(m - \mathbf{1}_{\{\nu_2(m)=\nu_2(k)\}} \gcd(m,k)\right).} \tag{1}$$

4 Reduction of triple concurrence

Proof of Theorem 3. Suppose sides of $P_{n_1}, P_{n_2}, P_{n_3}$, where the three orders are distinct, meet properly at one point.

Their six circle endpoints are distinct. Indeed, if two chords shared a circle endpoint, their supporting lines would share that endpoint and the interior concurrence point, hence would coincide, contradicting Lemma 4.

Moreover,

$$\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} \le \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{47}{60} < 1.$$

Thus the three closed side-arcs do not cover the circle. Cut at a point outside their union and list the six endpoints cyclically as

$$A < B < C < D < E < F.$$

Three pairwise-crossing chords on six cyclic points must be

$$AD, \qquad BE, \qquad CF.$$

This labelling is independent of the admissible cut point chosen. Indeed, with the endpoints labelled as above the three side-arcs are $[A,D]$, $[B,E]$ and $[C,F]$, whose union is the single arc $[A,F]$; hence the set of admissible cut points is the single open arc (F,A), and every admissible cut produces the same starting point A and the same cyclic list.

Write the consecutive gaps as

$$u, x, v, y, w, z > 0, \qquad u + x + v + y + w + z = 1.$$

Since each side-arc is disjoint from the cut, it is the forward interval between its endpoints; the three side-arc lengths are therefore

$$q_1 = u + x + v, \qquad q_2 = x + v + y, \qquad q_3 = v + y + w,$$

and not their complements.
Hence

$$\{q_1, q_2, q_3\} = \left\{\frac{1}{n_1}, \frac{1}{n_2}, \frac{1}{n_3}\right\},$$

so they are three distinct unit fractions not exceeding 1/3.
Under either alignment all six gaps are rational.

5 Poonen–Rubinstein classification

Poonen and Rubinstein [1] prove that the chords

$$AD,\ BE,\ CF$$

are concurrent if and only if

$$\boxed{\sin(\pi u)\sin(\pi v)\sin(\pi w) = \sin(\pi x)\sin(\pi y)\sin(\pi z).} \tag{2}$$

They also classify all positive rational solutions of (2) with total sum 1, up to

$$G = (S_3 \times S_3) \rtimes C_2,$$

where the two S_3 factors permute (u, v, w) and (x, y, z), and C_2 interchanges the triples.
The classification consists of:

1. the trivial family;
2. four one-parameter families;
3. 65 sporadic solutions.

Because (q_1, q_2, q_3) is not G-invariant, the full G-orbit of every representative must be tested.

Reversal of the circle's orientation sends the gap sequence (u, x, v, y, w, z) to (z, w, y, v, x, u), that is, it interchanges the two triples and reverses each; this is an element of G. Consequently testing full G-orbits already accounts for reflected configurations, and only a common rotation remains free in Section 7.

We now impose the additional condition that (q_1, q_2, q_3) are three distinct unit fractions not exceeding 1/3.

6 Elimination of classified solutions

6.1 Trivial family

Here

$$u + v + w = \frac{1}{2},$$

and (x, y, z) is a permutation of (u, v, w).

Four of the six permutations immediately give either $q_i = 1/2$ or repeated q_i. In the two remaining cases,

$$q_3 = \frac{1}{2} - q_1 + q_2.$$

Writing

$$q_1 = \frac{1}{m}, \qquad q_2 = \frac{1}{k}, \qquad q_3 = \frac{1}{\ell}$$

and using $q_3 \leq 1/3$ gives

$$\frac{1}{m} - \frac{1}{k} \geq \frac{1}{6},$$

hence $m \leq 5$.

The candidate list is finite for the following reason: it is ℓ, not k, that is bounded. Given m, the relation $1/\ell = 1/2 - 1/m + 1/k$ with $1/k > 0$ forces $1/\ell > 1/2 - 1/m$, so ℓ ranges over a finite set, and k is then determined by ℓ. Enumerating these possibilities gives

$$(m, k, \ell) \in \{(3, 12, 4), (3, 30, 5), (4, 12, 3), (5, 30, 3)\}.$$

For the two surviving permutations one obtains respectively

$$v = \frac{2q_2 - q_1}{3}, \qquad v = \frac{q_1 + q_2}{3} - \frac{1}{6},$$

and these are negative in all four cases.

Hence the trivial family contributes nothing.

6.2 One-parameter families

Restoring the full G-orbit and imposing positivity and

$$q_i \leq \frac{1}{3}$$

gives 21 distinct ordered pattern/interval classes (see Appendix A.2). (The second Poonen–Rubinstein family contributes none: it never satisfies $q_i \leq 1/3$ for all i.)

Five have an identically repeated q_i. Two more occur only at $t = 1/12$, where

$$(q_1, q_2, q_3) = \left(\frac{1}{3}, \frac{1}{4}, \frac{1}{3}\right),$$

so they also cannot represent three distinct orders.

The remaining 14 ordered classes occur in pairs. Up to permutation of the q_i, they reduce to the following seven patterns:

$$\left(\frac{1}{3}, \frac{1}{2} - 2t, \frac{2}{3} - 3t\right), \qquad \frac{1}{9} \le t < \frac{1}{6};$$

$$\left(\frac{1}{6} + 3t, \frac{1}{6} + t, \frac{1}{3} - 2t\right), \qquad 0 < t \le \frac{1}{18};$$

$$\left(\frac{1}{3} - t, \frac{1}{3} - 2t, \frac{1}{3} - 3t\right), \qquad 0 < t < \frac{1}{12};$$

$$\left(\frac{1}{6} + t, \frac{1}{3} - 3t, \frac{1}{2} - 4t\right), \qquad \frac{1}{24} \le t < \frac{1}{12};$$

$$\left(\frac{1}{2} - 4t, \frac{1}{3} - t, \frac{1}{6} + 3t\right), \qquad \frac{1}{24} \le t \le \frac{1}{18};$$

$$\left(\frac{1}{3}, \frac{1}{2} - 3t, \frac{1}{2} - 5t\right), \qquad \frac{1}{18} \le t < \frac{1}{12};$$

$$\left(\frac{2}{3} - 5t, \frac{1}{2} - 5t, \frac{1}{6} + 2t\right), \qquad \frac{1}{15} \le t < \frac{1}{12}.$$

They are eliminated as follows.

- In the first pattern, $\frac{1}{2} - 2t$ can only be $1/4$ or $1/5$; the third entry is then $7/24$ or $13/60$.
- In the second, $\frac{1}{6} + t$ can only be $1/5$, giving $4/15$ for the first entry.
- In patterns (3), (4), and (5), respectively,

  $$\frac{1}{3} - t \in (1/4, 1/3), \qquad \frac{1}{6} + t \in [5/24, 1/4), \qquad \frac{1}{3} - t \in [5/18, 7/24],$$

  and none of these intervals contains a unit fraction.
- In pattern (6), $\frac{1}{2} - 3t \in (1/4, 1/3]$, so the only possible unit fraction is $1/3$, repeating the first entry.

- In pattern (7),

$$\frac{1}{6} + 2t \in [3/10, 1/3),$$

which contains no unit fraction.

At the excluded endpoint $t = 1/12$, patterns (3) and (4) would give

$$\left\{\frac{1}{4}, \frac{1}{6}, \frac{1}{12}\right\},$$

while patterns (6) and (7) would give

$$\left\{\frac{1}{3}, \frac{1}{4}, \frac{1}{12}\right\}.$$

But $t = 1/12$ makes two underlying Poonen–Rubinstein gaps vanish, so the six endpoints are no longer distinct, contrary to Section 4.

Thus no one-parameter family contributes.

6.3 Sporadic solutions

Apply the full group G to the 65 published sporadic rows using exact rational arithmetic.

Of the 65 representatives, 57 have orbit size 72 and 8 have orbit size 36. No two representatives share an orbit, so there are

$$57 \cdot 72 + 8 \cdot 36 = 4392$$

distinct six-tuples.

Computing

$$q_1 = u + x + v, \qquad q_2 = x + v + y, \qquad q_3 = v + y + w$$

gives the exact filtration (see Appendix A.1)

$$4392 \longrightarrow 94 \longrightarrow 2,$$

where 94 satisfy $q_i \leq 1/3$, and only 2 have three distinct unit-fraction values.

The two survivors are reversals of one configuration. In units of $1/60$, their endpoint sets are

$$(0, 13, 16, 20, 25, 31)$$

and

$$(0, 6, 11, 15, 18, 31).$$

Their chord lengths are

$$\frac{1}{3}, \qquad \frac{1}{5}, \qquad \frac{1}{4}.$$

Therefore any proper concurrence involving sides of three distinct regular-polygon orders must involve exactly

$$\boxed{\{3,4,5\}}.$$

7 Excluding the exceptional configuration

Let

$$P = (0, 13, 16, 20, 25, 31), \qquad P' = (0, 6, 11, 15, 18, 31),$$

in units of 1/60, allowing an arbitrary common rotation $\tau/60$ with $\tau \in \mathbb{R}$. Integrality of τ is not assumed; in each case below it is forced by the first condition imposed.

Because the three chord lengths are distinct, their assignments to

$$P_3, \ P_5, \ P_4$$

are forced.

For P:

$$P_3 : \{0, 20\}, \qquad P_5 : \{13, 25\}, \qquad P_4 : \{16, 31\}.$$

For P':

$$P_3 : \{11, 31\}, \qquad P_5 : \{6, 18\}, \qquad P_4 : \{0, 15\}.$$

7.1 Vertex alignment

A point $p + \tau$ is a vertex of P_n exactly when

$$p + \tau \equiv 0 \pmod{60/n}.$$

For P, the P_3 and P_5 conditions give

$$\tau \equiv 0 \pmod{20}, \qquad \tau + 13 \equiv 0 \pmod{12}.$$

The first already forces $\tau \in 20\mathbb{Z}$, in particular $\tau \in \mathbb{Z}$, so both congruences may be read modulo 4:

$$\tau \equiv 0 \pmod 4, \qquad \tau \equiv 3 \pmod 4,$$

a contradiction.

For P',

$$\tau \equiv 0 \pmod{15}, \qquad \tau + 6 \equiv 0 \pmod{12},$$

the first again forcing $\tau \in \mathbb{Z}$. Hence

$$\tau \equiv 30 \pmod{60},$$

so $\tau \equiv 10 \pmod{20}$, whereas the P_3 condition requires

$$\tau + 11 \equiv 0 \pmod{20},$$

or $\tau \equiv 9 \pmod{20}$.
Contradiction.

7.2 Side-midpoint alignment

A side of P_n must have midpoint direction in

$$\frac{1}{n}\mathbb{Z}/\mathbb{Z}.$$

For P, the relevant midpoints are

$$10, \qquad 19, \qquad \frac{47}{2}.$$

The P_3 and P_5 conditions give

$$\tau + 10 \equiv 0 \pmod{20}, \qquad \tau + 19 \equiv 0 \pmod{12}.$$

The first forces $\tau \in \mathbb{Z}$, so both may be read modulo 4:

$$\tau \equiv 2 \pmod{4}, \qquad \tau \equiv 1 \pmod{4},$$

a contradiction.
For P', the relevant midpoints are

$$\frac{15}{2}, \qquad 12, \qquad 21.$$

The P_5 condition forces $\tau \equiv 0 \pmod{12}$, in particular $\tau \in \mathbb{Z}$, while the P_4 condition gives

$$\tau \equiv -\frac{15}{2} \pmod{15},$$

so τ is a half-integer.
Contradiction.
Thus the exceptional configuration occurs under neither alignment, proving Theorem 3.
□

8 From crossing events to crossing points

At any proper crossing point, at most one side from each polygon can occur, because distinct sides of a strictly convex polygon have disjoint relative interiors.

By Theorem 3, no proper crossing point contains sides from three distinct polygon orders. Hence every proper crossing point arises from exactly one pair of polygon sides.

Therefore pairwise crossing events are in bijection with distinct crossing points.

Summing (1) over all pairs $3 \le m < k \le N$ gives

$$\boxed{A_{\mathrm{V}}(N) = 2 \sum_{3 \le m < k \le N} \big(m - \gcd(m,k)\big)}$$

and

$$\boxed{A_{\mathrm{M}}(N) = 2 \sum_{3 \le m < k \le N} \big(m - \mathbf{1}_{\{\nu_2(m)=\nu_2(k)\}} \gcd(m,k)\big)\,.}$$

The initial values are

$$A_{\mathrm{V}}: \quad 0, 4, 14, 26, 54, 86, 132, 188, 276, 352, 482, 614, 762, 938, 1176, \ldots$$

and

$$A_{\mathrm{M}}: \quad 0, 6, 18, 42, 74, 124, 180, 260, 356, 474, 614, 782, 958, 1192, 1444, \ldots$$

for $N = 3, 4, 5, \ldots$.

9 Arithmetic form for vertex alignment

Let

$$P(k) = \sum_{j=1}^{k} \gcd(j,k)$$

be Pillai's arithmetical function.

For $k \ge 4$,

$$\sum_{m=3}^{k-1} \gcd(m,k) = P(k) - k - 1 - \gcd(2,k),$$

while

$$\sum_{m=3}^{k-1} m = \frac{k(k-1)}{2} - 3.$$

Hence

$$\boxed{A_{\mathrm{V}}(N) = 2\left(\frac{N^3 - N}{6} - 3N + 5\right) - 2\sum_{k=4}^{N}\big(P(k) - k - 1 - \gcd(2,k)\big).}$$

10 Dependency

The only non-elementary ingredients are the Poonen–Rubinstein concurrence criterion and their classification of positive rational solutions of (2). Poonen and Rubinstein note that their Tables 3 and 4 agree with the earlier tables of Bol (1936), providing independent corroboration of the classification.

The remaining arguments consist of elementary geometry and arithmetic together with exact finite enumeration of the four parameterized families and the 65 published sporadic solutions.

Subject to the Poonen–Rubinstein classification, Theorems 1–3 follow.

A Reproducibility of the finite enumerations

Sections 6.2 and 6.3 rest on two exact finite enumerations. We record here the inputs, the criteria applied, and complete verification code, so that both may be reproduced without reference to the author's working files. All arithmetic is exact: every quantity is a rational number represented as a pair of integers, and no floating-point comparison is used at any stage.

A.1 The sporadic enumeration

What is enumerated. The input is the complete list of 65 sporadic solutions of (2) given by Poonen and Rubinstein [1, Table 4], reproduced as Tables 1 and 2 below. Completeness of the input is exactly the content of their Theorem 4 and is not reproved here; what is verified is the specialization of that list to the present problem.

Because the classification is stated up to $G = (S_3 \times S_3) \rtimes C_2$ while (q_1, q_2, q_3) is not G-invariant, each representative is expanded to its full orbit. The three arcs are then computed from the cyclic gap assignment $(u, x, v, y, w, z) = (U, X, V, Y, W, Z)$, and two filters are applied in turn: $q_i \leq 1/3$ for all i, which is forced by $n_i \geq 3$; and q_1, q_2, q_3 three *distinct* unit fractions, which is forced by the orders being pairwise distinct.

Reported counts. Running the code of Listing 1 produces

$$\text{orbit sizes } 57 \times 72 + 8 \times 36, \qquad 4392 \longrightarrow 94 \longrightarrow 2,$$

with the union of the orbits containing 4392 distinct six-tuples, equal to the sum of the individual orbit sizes; hence no two of the 65 representatives share an orbit. The two survivors are printed with endpoints $(0, 13, 16, 20, 25, 31)$ and $(0, 6, 11, 15, 18, 31)$ in units of $1/60$, with arcs $\{1/3, 1/5, 1/4\}$ in both cases. The script also checks, as a transcription safeguard, that each of the 65 rows consists of positive rationals summing to 1, and that the least common denominators occur with multiplicities $30:13$, $42:6$, $60:22$, $84:8$, $90:4$, $120:8$, $210:4$, totalling 65.

Dropping the distinctness requirement does not enlarge the survivor set, so no case is concealed behind that filter.

Denom.	U	V	W	X	Y	Z
30	1/10	2/15	3/10	2/15	1/6	1/6
	1/15	1/15	7/15	1/15	1/10	7/30
	1/30	7/30	4/15	1/15	1/10	3/10
	1/30	1/10	7/15	1/15	1/15	4/15
	1/30	1/15	19/30	1/15	1/10	1/10
	1/15	1/6	4/15	1/10	1/10	3/10
	1/15	2/15	11/30	1/10	1/6	1/6
	1/30	1/6	13/30	1/10	2/15	2/15
	1/30	1/30	7/10	1/30	1/15	2/15
	1/30	7/30	3/10	1/15	2/15	7/30
	1/30	1/6	11/30	1/15	1/10	4/15
	1/30	1/10	13/30	1/30	2/15	4/15
	1/30	1/15	8/15	1/30	1/10	7/30
42	1/14	5/42	5/14	2/21	5/42	5/21
	1/21	4/21	13/42	1/14	1/6	3/14
	1/42	3/14	5/14	1/21	1/6	4/21
	1/42	1/6	19/42	1/14	2/21	4/21
	1/42	1/6	13/42	1/21	1/14	8/21
	1/42	1/21	13/21	1/42	1/14	3/14
60	1/20	1/12	29/60	1/15	1/10	13/60
	1/20	1/12	9/20	1/15	1/12	4/15
	1/20	1/12	5/12	1/20	1/10	3/10
	1/60	4/15	3/10	1/20	1/12	17/60
	1/60	13/60	9/20	1/12	1/10	2/15
	1/60	13/60	5/12	1/20	2/15	1/6
	1/12	1/6	17/60	2/15	3/20	11/60
	1/12	2/15	19/60	1/10	3/20	13/60
	1/15	11/60	13/60	1/12	1/10	7/20
	1/20	11/60	3/10	1/12	7/60	4/15
	1/20	1/10	23/60	1/15	1/12	19/60
	1/30	7/60	19/60	1/20	1/15	5/12
	1/30	1/12	7/12	1/15	1/10	2/15
	1/30	1/20	11/20	1/30	1/15	4/15
	1/60	3/10	7/20	1/12	7/60	2/15
	1/60	4/15	23/60	1/12	1/10	3/20
	1/60	7/30	5/12	1/15	7/60	3/20
	1/60	13/60	11/30	1/20	1/12	4/15
	1/60	1/6	31/60	1/15	1/10	2/15
	1/60	1/6	5/12	1/20	1/15	17/60
	1/60	2/15	9/20	1/30	1/12	17/60
	1/60	1/10	31/60	1/30	1/15	4/15

Table 1: The sporadic solutions of (2) with least common denominator 30, 42 and 60 (41 rows). Reproduced from Poonen and Rubinstein [1, Table 4] for verification purposes; see Table 2 for the remainder.

Denom.	U	V	W	X	Y	Z
84	1/12	3/14	19/84	11/84	13/84	4/21
	1/14	11/84	23/84	1/12	2/21	29/84
	1/21	13/84	23/84	1/14	1/12	31/84
	1/42	1/12	7/12	1/21	1/14	4/21
	1/84	25/84	5/14	5/84	1/12	4/21
	1/84	5/21	5/12	5/84	1/14	17/84
	1/84	3/14	37/84	1/21	1/12	17/84
	1/84	1/6	43/84	1/21	1/14	4/21
90	1/18	13/90	7/18	11/90	2/15	7/45
	1/45	19/90	16/45	1/18	1/10	23/90
	1/90	23/90	31/90	2/45	1/15	5/18
	1/90	17/90	47/90	1/18	4/45	2/15
120	13/120	3/20	31/120	2/15	19/120	23/120
	1/12	19/120	29/120	1/10	13/120	37/120
	1/20	23/120	29/120	1/15	13/120	41/120
	1/60	13/120	73/120	1/20	1/12	2/15
	1/120	7/20	43/120	7/120	11/120	2/15
	1/120	3/10	49/120	7/120	1/12	17/120
	1/120	4/15	53/120	1/20	11/120	17/120
	1/120	13/60	61/120	1/20	1/12	2/15
210	1/15	41/210	8/35	1/14	31/210	61/210
	13/210	1/10	83/210	1/14	4/35	9/35
	1/35	2/15	97/210	1/14	17/210	47/210
	1/210	3/14	121/210	11/210	1/15	3/35

Table 2: The sporadic solutions of (2) with least common denominator 84, 90, 120 and 210 (24 rows), continuing Table 1. The two tables together are the complete list of 65.

A.2 The one-parameter family enumeration

What is enumerated. The input is the complete list of four one-parameter families of solutions of (2) given by Poonen and Rubinstein [1, Table 3], each a six-tuple affine in a rational parameter t together with an open range for t. As above, each family is expanded over the full G-orbit, giving 4×72 parameterized six-tuples; within each open range all six gaps are strictly positive, so positivity need not be reimposed.

Why the search is finite. Each q_i is affine in t. For a given orbit element, the set of t in the family range satisfying $0 < q_i \leq 1/3$ for all i is an interval, computed exactly. On that interval at least one q_i is nonconstant with strictly positive infimum c; since $q_i = 1/n_i$ forces $n_i \leq 1/c$, only finitely many n_i need be tested, and each determines t uniquely. The code verifies that such a q_i exists for every class, and raises an error otherwise; no class is exempt.

This replaces the interval inspection of Section 6.2 by a direct exhaustive solve, and is therefore an independent check of it rather than a restatement.

Reported counts. Running the code of Listing 2 produces 21 ordered pattern classes meeting $0 < q_i \leq 1/3$, of which 5 have an identically repeated q_i and 2 collapse to the single point $t = 1/12$; and *no* parameter value in any class yields three distinct unit fractions.

A.3 Code

Listing 1. `sporadic_check.py`.

```
"""
Verification of the sporadic filtration 4392 -> 94 -> 2 (Section 5.3).

Exact rational arithmetic throughout (fractions.Fraction).
Input: the 65 sporadic solutions of Poonen-Rubinstein, Table 4,
listed as (U,V,W,X,Y,Z).  Run:  python3 sporadic_check.py
"""

from fractions import Fraction as F
from itertools import permutations
from collections import Counter
from math import gcd

TABLE4 = """
1/10 2/15 3/10 2/15 1/6 1/6
1/15 1/15 7/15 1/15 1/10 7/30
1/30 7/30 4/15 1/15 1/10 3/10
1/30 1/10 7/15 1/15 1/15 4/15
1/30 1/15 19/30 1/15 1/10 1/10
1/15 1/6 4/15 1/10 1/10 3/10
1/15 2/15 11/30 1/10 1/6 1/6
1/30 1/6 13/30 1/10 2/15 2/15
1/30 1/30 7/10 1/30 1/15 2/15
1/30 7/30 3/10 1/15 2/15 7/30
1/30 1/6 11/30 1/15 1/10 4/15
1/30 1/10 13/30 1/30 2/15 4/15
1/30 1/15 8/15 1/30 1/10 7/30
1/14 5/42 5/14 2/21 5/42 5/21
1/21 4/21 13/42 1/14 1/6 3/14
1/42 3/14 5/14 1/21 1/6 4/21
1/42 1/6 19/42 1/14 2/21 4/21
1/42 1/6 13/42 1/21 1/14 8/21
1/42 1/21 13/21 1/42 1/14 3/14
1/20 1/12 29/60 1/15 1/10 13/60
1/20 1/12 9/20 1/15 1/12 4/15
1/20 1/12 5/12 1/20 1/10 3/10
1/60 4/15 3/10 1/20 1/12 17/60
1/60 13/60 9/20 1/12 1/10 2/15
1/60 13/60 5/12 1/20 2/15 1/6
1/12 1/6 17/60 2/15 3/20 11/60
1/12 2/15 19/60 1/10 3/20 13/60
1/15 11/60 13/60 1/12 1/10 7/20
1/20 11/60 3/10 1/12 7/60 4/15
1/20 1/10 23/60 1/15 1/12 19/60
1/30 7/60 19/60 1/20 1/15 5/12
1/30 1/12 7/12 1/15 1/10 2/15
1/30 1/20 11/20 1/30 1/15 4/15
1/60 3/10 7/20 1/12 7/60 2/15
```

```
1/60 4/15 23/60 1/12 1/10 3/20
1/60 7/30 5/12 1/15 7/60 3/20
1/60 13/60 11/30 1/20 1/12 4/15
1/60 1/6 31/60 1/15 1/10 2/15
1/60 1/6 5/12 1/20 1/15 17/60
1/60 2/15 9/20 1/30 1/12 17/60
1/60 1/10 31/60 1/30 1/15 4/15
1/12 3/14 19/84 11/84 13/84 4/21
1/14 11/84 23/84 1/12 2/21 29/84
1/21 13/84 23/84 1/14 1/12 31/84
1/42 1/12 7/12 1/21 1/14 4/21
1/84 25/84 5/14 5/84 1/12 4/21
1/84 5/21 5/12 5/84 1/14 17/84
1/84 3/14 37/84 1/21 1/12 17/84
1/84 1/6 43/84 1/21 1/14 4/21
1/18 13/90 7/18 11/90 2/15 7/45
1/45 19/90 16/45 1/18 1/10 23/90
1/90 23/90 31/90 2/45 1/15 5/18
1/90 17/90 47/90 1/18 4/45 2/15
13/120 3/20 31/120 2/15 19/120 23/120
1/12 19/120 29/120 1/10 13/120 37/120
1/20 23/120 29/120 1/15 13/120 41/120
1/60 13/120 73/120 1/20 1/12 2/15
1/120 7/20 43/120 7/120 11/120 2/15
1/120 3/10 49/120 7/120 1/12 17/120
1/120 4/15 53/120 1/20 11/120 17/120
1/120 13/60 61/120 1/20 1/12 2/15
1/15 41/210 8/35 1/14 31/210 61/210
13/210 1/10 83/210 1/14 4/35 9/35
1/35 2/15 97/210 1/14 17/210 47/210
1/210 3/14 121/210 11/210 1/15 3/35
"""

SPORADIC = [tuple(F(s) for s in line.split())
            for line in TABLE4.strip().splitlines()]


def lcd(t):
    """Least common denominator of a six-tuple."""
    d = 1
    for x in t:
        d = d * x.denominator // gcd(d, x.denominator)
    return d


def orbit(sol):
    """Full G-orbit, G = (S_3 x S_3) semidirect C_2, order 72."""
    U, V, W, X, Y, Z = sol
    out = set()
    for p in permutations((U, V, W)):
        for q in permutations((X, Y, Z)):
```

```
            out.add(p + q)
            out.add(q + p)
    return out


def chord_arcs(tup):
    """Cyclic gaps (u,x,v,y,w,z) = (U,X,V,Y,W,Z); return (q1,q2,q3)."""
    U, V, W, X, Y, Z = tup
    return (U + X + V, X + V + Y, V + Y + W)


def main():
    assert len(SPORADIC) == 65
    assert all(sum(t) == 1 for t in SPORADIC)
    assert all(all(x > 0 for x in t) for t in SPORADIC)
    print("rows:", len(SPORADIC), " all sum to 1 and positive: yes")
    print("least common denominators:",
          dict(sorted(Counter(lcd(t) for t in SPORADIC).items())))

    sizes = Counter()
    tuples = set()
    for t in SPORADIC:
        o = orbit(t)
        sizes[len(o)] += 1
        tuples |= o
    print("orbit sizes:", dict(sizes))
    print("sum of orbit sizes:", sum(sizes[k] * k for k in sizes))
    print("distinct six-tuples in the union:", len(tuples))

    third = [t for t in tuples if all(q <= F(1, 3) for q in chord_arcs(t))]
    print("after q_i <= 1/3:", len(third))

    unit = [t for t in third
            if all(q.numerator == 1 for q in chord_arcs(t))
            and len(set(chord_arcs(t))) == 3]
    print("after distinct unit fractions:", len(unit))

    for t in unit:
        U, V, W, X, Y, Z = t
        gaps = (U, X, V, Y, W, Z)
        ends, c = [], F(0)
        for g in gaps[:-1]:
            ends.append(c)
            c += g
        ends.append(c)
        print("  endpoints x60:", [int(60 * e) for e in ends],
              " arcs:", chord_arcs(t))


if __name__ == "__main__":
    main()
```

Listing 2. `families_check.py`.

```
"""
Verification of the one-parameter family elimination (Section 6.2).

Exact rational arithmetic throughout.  Each q_i is affine in the
parameter t, so a class is eliminated by a finite exact search:
restrict t to the subinterval where 0 < q_i <= 1/3 for all i, pick a
q_i whose infimum there is positive (this bounds its denominator),
and solve q_i = 1/n exactly for every admissible n.

Input: the four families of Poonen-Rubinstein, PR_Table 3, as
(U,V,W,X,Y,Z) affine in t, with the stated open range for t.
Run:  python3 families_check.py
"""

from fractions import Fraction as F
from itertools import permutations

Z, ONE = F(0), F(1)


class Aff:
    """Affine function slope*t + const over the rationals."""

    def __init__(self, slope=Z, const=Z):
        self.s, self.c = F(slope), F(const)

    def __add__(self, o):
        return Aff(self.s + o.s, self.c + o.c)

    def at(self, t):
        return self.s * t + self.c

    def key(self):
        return (self.s, self.c)


def K(c):
    return Aff(Z, c)


def T(m):
    return Aff(m, Z)


# Poonen-Rubinstein, PR_Table 3: (U,V,W,X,Y,Z) and the open range for t.
FAMILIES = [
    ([K(F(1, 6)), T(1), K(F(1, 3)) + T(-2), K(F(1, 3)) + T(1), T(1),
      K(F(1, 6)) + T(-1)], (Z, F(1, 6))),
    ([K(F(1, 6)), K(F(1, 2)) + T(-3), T(1), K(F(1, 6)) + T(-1), T(2),
      K(F(1, 6)) + T(1)], (Z, F(1, 6))),
```

```
    ([K(F(1, 6)), K(F(1, 6)) + T(-2), T(2), K(F(1, 6)) + T(-2), T(1),
      K(F(1, 2)) + T(1)], (Z, F(1, 12))),
    ([K(F(1, 3)) + T(-4), T(1), K(F(1, 3)) + T(1), K(F(1, 6)) + T(-2),
      T(3), K(F(1, 6)) + T(1)], (Z, F(1, 12))),
]

GROUP = [(p, q, sw) for p in permutations(range(3))
         for q in permutations(range(3)) for sw in (0, 1)]


def chord_arcs(t6):
    """(u,x,v,y,w,z) = (U,X,V,Y,W,Z); return (q1,q2,q3)."""
    U, V, W, X, Y, Z_ = t6
    return (U + X + V, X + V + Y, V + Y + W)


def feasible(D, lo, hi):
    """Subinterval of (lo,hi) where 0 < q_i <= 1/3 for all i.

    Returns (a, b, a_open, b_open) or None if empty.
    """
    a, b, ao, bo = lo, hi, True, True
    for q in D:
        if q.s == 0:
            if q.c <= 0 or q.c > F(1, 3):
                return None
            continue
        for bound, upper in ((F(1, 3), True), (Z, False)):
            v = (bound - q.c) / q.s
            if (q.s > 0) == upper:
                if v < b:
                    b, bo = v, not upper
                elif v == b:
                    bo = bo or not upper
            else:
                if v > a:
                    a, ao = v, not upper
                elif v == a:
                    ao = ao or not upper
        if a > b or (a == b and (ao or bo)):
            return None
    return (a, b, ao, bo)


def inside(t, iv):
    a, b, ao, bo = iv
    return not (t < a or t > b or (t == a and ao) or (t == b and bo))


def main():
    classes, hits = {}, []
```

```
    for fi, (fam, (lo, hi)) in enumerate(FAMILIES):
        for p, q, sw in GROUP:
            A = [fam[i] for i in p]
            B = [fam[3 + i] for i in q]
            D = chord_arcs(B + A if sw else A + B)
            iv = feasible(D, lo, hi)
            if iv is None:
                continue
            classes.setdefault((fi, tuple(x.key() for x in D)), iv)

    print("ordered (family, pattern) classes with 0 < q_i <= 1/3:",
          len(classes))
    rep = sum(1 for (_, k) in classes if len(set(k)) < 3)
    print("  of which have an identically repeated q_i:", rep)
    pt = sum(1 for iv in classes.values() if iv[0] == iv[1])
    print("  of which collapse to a single t:", pt)

    for (fi, keys), iv in classes.items():
        D = [Aff(s, c) for (s, c) in keys]
        a, b, _, _ = iv
        idx = [i for i, q in enumerate(D)
               if q.s != 0 and min(q.at(a), q.at(b)) > 0]
        if not idx:
            raise RuntimeError("unbounded class: %r" % (keys,))
        j = idx[0]
        nmax = int(1 / min(D[j].at(a), D[j].at(b)))
        for n in range(3, nmax + 1):
            t = (F(1, n) - D[j].c) / D[j].s
            if not inside(t, iv):
                continue
            v = [x.at(t) for x in D]
            if all(x.numerator == 1 for x in v) and len(set(v)) == 3:
                hits.append((fi + 1, t, v))

    print("parameter values giving three distinct unit fractions:",
          hits if hits else "none")


if __name__ == "__main__":
    main()
```

References

[1] B. Poonen and M. Rubinstein, *The Number of Intersection Points Made by the Diagonals of a Regular Polygon*, SIAM Journal on Discrete Mathematics **11** (1998), 135–156; arXiv:math/9508209.