TWENTY-ONE POINTS IN THREE-SPACE

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Abstract. We prove that every finite subscheme of length twenty-one in complex affine three-space is a limit of distinct points. So the Hilbert scheme of twenty-one points in $\mathbb{A}^3$ is irreducible, and the least number of points for which it is reducible lies between twenty-two and seventy-eight. The proof continues the induction of *Nineteen points* and *Twenty points in three-space*, whose main tool is linkage: a scheme of length d in a complete intersection of length less than $2d$ is linked to a shorter one. The new tool is linkage at infinity: we deform the tails of a scheme while magnifying it, and link the limit, which keeps part of the tails, instead of the tangent cone. When the tangent cone lies on a quadric, we move one equation in a flat family to cap the cost of its special directions.

1. Introduction

On a smooth surface every finite subscheme is a limit of distinct points; this is a theorem of Fogarty [4]. In $\mathbb{A}^n$ with $n \geq 4$ the statement fails already for eight points [10, 1]. Three-space lies between the two. Iarrobino proved that the statement fails there too, for long schemes [8]. He found schemes of length 78, each supported at a single point, that move in a family of dimension 235 [9, Example 4.3], while the limits of 78 distinct points form a family of dimension $3 \cdot 78 = 234$; see [11, Section 2.3]. The argument is a count of dimensions. It proves that a second component of the Hilbert scheme exists, but it does not exhibit one:

> *It is puzzling that despite proofs of reducibility, no components other than the smoothable one are known* [...] *for* $\mathrm{Hilb}_d(\mathbb{A}^3)$. [11, Section 2.1]

We approach the question from below. A finite subscheme is *smoothable* if it is a limit of reduced ones. The smoothable subschemes of $\mathbb{A}^3$ of length d form a component of dimension $3d$ of the Hilbert scheme $\mathrm{Hilb}_d(\mathbb{A}^3)$, and the Hilbert scheme is irreducible if and only if there is no other. The least d for which $\mathrm{Hilb}_d(\mathbb{A}^3)$ is reducible is unknown; this is Problem V of Jelisiejew's list [11, Section 2.1]. Mazzola showed that it is larger than 7 [13]; Cartwright, Erman, Velasco and Viray settled $d = 8$ [1]; Henni and Jardim $d = 9$ and 10 [6], after Šivic's work on commuting triples [17]; and Douvropoulos, Jelisiejew, Nødland and Teitler $d = 11$ [3]. The three preceding papers of this series reached $d = 14$, 19 and 20 [14], [15], [16]. Here we add one more point.

Theorem 1.1. *Finite subschemes of* $\mathbb{A}^3$ *of length at most twenty-one are smoothable. So* $\mathrm{Hilb}_{21}(\mathbb{A}^3)$ *is irreducible, of dimension* 63*, and the least* d *for which* $\mathrm{Hilb}_d(\mathbb{A}^3)$ *is reducible satisfies* $22 \leq d \leq 78$*.*

Smoothability can be tested locally, so the theorem holds on every smooth irreducible quasi-projective threefold [3, Section 1].

1.2. We argue by induction on the length, with the tools of [15] and [16] and their results up to length twenty. A scheme with two or more points of support is a disjoint union of shorter schemes, and a scheme of embedding dimension at most two is a scheme in the plane; so it suffices to treat a local algebra $A = S/I$ of length twenty-one, where $S = \mathbb{C}[x, y, z]$, $I \subseteq \mathfrak{m}^2$ and $\mathfrak{m} = (x, y, z)$ [15, Proposition 2.2(3)]. Its tangent cone has a Hilbert function $H_A = (1, 3, H_A(2), \dots)$, a sequence of positive integers with sum 21 that grows no faster than Macaulay's bound allows. There are 254 such sequences. General results of [14], [15] and [3] settle 196 of them, and the remaining 58 need individual arguments (Tables 1 and 2).

1.3. Linkage is our main tool, and its budget explains what changes at length twenty-one. If a scheme Z lies in a finite complete intersection Z' of three surfaces in an open subset of $\mathbb{A}^3$, then the residual scheme has length $\ell(Z') - \ell(Z)$, and Z is smoothable if the residual scheme is [12], [15, Theorem 2.22]. So a scheme of length twenty-one is smoothable whenever some complete intersection through it has local length at most 41. The budget is two more than at length twenty, and it is spent in two standard ways:

$$41 = 27 + 14 = 24 + 17.$$

After a deformation of Z that keeps its Hilbert function, three equations of Z with cubic leading forms cut out 27, plus at most $(k(p) - 3)\, e_p$ for each direction p common to the three cubic cones; here e_p is a multiplicity and $k(p)$ is the first degree in which the surfaces separate over p [15, Theorem 3.12]. When the tangent cone lies on the double plane $x^2 = 0$, its cubics are those divisible by x, and the transverse cubic of the quadratic equation vanishes (Section 2.8), three equations of orders 2, 3 and 4 cut out 24, plus at most $(\kappa(v) - 2)\, \mu_v$ for each root v of a binary quartic [16, Theorem 3.15]. Two bounds need no count over directions: a weighted Bézout bound $4^3/2 = 32$ [16, Proposition 3.12] and a global complete intersection of degrees 3, 3, 4 and length 36 [15, Proposition 3.18]. Their margins are now 9 and 5.

Of the 254 Hilbert functions, 59 escape the general results of [14] and [15], against 33 of 199 at length twenty, in keeping with Jelisiejew's remark that the work grows roughly exponentially with the length [11, Section 1.2]; their socle degrees reach nine. One of them cannot be treated by direct short linkage. An algebra with Hilbert function $(1, 3, 6, 10, 1)$ satisfies $\mathfrak{m}^5 \subseteq I \subseteq \mathfrak{m}^4$, so all its equations have order at least four, and every complete intersection that contains it has local length at least $4^3 = 64$. Such algebras are *very compressed*, and Douvropoulos, Jelisiejew, Nødland and Teitler proved, with

a computer calculation, that very compressed algebras of length at most 95 are smoothable [3, Proposition 1.2]. For the other 58 the old devices often stop at 42, one more than the budget, and the work is to find the last point. A member of an irreducible family is a *linkage anchor* if it lies locally in a complete intersection of length less than twice its own length; short linkage is an open condition and smoothability a closed one, so one anchor makes the whole family smoothable. Some help comes from length twenty. An anchor of length twenty serves again at length twenty-one if some cubic outside its cubic top forms is orthogonal to the cubic parts of its three equations: adding this cubic to the inverse system keeps the complete intersection [16, Remark 3.7].

1.4. An algebra $A = S/I$ supported at the origin is described by its *inverse system* $M \subseteq T = \mathbb{C}[X, Y, Z]$, the polynomials that I annihilates when x, y, z act as partial derivatives. The top forms of the elements of M span spaces $D_k \subseteq T_k$, the *graded data*, which describe the tangent cone; their lower parts, the *tails*, describe how the scheme bends away from it.

The new idea is to link a limit of the scheme that is not its tangent cone. As a scheme is magnified by the homotheties $p \mapsto cp$, $c \to \infty$, it tends to its tangent cone, and the tests above measure the tangent cone or a mild deformation of it. A direction common to many surfaces of the tangent cone then costs in full. We deform the tails of the scheme along a curve and magnify at the same time, at matched speeds. The limit is a scheme with the same Hilbert function that keeps a chosen part of the tails. If it lies in a complete intersection of local length less than $2d$, so does a general scheme of the curve, as short linkage is an open condition, and the original scheme is smoothable, as smoothability is a closed one (Lemma 3.2). We call this *linkage at infinity*. Keeping a cubic tail on the top forms gives a test that depends only on the graded data, contains excess linkage, and needs no finiteness of the base scheme (Corollary 3.4); when the base scheme is finite, its value is a multiplicity, computed one direction at a time (Proposition 3.7). Keeping a quartic tail as well gives Lemma 3.9.

For example, let A have Hilbert function $(1, 3, 6, 5, 4, 1, 1)$; its top form is a sixth power, X^6 in suitable coordinates. Suppose that the cubics of the tangent cone cut out the single point $[X]$ of the plane of directions, with a local ring of length five that is a complete intersection but not curvilinear. Then, after a change of y and z, these cubics are xyz, $xy^2 - z^3$, y^3, y^2z, yz^2. Excess linkage gives $27 + 3 \cdot 5 = 42$, and so does the multiplicity of Proposition 3.7 when all quartics of the tangent cone are singular at $[X]$. Now add the quartic tail $-180X^2Y^2$ to the top form and magnify. The limit has the inverse system

$$T_{\leq 2} \oplus D_3 \oplus D_4 \oplus \langle\, 6X^5 - 360XY^2,\ X^6 - 180X^2Y^2 \,\rangle,$$

where $T_{\leq 2}$ is the space of polynomials of degree at most two. It is annihilated by

$$xyz, \qquad xy^2 - z^3 + x^5, \qquad y^3 + x^4 z;$$

the coefficient is -180 because $(xy^2 + x^5) \circ (X^6 + aX^2Y^2) = (720 + 4a)X$. The cubic cones of the three equations meet only at $[X]$. On the blow-up of the origin their proper transforms meet in one point over $[X]$; there the first one splits into two branches, which meet the curve cut out by the other two in lengths 6 and 8, so the complete intersection has local length

$$27 + 6 + 8 = 41$$

(Proposition 3.11). The quartic tail contributes the square of the radial parameter to the second equation, and the quintic $x^4 z$ contributes it to the third; together they save the last point.

1.5. When the tangent cone lies on one quadric, the cost of a root v of the binary quartic grows with the degree $\kappa(v)$ in which the tangent cone first separates it, and at length twenty-one it often reaches the budget. We cap it by moving one equation in a flat family through A that keeps the tangent cone (Section 4). Turning the quadric equation by the power b^{s-1} of a linear form that vanishes at no root separates all roots in degree $s - 1$, and caps each cost at $s - 3$ (Proposition 4.4). Sliding a cubic equation by a form of order five costs $2 \cdot (5 - 3) = 4$ for each unit of multiplicity of a root, $24 + 16 = 40$ in all, whatever the cubic does at the roots (Proposition 4.7). For $(1, 3, 5, 4, 4, 2, 2)$, radial costs can reach $24 + 5 \cdot 3 + 3 = 42$, and the turning power gives $24 + 3 \cdot 4 = 36$ for every algebra (Theorem 8.1).

1.6. Where we compute, we compute exactly: SageMath, over the rational numbers, finds the local lengths of complete intersections, checks that inverse systems with indeterminate tails are closed under differentiation, and measures parameter spaces at chosen members. The text names each computation where it enters, and Appendix D collects them.

The tools borrowed from [15] and [16] are collected in Section 2. Section 3 proves linkage at infinity, and Section 4 the moves of one equation. Section 5 sorts the Hilbert functions of length twenty-one by method. Sections 6, 7 and 8 treat the three groups that remain, with no quadric and socle degree at most five, with no quadric and socle degree at least six, and with one quadric; each proves the most instructive cases in full, and Appendices A–C give the rest. From Section 6 on, whenever we treat algebras of length d, we assume without saying so that all finite subschemes of $\mathbb{A}^3$ of length less than d are smoothable; for $d = 21$ this holds by [16].

2. Tools recalled

This section recalls, with their numbers there, the results of [15] and [16] that we use. Throughout, $A = S/I$ is a local algebra of length d, and each statement that A is smoothable assumes that all finite subschemes of $\mathbb{A}^3$ of length less than d are smoothable.

2.1. We work over $\mathbb{C}$, with $S = \mathbb{C}[x, y, z]$, $\mathfrak{m} = (x, y, z)$ and S_k the forms of degree k. The ring S acts on $T = \mathbb{C}[X, Y, Z]$ by differentiation, $g \circ F$, and $\langle g, F\rangle = (g \circ F)(0)$. The algebra A is supported at the origin. Its *inverse system* $M = \{F \in T : I \circ F = 0\}$ is closed under differentiation, of dimension d, and I is its orthogonal complement. The *leading form* of a polynomial in S is its lowest nonzero homogeneous part, and the *top form* of a polynomial in T its highest. The leading forms of the elements of I span the ideal I^* of the tangent cone, $H_A(k) = \dim S_k/I^*_k$, and the *socle degree* s is the largest k with $H_A(k) \neq 0$. The top forms of degree k of the elements of M form the orthogonal complement $D_k \subseteq T_k$ of I^*_k, the *graded data* of A [15, Lemma 2.5]. A nonzero $P \in T_1$ is a point $p = [P]$ of the plane $\mathbb{P}^2 = \mathbb{P}(T_1)$ of directions, $N_P \subseteq S_1$ is the plane of linear forms vanishing at p, and

$$P^k \in D_k \iff \text{every form of } I^*_k \text{ vanishes at } p \qquad [15, \text{equation (1)}]. \tag{1}$$

A form has r *essential variables* if its first derivatives span a space of dimension r; it is *concise* if $r = 3$ and a *power* if $r = 1$.

An element of M with top form F is a *lift* of F, and its homogeneous parts below F are its *tails*. When $H_A(2) = 6$, we have $I \subseteq \mathfrak{m}^3$ and $T_{\leq 2} \subseteq M$; tails of degree at most two lie in M, and we omit them. In socle degree five,

$$M = T_{\leq 2} \oplus D_3 \oplus \{u + \tau(u) : u \in D_4\} \oplus \{F + R(F) + E(F) : F \in D_5\} \tag{2}$$

with linear maps $\tau\colon D_4 \to T_3$, $R\colon D_5 \to T_4$, $E\colon D_5 \to T_3$. Given $S_1 \circ D_5 \subseteq D_4$ and $S_1 \circ D_4 \subseteq D_3$, this space is closed under differentiation if and only if

$$v \circ R(F) - \tau(v \circ F) \in D_3 \qquad \text{for all } v \in S_1 \text{ and } F \in D_5, \tag{3}$$

and the cubic tails E are free [15, Lemma E.8 and its proof]. So for fixed graded data the algebras are parametrized by a vector space of tails.

For a finite $\Gamma \subset \mathbb{P}^2$ let $\nu_k(\Gamma) = (I_\Gamma)^\perp_k \subseteq T_k$, spanned by the powers P^k, $P \in \Gamma$, when Γ is reduced [15, Section 2.13]. A plateau $H_A(k) = H_A(k+1) = n \leq k$ gives such a Γ of length n with $D_j = \nu_j(\Gamma)$ for $j = k, k+1$ and $D_j \subseteq \nu_j(\Gamma)$ for $j > k$ [15, Lemma 2.16].

2.2. Inverse systems with polynomial bases give families [15, Lemma 2.3(2)], and smoothability passes to limits [15, Proposition 2.2(1)]. So in an irreducible family one member suffices for an open condition: being Gorenstein [15, Proposition 2.20], or lying in a short complete intersection. If data are cut out by N equations in an irreducible variety $\mathcal{Q}$, strata of dimension less than $\dim \mathcal{Q} - N$ are not components [15, Lemmas 2.17 and 2.18]; strata of exactly this dimension may be.

2.3. For polynomial coordinates a, b, c centered at the origin, a *ray* of I is the ideal $L = (b, c)$ of a curve with

$$(I : L) \not\subseteq I + L; \tag{4}$$

along it one point leaves the scheme, while a scheme of length $d - 1$ stays behind, and A is smoothable [15, Theorem 2.11(1)]. The ideal (N_P) is a ray, along the line of p, if $P^k \in M$ but $P^k \notin N_P \circ M$ for some k [15,

Theorem 2.11(2)]. More generally, (b, c) is a ray if some $u \in S$ has $ub, uc \in I$ and some $\Psi \in M$ has $b \circ \Psi = c \circ \Psi = 0 \neq \langle u, \Psi \rangle$: then $\langle \cdot, \Psi \rangle$ vanishes on $I + L$ but not at $u \in (I : L)$; for $u = a^k$ this is [15, Lemma 5.4]. Such rays run along parabolas and curves of higher order [15, Lemmas 5.11 and 5.13].

2.4. Linkage replaces a scheme by the residual scheme in a complete intersection, which is shorter when the complete intersection is short. *If $V(I)$ lies, in some open $X \subseteq \mathbb{A}^3$, in a finite subscheme whose ideal is generated by three elements at each point, of length less than $2d$, then A is smoothable, and so is every member of an irreducible family that contains $V(I)$* [15, Corollary 2.23 and Proposition 2.24], from [12]. We call such a member a *linkage anchor* of the family. Elements of I whose leading forms of degrees k_1, k_2, k_3 have no common zero in $\mathbb{P}^2$ cut out a complete intersection of local length at most $k_1k_2k_3$ [15, Lemma 3.3]. Local lengths are certified by stabilization, $\dim S/(J + \mathfrak{m}^N) = \dim S/(J + \mathfrak{m}^{N+1})$, and a bound on them is an open condition on parameters [15, Section 2.21].

2.5. Three equations whose leading cubics meet in finitely many directions cut out 27 plus an excess. The excess lies over these directions, and it grows with the degree in which the surfaces through the scheme separate there. Suppose $H_A(2) = 6$. The *cubic base scheme* $\mathfrak{B} \subset \mathbb{P}^2$ is defined by the cubics of I_3^*. At a point p of a finite $\mathfrak{B}$, the *multiplicity* e_p is the Hilbert–Samuel multiplicity of the local ideal of $\mathfrak{B}$, the length when this ideal has two generators [15, Section 3.11]. The *separation degree* is $k(p) = \min\{k : 4 \leq k \leq s,\ P^k \notin D_k\}$, and $k(p) = s$ if there is no such k [15, equation (4)].

Theorem 2.6 (Excess linkage, [15, Theorem 3.12])**.** *Let $H_A(2) = 6$ and $s \geq 4$. If $\dim I_3^* \geq 3$, $\mathfrak{B}$ is finite, and*

$$27 + \sum_{p \in \mathfrak{B}} \big(k(p) - 3\big)\, e_p < 2d,$$

then A is smoothable.

The proof deforms a cubic tail, whence the cap at s, and measures three surfaces on the blow-up of the origin [5, Theorem 3(A) and its proof]. If $\ell(\mathfrak{B}) \leq 5$, then $\sum_p e_p \leq \ell(\mathfrak{B}) + 1$ [16, Lemma 3.2]; so in socle degree five, where every weight $k(p) - 3$ is at most 2, the left side is at most 39 [16, Corollary 3.3].

2.7. Three devices of [15] and [16] recur. *Free quartic tails* [15, Lemma D.3]: let $s \geq 6$ and $D_s = \mathbb{C}X^s$, and let $\mathcal{K}$ be the space of quartics R with $y \circ R, z \circ R \in D_3$; if the cubics $x \circ R$, $R \in \mathcal{K}$, span at least three dimensions modulo D_3, every $k(p)$ in Theorem 2.6 may be replaced by $\min(k(p), s - 1)$. *Global linkage* [15, Proposition 3.18]: if $H_A(2) = 6$, $d \geq 19$, $\mathfrak{m}^5 \subseteq I$, $\dim I_3^* \geq 3$, the cubics of I_3^* have no common factor, and every finite subscheme of $\mathbb{A}^3$ of length less than d is smoothable, then A is smoothable. The algebras with the graded data of A form an affine space of cubic tails, and one of

them lies in a complete intersection of degrees 3, 3, 4 and global length 36; a variant holds in socle degree five, where the tails must still prescribe the values needed [15, Lemma E.8]. *Adding a cubic* [16, Remark 3.7]: if $H(2) = 6$ and f_1, f_2, f_3 annihilate an inverse system M', and $G \in T_3$ lies outside the cubic top forms of M' and is orthogonal to the cubic parts of the f_i, then $M' \oplus \mathbb{C}G$ is an inverse system [15, Lemma A.14] annihilated by the f_i; so an anchor of length d gives one of length $d+1$ in the same complete intersection.

2.8. When the tangent cone lies on exactly one quadric and this quadric is a square, $I_2^* = \mathbb{C}x^2$ in suitable coordinates. For $f \in I$ with leading form x^2, the *transverse cubic* γ_f is the cubic part of $f(0, y, z)$. We write $\bar{T}_k = \mathbb{C}[Y, Z]_k$ and $\bar{S}_k = \mathbb{C}[y, z]_k$ for the binary forms on the line $x = 0$ of $\mathbb{P}^2$, and π for the restriction to the plane $x = 0$. In the *binary case* $I_3^* \subseteq xS_2$, that is, $I_3^* = x^2S_1 \oplus xW_2$ with $W_2 \subseteq \bar{S}_2$, the cubic γ_f does not depend on f. The binary case is a condition on the cubic equations; by itself it does not make all higher top forms binary. For a point $v = [V]$ of the line $x = 0$ let $\kappa(v) = \min\{k \geq 4 : V^k \notin D_k\}$, and $\kappa(v) = s + 1$ if there is no such $k \leq s$; by (1), $\kappa(v)$ is the first degree $k \geq 4$ in which some form of I_k^* does not vanish at v. Section 4.1 recalls how surfaces with rational double points and turning reduce the tangent cones on one quadric to the binary case with $\gamma_f = 0$ in socle degree at most six.

2.9. On this double plane, equations of orders 2, 3 and 4 meet in 24 plus an excess over the roots of a quartic; a root costs more the later the tangent cone separates it. Suppose that $I_2^* = \mathbb{C}x^2$, $I_3^* = xS_2$ and $\gamma_f = 0$. Then $D_k \subseteq \bar{T}_k$ and $I_k^* = xS_{k-1} \oplus J_k$ for $k \geq 4$, with $J_k = D_k^\perp \cap \bar{S}_k$, so $\kappa(v)$ is the first degree $k \geq 4$ in which some form of J_k does not vanish at v. Unlike $k(p)$, it is not capped at s.

Theorem 2.10 (Radial costs, [16, Theorem 3.15])**.** *Let $I_2^* = \mathbb{C}x^2$, $I_3^* = xS_2$ and $\gamma_f = 0$, and let $p \in J_4$ be nonzero. Then $V(I)$ lies, in an open set $X \ni 0$, in a finite complete intersection cut out by three elements of I of orders 2, 3 and 4, of length at most*

$$24 + \sum_{p(v)=0} (\kappa(v) - 2)\, \mu_v,$$

where μ_v is the multiplicity of the root v. If this number is less than $2d$, then A is smoothable.

2.11. With weights $2, 1, 1$ on x, y, z, the forms x^2, $x\,m$ with $m \in \bar{S}_2$, and the binary quartics all have weighted degree four; three equations with such weighted initial forms and no common zero other than the origin cut out at most $4^3/2 = 32$ [16, Lemma 3.11]. In the binary case assume $\gamma_f = 0$. For $m \in W_2$ choose $K_m \in I$ with cubic part xm, and let θ_m be π of its quartic part; it is well defined modulo $J_4 = \pi(I_4^*)$. A point v of the line $x = 0$ is *bad* if $g(v) = 0$ for all $g \in J_4$ and $m(v) = \theta_m(v) = 0$ for all $m \in W_2$.

Proposition 2.12 (Weighted linkage, [16, Proposition 3.12])**.** *Let $d \geq 17$, $I_2^* = \mathbb{C}x^2$ and $\mathfrak{m}^6 \subseteq I$, in the binary case with $\gamma_f = 0$. If $J_4 \neq 0$ and there is no bad point, then A is smoothable. This holds if $J_4 \neq 0$ and W_2 has no common zero on the line $x = 0$, for instance if $I_3^* = xS_2$ and $H_A(4) \leq 4$.*

2.13. In socle degree five, moving the tails within (3) bends the surfaces through the scheme but not their tangent cones, and prescribes the higher terms of three equations.

Lemma 2.14 (Prescribed equations, [16, Lemma 3.5])**.** *Let $H_A(2) = 6$ and $s = 5$. Let $w_1, w_2, w_3 \in I_3^*$ be linearly independent, let $p_1, p_2, p_3 \in S_4$ satisfy $p_i \circ D_5 = 0$, and let $r_1, r_2, r_3 \in S_5$. Then some algebra with the graded data of A has $w_i + p_i + r_i$, $i = 1, 2, 3$, in its ideal. If these three elements cut out a scheme of local length less than $2d$ at the origin, then every algebra with the graded data of A is smoothable.*

2.15. To find a short complete intersection, it suffices to bound the multiplicity of a suitable ideal.

Proposition 2.16 (Multiplicity principle, [16, Proposition 3.17])**.** *Let $L \subseteq S$ be an $\mathfrak{m}$-primary ideal containing three elements that generate an ideal of finite colength e in $S_\mathfrak{m}$.*

(a) *If $N \subseteq L$ is a reduction of L, then three general elements of N generate an ideal of colength at most e in $S_\mathfrak{m}$.*
(b) *If L is homogeneous, $I^* \subseteq L$, $L^2 = I^*L$ and $e < 2d$, then A is smoothable.*

3. Linkage at infinity

This section proves linkage at infinity, sketched in Section 1.4: we first deform the tails of the scheme and then magnify. The general principle is Lemma 3.2. Its first instance, Corollary 3.4, keeps a cubic tail on each top form; when the cubic base scheme is finite, its best value is a multiplicity, computed by a local formula (Proposition 3.7). Its second instance, Lemma 3.9, keeps a quartic tail as well, and Proposition 3.11 shows it at work. Lemma 3.13 extends all of this to families.

3.1. For $c \in \mathbb{C}^*$ let h_c be the homothety $p \mapsto cp$ of $\mathbb{A}^3$. The image $h_c(V(I))$ has the ideal $\{g(x/c) : g \in I\}$ and the inverse system $\{G(cX) : G \in M\}$, since a monomial x^a pairs with X^a and the two rescalings cancel. For $\varepsilon \neq 0$ write $G^{[\varepsilon]} = G(X/\varepsilon)$. If G has degree k, then $\varepsilon^k G^{[\varepsilon]}$ is a polynomial in ε whose value at $\varepsilon = 0$ is the top form of G. So, as $\varepsilon \to 0$, the schemes $h_{1/\varepsilon}(V(I))$, the magnifications of $V(I)$, tend to the scheme defined by the tangent cone ideal I^*, whose inverse system $\bigoplus_k D_k$ is spanned by the top forms.

Lemma 3.2 (Linkage at infinity)**.** *Let A be a local algebra of length d, and suppose that every finite subscheme of $\mathbb{A}^3$ of length less than d is smoothable. Let U be an irreducible variety with a morphism $U \to \mathrm{Hilb}_d(\mathbb{A}^3)$, $u \mapsto Z_u$,*

such that $Z_{u_0} = V(I)$ for some $u_0 \in U$. If some point of the closure of $\{h_c(Z_u) : c \in \mathbb{C}^*,\ u \in U\}$ in $\mathrm{Hilb}_d(\mathbb{A}^3)$ lies, in some open subset of $\mathbb{A}^3$, in a finite local complete intersection of length less than $2d$, then A is smoothable.

Proof. The map $\mathbb{C}^* \times U \to \mathrm{Hilb}_d(\mathbb{A}^3)$, $(c, u) \mapsto h_c(Z_u)$, is a morphism, so the closure $\mathcal{W}$ of its image is irreducible. Let $\mathcal{O} \subseteq \mathrm{Hilb}_d(\mathbb{A}^3)$ be the set of schemes that lie, in some open subset of $\mathbb{A}^3$, in a finite local complete intersection of length less than $2d$. It is open, by the proof of [15, Proposition 2.24], and its members are smoothable, as they are linked to schemes of length less than d (Section 2.4). By hypothesis $\mathcal{O} \cap \mathcal{W}$ is a nonempty open subset of $\mathcal{W}$, hence dense, and smoothability is a closed condition [15, Proposition 2.2(1)]. So every member of $\mathcal{W}$ is smoothable, among them $V(I) = h_1(Z_{u_0})$. □

The lemma is useful when the family is a deformation of the tails of A that grows as we magnify, so that the two effects balance. The argument never uses the smoothability of the limit: the openness of short linkage carries the conclusion from the limit to a general member of U, and the closedness of smoothability carries it to A.

3.3. The first instance keeps a cubic tail on each top form of the highest degree. Excess linkage [15, Theorem 3.12] deforms these cubic tails; here we deform them and magnify at the same time.

Corollary 3.4 (Cubic tails at infinity)**.** *Let $A = S/I$ be a local algebra of length d with $H_A(2) = 6$, socle degree $s \geq 4$ and graded data $D_3, \dots, D_s$, and suppose that every finite subscheme of $\mathbb{A}^3$ of length less than d is smoothable. Let $Q\colon D_s \to T_3$ be linear, and*

$$M_Q = T_{\leq 2} \oplus D_3 \oplus \cdots \oplus D_{s-1} \oplus \{F + Q(F) : F \in D_s\}.$$

Then M_Q is the inverse system of a local algebra of length d with the graded data of A. If three elements of $\mathrm{Ann}(M_Q)$ generate an ideal of colength less than $2d$ in $S_{\mathfrak{m}}$, then every local algebra with the graded data $D_3, \dots, D_s$ is smoothable.

No hypothesis on the cubic base scheme is needed: it may contain a line.

Proof. The space M_Q is closed under differentiation, since $S_1 \circ D_{k+1} \subseteq D_k$ and $v \circ (F + Q(F)) \in D_{s-1} \oplus T_2$ for $v \in S_1$. Its spaces of top forms are the D_k, so its dimension is d.

Choose $\Phi_F \in M$ with top form F, linearly in $F \in D_s$, and let $M_{<s}$ be the space of elements of M of degree less than s, so that $M = M_{<s} \oplus \{\Phi_F\}$. For $\lambda \in \mathbb{C}$ the space $M_\lambda = M_{<s} \oplus \{\Phi_F + \lambda Q(F)\}$ is closed, because the derivatives of the cubics $Q(F)$ lie in $T_2 \subseteq M_{<s}$, and it has the graded data of A. By [15, Lemma 2.3(2)] the schemes $Z_\lambda = V(\mathrm{Ann}\, M_\lambda)$ form a family over $\mathbb{A}^1$ with $Z_0 = V(I)$; this is the line of Step 1 of the proof of [15, Theorem 3.12].

Choose a basis $G_1, \dots, G_r$ of $M_{<s}$ with linearly independent top forms; by dimension these top forms span $T_{\leq 2} \oplus D_3 \oplus \cdots \oplus D_{s-1}$. Choose a basis

$F_1, \dots, F_m$ of D_s. For $\varepsilon \in \mathbb{C}$ put

$$\varepsilon^{\deg G_i} G_i^{[\varepsilon]}, \qquad \varepsilon^s \big(\Phi_{F_j} + \varepsilon^{-(s-3)} Q(F_j)\big)^{[\varepsilon]} = \sum_{k \le s} \varepsilon^{s-k} (\Phi_{F_j})_k + Q(F_j),$$

where $(\cdot)_k$ is the part of degree k. These are polynomials in ε. At $\varepsilon = 0$ they are the top forms of the G_i and the $F_j + Q(F_j)$: linearly independent, spanning the closed space M_Q. For $\varepsilon \neq 0$ they span $\{G^{[\varepsilon]} : G \in M_\lambda\}$ with $\lambda = \varepsilon^{-(s-3)}$, the inverse system of $h_{1/\varepsilon}(Z_\lambda)$. By [15, Lemma 2.3(2)] they define a morphism $\mathbb{A}^1 \to \mathrm{Hilb}_d(\mathbb{A}^3)$, so $V(\operatorname{Ann} M_Q)$ lies in the closure of $\{h_c(Z_\lambda)\}$. If three elements of $\operatorname{Ann}(M_Q)$ generate an ideal of colength $e < 2d$ in $S_{\mathfrak{m}}$, their zero set is, in an open set around the origin, a finite local complete intersection of length e that contains $V(\operatorname{Ann} M_Q)$. Lemma 3.2, with U the λ-line, shows that A is smoothable. As M_Q depends only on the graded data and on Q, the same holds for every algebra with these graded data. (Script `l21_general_s6`, class S, follows the limit on actual algebras.) □

3.5. The equations available in $\operatorname{Ann}(M_Q)$ are easy to describe. Let $w_1, w_2, w_3 \in I_3^*$ be linearly independent, let $c_1, c_2, c_3 \in I_4^* + \dots + I_{s-1}^*$, and let $q_1, q_2, q_3 \in S_s$ and $r_1, r_2, r_3 \in \mathfrak{m}^{s+1}$ be arbitrary. Choose $C_j \in T_3$ with $\langle w_i, C_j \rangle = \delta_{ij}$ and put $Q(F) = -\sum_j \langle q_j, F \rangle C_j$. Then

$$f_i = w_i + c_i + q_i + r_i \in \operatorname{Ann}(M_Q) \qquad \text{for } i = 1, 2, 3, \tag{5}$$

because f_i is orthogonal to $T_{\le 2}$ and to $D_3, \dots, D_{s-1}$ by degrees, and $\langle f_i, F + Q(F) \rangle = \langle q_i, F \rangle + \langle w_i, Q(F) \rangle = 0$; for a closed space, orthogonality is annihilation [15, Lemma 2.5(1)]. Conversely, every triple of elements of the ideal

$$L_{\mathrm{def}} = I_3^* + I_4^* + \dots + I_{s-1}^* + \mathfrak{m}^s$$

with linearly independent cubic parts has this form, for a suitable Q; and $\operatorname{Ann}(M_Q) \subseteq L_{\mathrm{def}}$. So Corollary 3.4 applies whenever three elements of L_{def} with linearly independent cubic parts generate an ideal of colength less than $2d$. The tangent cone ideal I^*, which serves the multiplicity principle (Proposition 2.16(b) with $L = I^*$), is contained in L_{def} and differs from it only in degree s, where $I_s^* = D_s^\perp$: the moving cubic tail frees the top layer.

3.6. When the cubic base scheme $\mathfrak{B}$ is finite, the best value of Corollary 3.4 is the multiplicity of L_{def}, and it can be computed one base point at a time. Fix $v \in \mathfrak{B}$, choose coordinates with $v = [1 : 0 : 0]$, and use the chart $x = \rho$, $y = \rho\eta$, $z = \rho\zeta$ of the blow-up of the origin, with local ring $\mathcal{O}_v = \mathbb{C}\{\rho, \eta, \zeta\}$ at v. For a homogeneous ideal $L \subseteq S$ with $L_k = 0$ for $k < 3$, $L_3 = I_3^*$ and $\mathfrak{m}^N \subseteq L$ for some N, let

$$K_v(L) = \big(\rho^{k-3} c(1, \eta, \zeta) : \ c \in L_k, \ k \ge 3\big) \subseteq \mathcal{O}_v.$$

If $\mathfrak{b}_k$ is the local ideal at v of the base scheme of the linear system L_k on $\mathbb{P}^2$, then $K_v(L) = (\mathfrak{b}_3, \rho\,\mathfrak{b}_4, \rho^2 \mathfrak{b}_5, \dots, \rho^{N-3})$; here $\mathfrak{b}_3$ is the local ideal of $\mathfrak{B}$. Let $e(\cdot)$ denote the Hilbert–Samuel multiplicity.

Proposition 3.7 (Multiplicities at infinity)**.** *Let $A = S/I$ be a local algebra of length d with $H_A(2) = 6$, socle degree $s \geq 4$, $\dim I_3^* \geq 3$ and finite cubic base scheme $\mathfrak{B}$.*

(a) *For every L as above, $e(L) = 27 + \sum_{v\in\mathfrak{B}} e(K_v(L))$.*
(b) *$e\big(K_v(L_{\mathrm{def}})\big) \leq \big(k(v) - 3\big)e_v$ for every $v \in \mathfrak{B}$.*
(c) *If every finite subscheme of $\mathbb{A}^3$ of length less than d is smoothable and $e(L_{\mathrm{def}}) < 2d$, then every local algebra with the Hilbert function of A and the graded data $D_3, \dots, D_{s-1}$ of A is smoothable.*

Proof. (a) Fix bases of $L_3, \dots, L_N$. Three general linear combinations F_1, F_2, F_3 of these elements generate a reduction of $LS_{\mathfrak{m}}$, and the same coefficients applied to the elements $\rho^{k-3}c(1, \eta, \zeta)$ give, for each of the finitely many v, three generators of a reduction of $K_v(L)$ [7, Theorem 8.6.6 and Corollary 8.3.9]; each of these conditions holds on a nonempty open set of coefficients. We may also take the cubic parts of the F_i general in I_3^*, so that they meet in $\mathbb{P}^2$ exactly at the points of $\mathfrak{B}$. If three elements generate a reduction J of an ideal K of finite colength in a regular local ring of dimension three, then the colength of J is $e(J) = e(K)$ [7, Corollary 1.2.5, Propositions 11.2.1 and 11.1.10(2)]. As $F_i(\rho, \rho\eta, \rho\zeta) = \rho^3 \cdot \sum_k \rho^{k-3}c_{ik}(1, \eta, \zeta)$, the second factors define the proper transforms of the surfaces $V(F_i)$, and over v they meet in length $e(K_v(L))$. So [16, Lemma 3.14(a)] gives $e(L) = \ell(S_{\mathfrak{m}}/(F_1, F_2, F_3)) = 27 + \sum_v e(K_v(L))$.

(b) If $k(v) < s$, some form of $I^*_{k(v)}$ does not vanish at v, by (1), so $\rho^{k(v)-3} \in K_v(L_{\mathrm{def}})$; if $k(v) = s$, then $\rho^{s-3} \in K_v(L_{\mathrm{def}})$ because $\mathfrak{m}^s \subseteq L_{\mathrm{def}}$. If (a_1, a_2) is a reduction of $\mathfrak{b}_3$, the ideal $K_v(L_{\mathrm{def}})$ contains $(a_1, a_2, \rho^{k(v)-3})$, of colength $(k(v) - 3)\, e_v$, and the multiplicity of an ideal is at most that of a parameter ideal inside it.

(c) Three general elements of L_{def} generate an ideal of colength $e(L_{\mathrm{def}})$ in $S_{\mathfrak{m}}$, as in (a), and their cubic parts are linearly independent. By Section 3.5, Corollary 3.4 applies to every algebra with the Hilbert function of A and its graded data in degrees $3, \dots, s - 1$, as L_{def} depends only on these. □

By (a) and (b), part (c) contains excess linkage [15, Theorem 3.12]: it uses the whole flag of base schemes $\mathfrak{B} = \mathfrak{B}_3 \supseteq \mathfrak{B}_4 \supseteq \cdots$ at v, not only the first degree that separates v. For $L = I^*$ the top power in $K_v(I^*)$ is ρ^{s-2} instead of ρ^{s-3}, so $e(L_{\mathrm{def}}) \leq e(I^*)$. In practice, $e(K_v(L))$ is bounded by the local length of any three explicit elements of $K_v(L)$ that generate an ideal of finite colength, and it depends only on the germs at v of the flag $\mathfrak{B}_3 \supseteq \mathfrak{B}_4 \supseteq \cdots$. So a stratum on which these germs are fixed up to an analytic isomorphism is settled by one local computation. (Script `l21_general_WM` compares (a) with a global computation on three graded data, class S, and bounds the local values on fourteen graded data, class A.)

3.8. The second instance keeps a quartic tail as well. A cubic tail can be added to a top lift freely, as its derivatives lie in $T_{\leq 2} \subseteq M$. A quartic tail R has cubic derivatives, so it can ride on the lift Φ of the top form only if these

derivatives stay in M: we ask this along a plane N_ℓ of linear forms, and we move the remaining derivative $\ell \circ \Phi$ together with Φ. This is the magnified form of the free quartic tails of [15, Lemma D.3], with two new freedoms: the top form need not be a power, and a cubic tail rides on top. Let $D_s = \mathbb{C}F$, and let $\ell \in S_1$ have $\langle \ell^s, F\rangle \neq 0$. Put $N_\ell = \{v \in S_1 : \langle \ell^{s-1}v, F\rangle = 0\}$, $D'_{s-1} = D_{s-1} \cap (\ell^{s-1})^\perp$ and $K_\ell = \{R \in T_4 : N_\ell \circ R \subseteq D_3\}$.

Lemma 3.9 (A quartic tail at infinity)**.** *Let $A = S/I$ be a local algebra of length d with $H_A(2) = 6$, socle degree $s \geq 5$ and $D_s = \mathbb{C}F$, and suppose that every finite subscheme of $\mathbb{A}^3$ of length less than d is smoothable. Let ℓ be as above, $R \in K_\ell$ and $C \in T_3$, and put*

$$M_\infty = T_{\leq 2} \oplus D_3 \oplus \cdots \oplus D_{s-2} \oplus D'_{s-1} \oplus \mathbb{C}\,\ell \circ (F + R + C) \oplus \mathbb{C}\,(F + R + C).$$

Then M_∞ is the inverse system of a local algebra with the graded data of A. If three elements of $\operatorname{Ann}(M_\infty)$ generate an ideal of colength less than $2d$ in $S_{\mathfrak{m}}$, then every local algebra with the graded data $D_3, \ldots, D_s$ is smoothable.

Proof. Let $c_0 = \langle \ell^s, F\rangle \neq 0$. Then N_ℓ is a plane not containing ℓ, and D'_{s-1} is a hyperplane of D_{s-1} not containing $\ell \circ F$, as $\langle \ell^{s-1}, \ell \circ F\rangle = c_0$. Let $\Phi \in M$ have top form F, and $M'' = \{m \in M : \langle \ell^{s-1}, m\rangle = \langle \ell^s, m\rangle = 0\}$. The two functionals take the values $(c_0, 0)$ on $\ell \circ \Phi$ and $(*, c_0)$ on Φ, so $M = M'' \oplus \mathbb{C}\ell \circ \Phi \oplus \mathbb{C}\Phi$. The elements of M'' have degree at most $s-1$, so the two functionals vanish on their derivatives, and M'' is closed. It contains all elements of M of degree at most $s-2$, among them $T_{\leq 2} \oplus D_3$ and the derivatives of $\ell \circ \Phi$, and it contains $v \circ \Phi$ for $v \in N_\ell$, as $\langle \ell^{s-1}v, \Phi\rangle = \langle \ell^{s-1}v, F\rangle = 0$. For $(\lambda, \mu) \in \mathbb{A}^2$ put

$$M_{\lambda,\mu} = M'' \oplus \mathbb{C}\big(\ell \circ \Phi + \lambda\, \ell \circ R + \mu\, \ell \circ C\big) \oplus \mathbb{C}\big(\Phi + \lambda R + \mu C\big).$$

For $v \in N_\ell$, $v \circ (\Phi + \lambda R + \mu C) \in M'' + D_3 + T_2 = M''$; the derivative by ℓ is the middle element, and $N_\ell + \mathbb{C}\ell = S_1$; the derivatives of the middle element lie in $M'' + T_2 + T_1 = M''$. So $M_{\lambda,\mu}$ is closed. The added terms have degrees at most $4 < s$ and $3 < s-1$, so it has the graded data of A, and by [15, Lemma 2.3(2)] these spaces form a family over $\mathbb{A}^2$ with $V(I)$ at $(0,0)$. The same argument with the top forms shows that M_∞ is closed, as $v \circ F \in D'_{s-1}$ for $v \in N_\ell$; its top forms are those of the D_k, as $D'_{s-1} \oplus \mathbb{C}\ell \circ F = D_{s-1}$.

Now rescale a basis of M'' with independent top forms as in the proof of Corollary 3.4, and add $\varepsilon^{s-1}(\ell \circ \Phi + \lambda\, \ell \circ R + \mu\, \ell \circ C)^{[\varepsilon]}$ and $\varepsilon^s(\Phi + \lambda R + \mu C)^{[\varepsilon]}$ with $\lambda = \varepsilon^{-(s-4)}$ and $\mu = \varepsilon^{-(s-3)}$. As $R^{[\varepsilon]} = \varepsilon^{-4}R$ and $C^{[\varepsilon]} = \varepsilon^{-3}C$, the last two elements are $\ell \circ (F + R + C) + O(\varepsilon)$ and $F + R + C + O(\varepsilon)$, polynomial in ε. At $\varepsilon = 0$ all these elements span M_∞; so, by [15, Lemma 2.3(2)], $V(\operatorname{Ann} M_\infty)$ lies in the closure of the magnifications of the family $M_{\lambda,\mu}$, and Lemma 3.2 with $U = \mathbb{A}^2$ concludes. The space M_∞ depends only on $D_3, \ldots, D_s$, ℓ, R and C. □

For $R = 0$ this is the case $\dim D_s = 1$ of Corollary 3.4, as $\ell \circ C \in T_2$. For $F = X^s$, $\ell = x$ and $C = 0$, the space K_ℓ is the space of quartics of [15, Lemma D.3], but no rank condition is needed: the test is one explicit ideal.

3.10. The following case shows why the graded tests can stop just above the budget, and how a quartic tail at infinity brings the count below it. Let $H_A = (1,3,6,5,4,1,1)$. Then $D_6 = \mathbb{C}X^6$ and $D_5 = \mathbb{C}X^5$ in suitable coordinates, as $S_1 \circ D_6 \subseteq D_5$ forces the top form to be a power. Every point of the cubic base scheme has weight $k(p) - 3 \leq 3$, and $[X]$ lies in it and has weight exactly three, as $X^k \in D_k$ for $3 \leq k \leq 6$.

Proposition 3.11 (A non-curvilinear base point)**.** *Let A be a local algebra with Hilbert function $(1,3,6,5,4,1,1)$ whose cubic base scheme $\mathfrak{B}$ is a single point of length five, with a local ideal that is a complete intersection but not curvilinear. Then A is smoothable.*

Excess linkage gives $27 + 3 \cdot 5 = 42$ here, one more than the budget.

Proof. We first put $\mathfrak{B}$ in normal form, then apply Lemma 3.9 with a quartic tail chosen so that the limit contains three equations of local length $27+6+8$.

Normal form. Let $\mathfrak{b} \subseteq \mathcal{O} = \mathbb{C}\{u,w\}$ be the local ideal of $\mathfrak{B}$ at $[X]$, with $u = y/x$ and $w = z/x$. As $\mathcal{O}/\mathfrak{b}$ has length five and $\mathfrak{b} \subseteq \mathfrak{m}_{\mathcal{O}}^2$, its Hilbert function is $(1,2,2)$ or $(1,2,1,1)$. A complete intersection is Gorenstein, while $(1,2,2)$ would put a space of dimension two into the socle; so it is $(1,2,1,1)$. The quadratic initial forms of $\mathfrak{b}$ form a pencil with a common factor, as two coprime ones would bound the length by four; we take it to be u, so the initial ideal is (u^2, uw, w^4), and $1, u, w, w^2, w^3$ is a basis of $\mathcal{O}/\mathfrak{b}$. Hence $u^2 \equiv \alpha w^3$ and $uw \equiv \beta w^3$ modulo $\mathfrak{b}$, and $\alpha \neq 0$, since otherwise w^3 and $u - \beta w^2$ would be two independent socle elements. As $\mathfrak{m}_{\mathcal{O}}^4 \subseteq \mathfrak{b}$, the new coordinates $u' = \alpha u$ and $w' = \alpha w - \beta u$ satisfy $u'w' \equiv 0$ and $u'^2 \equiv \alpha^3 w^3 \equiv w'^3$; so $(u'w', u'^2 - w'^3) \subseteq \mathfrak{b}$, with equality, as both have colength five. So, after a linear change of y and z, $\mathfrak{b} = (uw,\ u^2 - w^3)$, a complete intersection of colength $2 + 3 = 5$. Only y and z have changed, so $D_6 = \mathbb{C}X^6$ and $D_5 = \mathbb{C}X^5$ still hold. The five cubics below dehomogenize into $\mathfrak{b}$ (for instance $u^3 \equiv uw^3 = w^2 \cdot uw \equiv 0$). The restrictions of $x^3, x^2y, x^2z, xz^2, z^3$ are the basis $1, u, w, w^2, w^3$ of $\mathcal{O}/\mathfrak{b}$, so $S_3 \to \mathcal{O}/\mathfrak{b}$ is onto and $\dim(I_{\mathfrak{B}})_3 = 10 - 5 = 5 = \dim I_3^*$; so

$$I_3^* = (I_{\mathfrak{B}})_3 = \langle\, xyz,\ xy^2 - z^3,\ y^3,\ y^2z,\ yz^2 \,\rangle,$$

and $D_3 = \langle X^3, X^2Y, X^2Z, XZ^2, Z^3 + 3XY^2 \rangle$.

The limit. Apply Lemma 3.9 with $s = 6$, $F = X^6$, $\ell = x$, $C = 0$ and $R = -180X^2Y^2$; the coefficient is -180 because $(xy^2 + x^5) \circ (X^6 + aX^2Y^2) = (720 + 4a)X$. Then $N_\ell = \langle y, z \rangle$, and $R \in K_\ell$ because $y \circ R = -360X^2Y \in D_3$ and $z \circ R = 0$; moreover $D_5' = 0$, since $\langle x^5, X^5 \rangle \neq 0$. So

$$M_\infty = T_{\leq 2} \oplus D_3 \oplus D_4 \oplus \langle\, 6X^5 - 360XY^2,\ X^6 - 180X^2Y^2 \,\rangle.$$

It is annihilated by

$$J = (\, xyz,\ \ xy^2 - z^3 + x^5,\ \ y^3 + x^4z \,).$$

Indeed the cubic parts lie in I_3^*; the second generator pairs with $6X^5 - 360XY^2$ to $6 \cdot 120 - 360 \cdot 2 = 0$; and no other pairing involves a nonzero coefficient.

As M_∞ is closed, orthogonality is annihilation. Note that J does not depend on D_4.

The length. The cubic cones xyz, $xy^2 - z^3$, y^3 meet only at $[X]$. In the chart $x = \rho$, $y = \rho\eta$, $z = \rho\zeta$ the proper transforms are

$$\eta\zeta, \qquad \eta^2 - \zeta^3 + \rho^2, \qquad \eta^3 + \rho^2\zeta.$$

The last two define a complete intersection curve germ Γ. Its quotients by ζ and by η are finite, so ζ and η are nonzerodivisors on Γ, and length is additive on their product: the length is the sum of two lengths. On $\zeta = 0$ we get $\mathbb{C}\{\rho, \eta\}/(\eta^2 + \rho^2, \eta^3)$, of length $2 \cdot 3 = 6$. On $\eta = 0$ the third equation minus ζ times the second is ζ^4, so we get $\mathbb{C}\{\rho, \zeta\}/(\zeta^4, \rho^2 - \zeta^3)$, of length $4 \cdot 2 = 8$. By [16, Lemma 3.14(a)], $V(J)$ has local length

$$27 + 6 + 8 = 41 < 42$$

at the origin, and Lemma 3.9, with all schemes of length at most twenty smoothable by [16], concludes for every algebra with these graded data. (Script `l21_1365411_qt`, part (J): local length 41 certified by a local standard basis and by stabilization, also without the normalization, for seven pairs (α, β); part (E) checks the whole argument on ten actual algebras. Script `l21_explicit_triples`, parts (N) and (J41), checks the normal form, and that J annihilates M_∞ and has local length 41 after the normalization.) $\square$

The last point is saved by two correlated corrections. The quartic tail $-180X^2Y^2$ adds ρ^2 to the second proper transform, and the quintic x^4z, which annihilates M_∞, adds $\rho^2\zeta$ to the third. On the branch $\zeta = 0$ the first turns η^2 into the two transverse branches of $\eta^2 + \rho^2$; on the branch $\eta = 0$ the two together leave ζ^4 and $\rho^2 - \zeta^3$.

The quartic tail works for every possible D_4; the cubic tails of Proposition 3.7(c) are more sensitive to it. In the normal form above, $\nu_4(\mathfrak{B}) = \langle X^4, X^3Y, X^3Z, X^2Z^2, Q_4\rangle$ with $Q_4 = \frac{3}{2}X^2Y^2 + XZ^3$, and the possible D_4 are the hyperplanes of $\nu_4(\mathfrak{B})$ that contain X^4, as $S_1 \circ D_4 \subseteq D_3$ and $X^4 \in x^2 \circ D_6$. They form a $\mathbb{P}^3$; those that contain X^3Z form a plane, and those that contain X^3Y and X^3Z a line. For $D_4 = \langle X^4, X^3Y, X^2Z^2, Q_4\rangle$ the elements

$$xyz + x^6, \qquad xy^2 - z^3, \qquad y^3 + x^3z + x^6$$

of L_{def} annihilate $M_Q = T_{\leq 2} \oplus D_3 \oplus D_4 \oplus \mathbb{C}X^5 \oplus \mathbb{C}(X^6 - 720XYZ - 120Y^3)$ of Corollary 3.4, and have local length 38. For $D_4 = \langle X^4, X^3Z, X^2Z^2, Q_4\rangle$ the elements $xyz + x^6$, $xy^2 - z^3$, $y^3 + x^3y + x^4z$ annihilate the space with top lift $X^6 - 720XYZ$, and have local length 41 (script `l21_explicit_triples`, parts (A38), (A41)). As the bound is an open condition on D_4 (as in the proof of Lemma 3.13), Proposition 3.7(c) gives at most 38 on a dense open subset of the $\mathbb{P}^3$, and at most 41 on a dense open subset of the plane. On the line, where all quartics of I_4^* are singular at $[X]$, it gives exactly 42: there $\mathfrak{b}_4 \subseteq (\eta, \zeta)^2$ and $\mathfrak{b}_5 = (\eta, \zeta)$, so $K_{[X]}(L_{\mathrm{def}})$ lies in the monomial ideal $K' = (\eta^2, \eta\zeta, \zeta^3, \rho\zeta^2, \rho^2\eta, \rho^2\zeta, \rho^3)$, of multiplicity 15 by its Newton

polyhedron, and also by a parameter ideal $J' \subseteq K'$ of local colength 15 with $(K'^2)_{\mathfrak{n}} = (J'K')_{\mathfrak{n}}$ at the origin $\mathfrak{n}$, a reduction after localization (script `l21_1365411_testc`, part (U42)), and Proposition 3.7(b) bounds it by $3 \cdot 5$.

3.12. The tests of this section depend only on graded data. In a family of actual algebras one certified member suffices.

Lemma 3.13 (One member suffices)**.** *Let U be an irreducible variety with a morphism $U \to \mathrm{Hilb}_d(\mathbb{A}^3)$ whose members are local algebras A_u with $H_{A_u}(2) = 6$, a fixed Hilbert function and socle degree $s \geq 4$, such that, for $3 \leq k \leq s-1$, the spaces $I_k^*(A_u)$ are the fibres of a subbundle of $U \times S_k$, and suppose that every finite subscheme of $\mathbb{A}^3$ of length less than d is smoothable. If for one $u_0 \in U$ three elements of $L_{\mathrm{def}}(A_{u_0})$ with linearly independent cubic parts generate an ideal of colength less than $2d$ in $S_{\mathfrak{m}}$, then every A_u is smoothable.*

Proof. Near u_0 choose frames of the bundles $I_3^*, \dots, I_{s-1}^*$, and write the three elements with constant coefficients in these frames and in the monomials of degree at least s. This gives triples $f_i(u) \in L_{\mathrm{def}}(A_u)$, depending regularly on u in an open neighbourhood of u_0. Linear independence of the cubic parts, and a local length less than $2d$, are open conditions on u (the second by the criterion after [15, Proposition 2.24]). So they hold on a nonempty open subset of U, where Section 3.5 and Corollary 3.4 make A_u smoothable. This subset is dense, and smoothability is a closed condition [15, Proposition 2.2(1)]. □

The lemma turns a classification of graded data into a classification of irreducible families: it suffices to know that the algebras with given discrete invariants form a few irreducible families, and to find one member of each whose graded data pass the test. We use it in this form for $(1, 3, 6, 5, 3, 2, 1)$ (Appendix B).

4. Moving one equation

When the tangent cone lies on exactly one quadric, three equations of orders 2, 3 and 4 cost 24, plus the radial cost of each root of a binary quartic (Theorem 2.10). A root that the tangent cone separates late is expensive. This section lowers these costs by moving one of the three equations in a flat family through A, without changing the tangent cone: we turn the quadric equation by a power of a linear form (Proposition 4.4), or slide a cubic equation by a form of higher order (Lemma 4.6 and Proposition 4.7). On the blow-up of the origin the added term becomes a unit times a power of the radial parameter, and this power, not the separation degree, caps the cost of a root. A third device, turning past a quadratic defect, is used only in Appendix C and is proved there (Lemma C.3).

4.1. We recall the setting of [16, Section 2.8]. On a quadric we look for a surface through the scheme with a rational double point, where Zheng's theorem applies [18, Theorem 3.3], see also [2]. So A is smoothable if

$H_A(2) \leq 4$, or $H_A(2) = 5$ and $H_A(3) \leq 2$, or the tangent cone lies on a quadric that is not a square, or $I_2^* = \mathbb{C}x^2$ and γ_f is not a cube [14], [15, Theorem 2.9]; here $0 = 0^3$ counts as a cube. Cubes can be *turned*. If $u \in \mathfrak{m}^3$ and $\mathfrak{m}^2 u \subseteq I$, then I is the special member of a flat family I_t whose members contain $f + tu$ and $I \cap \mathfrak{m}^4$ [15, Lemma 4.2]; by its proof they also contain every element of $I \cap \mathfrak{m}^3$ whose cubic part lies in a fixed complement of $x^2 S_1$ in I_3^*. We call this family the *turning family* of u. Its members have the tangent cone of A: the leading forms of these elements and of the $l(f + tu)$, $l \in S_1$, span I^*, and the colength is constant. An element $u \in \mathfrak{m}^3$ lies in $(I : \mathfrak{m}^2)$ if and only if $u \circ M \subseteq T_{\leq 1}$. We call u a *turning operator*, of type (a) or (b), if

(a) $u \in (I : \mathfrak{m}^2) \cap \mathfrak{m}^3$ and π of its cubic part is not a cube [15, Corollary 4.6];
(b) $\mathfrak{m}^7 \subseteq I$, $u \in (I : \mathfrak{m}^2) \cap \mathfrak{m}^3$ and π of its cubic part is not zero [15, Corollary 4.8].

In both cases A is smoothable, without induction.

4.2. As $I \cap \mathfrak{m}^3 \subseteq (I : \mathfrak{m}^2)$, type (b) leaves only the binary case of Section 2.8 when $\mathfrak{m}^7 \subseteq I$; and then $\gamma_f \neq 0$ makes A smoothable [16, Lemma 3.9]. With $\gamma_f = 0$, write $f = x^2 + xa + x^2 l + O(\mathfrak{m}^4)$ with $a \in \bar{S}_2$ and $l \in S_1$; the shear $x \mapsto x - a/2$ of [16, Lemma B.4] keeps every leading form and gives f with cubic part in $x^2 S_1$. These reductions use no induction, and we make them without comment when $\mathfrak{m}^7 \subseteq I$. For $m \in W_2$ we choose $K_m \in I$ with cubic part exactly xm, linearly in m, and θ_m is the binary part of its quartic part, as in Section 2.11.

4.3. In the proof of Theorem 2.10, a root v of the quartic costs $\kappa(v) - 2$ layers because the quadric equation only meets an equation that separates v in degree $\kappa(v)$. A power of a linear form that vanishes at no root separates every root at once, in degree $s - 1$, and it can be added to the quadric equation by turning.

Proposition 4.4 (Radial costs with a turning power)**.** *Let $A = S/I$ be a local algebra of length d with $I_2^* = \mathbb{C}x^2$ and socle degree $s \geq 5$, in the binary case with $\gamma_f = 0$. Let $p \in I_4^*$ have $\pi(p) \neq 0$, with roots v of multiplicities μ_v on the line $x = 0$, and let $r \in W_2$ and $b \in \bar{S}_1$ vanish at no root of $\pi(p)$. Then $b^{s-1} \in (I : \mathfrak{m}^2)$, and for general t the member I_t of its turning family lies, in an open set around the origin, in a finite complete intersection of local length at most*

$$24 + \sum_{\pi(p)(v)=0} \big(\min(\kappa(v), s-1) - 2\big)\,\mu_v \;\leq\; 24 + 4(s-3).$$

If this number is less than $2d$ and every finite subscheme of $\mathbb{A}^3$ of length less than d is smoothable, then A is smoothable.

Proof. We follow the proof of Theorem 2.10 in [16, Theorem 3.15], with the turning power as an extra separating term. As $s - 1 \geq 4$, the form $u = b^{s-1}$

lies in $\mathfrak{m}^3$, and $\mathfrak{m}^2 u \subseteq \mathfrak{m}^{s+1} \subseteq I$. Take the turning family of u with the complement xW_2 of x^2S_1 in I_3^*. Its member I_t contains $f + tu$, an element K_r with cubic part xr, an element $P \in I \cap \mathfrak{m}^4$ with leading form p, and, for each root v, an element $Q_v \in I \cap \mathfrak{m}^4$ whose leading form, of degree $\kappa(v)$, does not vanish at v (or $Q_v = l_0^{s+1}$ with $l_0(v) \neq 0$ if $\kappa(v) = s+1$). Put $\tilde{f} = f + tu + \sum_v \sigma_v Q_v$. The tangent cones x^2, xr, p of $\tilde{f}$, K_r, P meet exactly at the roots of $\pi(p)$, so by [16, Lemma 3.14(a)] the local length is $24 + \sum_v \lambda_v$ once each local contribution λ_v is finite.

Choose y with $y(v) \neq 0$ at every root, and use the chart $x = \rho u'$, $y = \rho$, $z = \rho w$ of [16, Theorem 3.15]. For g of order k write $g(\rho u', \rho, \rho w) = \rho^k g^\natural(u', w, \rho)$, so that $V(g^\natural)$ is the proper transform of $V(g)$. Let v have the coordinate $w = c$. As $r(v) \neq 0$, the equation $K_r^\natural = 0$ gives $u' = \rho\chi(w, \rho)$ near v; put $g'' = g^\natural(\rho\chi, w, \rho)$. Then $B = \mathbb{C}\{w - c, \rho\}/(P'')$ is free of rank μ_v over $\mathbb{C}\{\rho\}$. As the cubic part of f is divisible by x, $f'' = \rho^2\psi$, and $u(\rho u', \rho, \rho w) = \rho^{s-1} b(1, w)^{s-1}$, normalized by the order two of $\tilde{f}$. Hence

$$\tilde{f}'' = \rho^2\Big(\psi + t\rho^{s-5} b(1, w)^{s-1} + \sum_{v'} \sigma_{v'} \rho^{\kappa(v')-4} Q''_{v'}\Big),$$

where $b(1, w)^{s-1}$ and Q''_v are units of B. So at each root the turning term and the separating equation are units times powers of the radial parameter, and [16, Lemma 3.14(b)], with $a = 2$ and with the unit of smaller exponent, $s - 5$ or $\kappa(v) - 4$, as u_1, gives $\lambda_v \leq (\min(\kappa(v), s-1) - 2)\mu_v$ on a nonempty open set of (t, σ). We intersect these open sets over the finitely many roots; the projection of the intersection to the t-line is nonempty and open, so for general t a suitable σ exists. Each weight is at most $s - 3$, and $\sum_v \mu_v = 4$. The members I_t with general t are linked to shorter schemes (Section 2.4), and A is their limit [15, Proposition 2.2(1)]. □

The bound is 32, 36 and 40 for $s = 5$, 6, 7, and model triples have exactly these local lengths (script `l21_map_135_turn`, part (T1); for $s = 6$ also on graded algebras, `l21_general_N1`, part (G)). Only r and b need to avoid the roots: neither $W_2 = \bar{S}_2$ nor $D_4 \subseteq \bar{T}_4$ is used.

4.5. Turning moves the quadric equation by an element of $(I : \mathfrak{m}^2)$. Sliding moves the cubic equations instead, by forms of order at least four. The price is that the moved forms must be compatible with the linear syzygies of W_2.

Lemma 4.6 (Sliding a cubic equation)**.** *Let $A = S/I$ be a local algebra in the binary case, and let $f \in I$ have leading form x^2 and cubic part in x^2S_1. Let $\varphi\colon W_2 \to \mathfrak{m}^4$ be linear, with*

(i) *$x\varphi(m) \in I$ and $\mathfrak{m}^2\varphi(m) \subseteq I$ for every $m \in W_2$;*
(ii) *$\sum_i c_i\varphi(m_i) \in I$ whenever $c_i \in \bar{S}_1$, $m_i \in W_2$ and $\sum_i c_i m_i = 0$.*

Let $E \subseteq S_4$ be a subspace with $E \cap (x^2S_2 + x\bar{S}_1W_2) \subseteq x^2S_2$. Then there is an ideal $\mathcal{I} \subseteq S[\tau]$ such that $S[\tau]/\mathcal{I}$ is a free $\mathbb{C}[\tau]$-module of rank $\ell(A)$, $\mathcal{I}|_{\tau=0} = I$, and for every $\tau_0 \in \mathbb{C}$ the ideal $\mathcal{I}|_{\tau=\tau_0}$ contains f, every $K_m + \tau_0\varphi(m)$, and

every element of $I \cap \mathfrak{m}^4$ whose quartic part lies in $x^2S_2 + E$; in particular it contains $I \cap \mathfrak{m}^5$, and it has the tangent cone of A.

Condition (ii) says that the map $c \otimes m \mapsto c\,\varphi(m)$ modulo I, on $\bar{S}_1 \otimes W_2$, kills the linear syzygies $\ker(\bar{S}_1 \otimes W_2 \to \bar{S}_3)$ of W_2; it is exactly the compatibility of the slide with multiplication by y and z, while (i) handles multiplication by x and by forms of degree at least two.

Proof. We split I into a fixed ideal and finitely many classes, and move the classes. Let $H_4 = x\bar{S}_1W_2$ and $V = x^2S_2 + E$. Then $H_4 \cap V = 0$, as an element $a+e$ of H_4 with $a \in x^2S_2$ and $e \in E$ has $e \in x^2S_2$ by hypothesis, and $x^2S_2 \cap x\bar{S}_3 = 0$. Choose a linear map $\lambda_H \colon S_4 \to H_4$ that is the identity on H_4 and vanishes on V, and let $\mathcal{N} = \{g \in I \cap \mathfrak{m}^3 :\ g_3 \in x^2S_1,\ \lambda_H(g_4) = 0\}$, where g_k is the part of degree k. It is an ideal, as $(hg)_3 = h(0)g_3$ and $(hg)_4 = h(0)g_4 + h_1g_3$ with $h_1g_3 \in x^2S_2$; it contains $I \cap \mathfrak{m}^5$ and every element of $I \cap \mathfrak{m}^4$ with quartic part in V. Fix a basis $m_1, \dots, m_n$ of W_2, and choose $u_k = \sum_i a_{ki}K_{m_i}$, with $a_{ki} \in \bar{S}_1$, whose quartic parts form a basis of H_4. Modulo $\mathcal{N}$, the classes of f, the K_{m_i} and the u_k are detected successively in degrees two, three and four: by the leading form x^2, by the cubic parts xm_i modulo x^2S_1, and by λ_H of the quartic parts. Subtracting multiples of them in this order gives $I = \mathcal{N} \oplus \mathbb{C}f \oplus \bigoplus_i \mathbb{C}K_{m_i} \oplus \bigoplus_k \mathbb{C}u_k$. Let $\mathcal{I} \subseteq S[\tau]$ be the $\mathbb{C}[\tau]$-span of $\mathcal{N}$, f, the $K_{m_i} + \tau\varphi(m_i)$ and the $u_k + \tau \sum_i a_{ki}\varphi(m_i)$.

It is an ideal. Indeed $\mathfrak{m}f \subseteq \mathcal{N}$, as $f_3 \in x^2S_1$; $xK_m \in \mathcal{N}$, and $x\varphi(m) \in I \cap \mathfrak{m}^5$ by (i); for $l \in \bar{S}_1$, lK_m differs from the element u of the span of the u_k with the same quartic part by $\sum_i c_iK_{m_i}$ with $\sum_i c_im_i = 0$, and by (ii) the corresponding combination of the $\varphi(m_i)$ lies in $I \cap \mathfrak{m}^5 \subseteq \mathcal{N}$; and $\mathfrak{m}^2K_m$, $\mathfrak{m}\,u_k$ and, by (i), $\mathfrak{m}^2\varphi(m)$ lie in $I \cap \mathfrak{m}^5 \subseteq \mathcal{N}$. For every τ_0 the sum $\mathcal{N} + \mathbb{C}f + \sum_i \mathbb{C}(K_{m_i} + \tau_0\varphi(m_i)) + \sum_k \mathbb{C}(\cdots)$ is direct by the same comparison: the changes to the K_{m_i} begin in degree four and do not change cubic parts, so the coefficients of the K_{m_i} are fixed in degree three before quartic parts are compared, and the changes to the u_k begin in degree five. So every fibre has colength $\ell(A)$, and [15, Lemma 2.3(1)] gives freeness. By the same comparison, an element of a member has the leading form of the element of I with the same coefficients, as the moved terms never reach the leading form; so the tangent cone ideal of a member lies in I^*, and equals it, as the colengths agree. □

Condition (ii) is empty when W_2 is a line, or a pencil without a common factor; it is not empty when $W_2 = l_0\bar{S}_1$ or $W_2 = \bar{S}_2$ (for instance $y\cdot yz - z\cdot y^2 = 0$). For $W_2 = l_0\bar{S}_1$ the choice $\varphi(l_0l) = l\,\ell$, with any form ℓ of order at least three, satisfies it identically. In socle degree s, condition (i) holds when $\varphi(m)$ has order at least s, and when it has order $s-1$ and $x\varphi(m)$ is orthogonal to D_s. If the top forms are binary, every linear map $\varphi \colon W_2 \to S_{s-1}$ satisfies (i); the syzygy condition (ii) must still be checked.

Proposition 4.7 (Slid radial costs)**.** *Let $A = S/I$ be a local algebra of length d in the binary case with $\gamma_f = 0$, and let f have cubic part in x^2S_1. Let*

$P \in I$ have a quartic leading form p_4 with $p = \pi(p_4) \neq 0$, let φ be as in Lemma 4.6 with $E = \mathbb{C}p_4$, and suppose that some $m \in W_2$ has $\varphi(m)$ of order k, with a leading form that vanishes at no root of p on the line $x = 0$. Then for general τ the ideal $\mathcal{I}|_\tau$ lies in a finite complete intersection of local length at most $24 + 8(k-3)$. If this is less than $2d$ and every finite subscheme of $\mathbb{A}^3$ of length less than d is smoothable, then A is smoothable.

Proof. As $p \neq 0$, $E \cap (x^2 S_2 + x\bar{S}_1 W_2) = 0$, so Lemma 4.6 applies, and f, P and $K_m + \tau\varphi(m)$ lie in $\mathcal{I}|_\tau$. Their tangent cones x^2, p_4, xm meet only at the roots of p on the line $x = 0$, so the local length is $2 \cdot 4 \cdot 3 + \sum_v \lambda_v$ [16, Lemma 3.14(a)]. In the chart of the proof of Proposition 4.4, the local ring B at v of $V(f^\natural, P^\natural)$ is, modulo ρ, the ring $\mathbb{C}\{u', w - c\}/(u'^2,\ p(1,w) + u'h(w))$ for some $h(w)$, of length $2\mu_v$, as the determinant of the two relations on the basis $1, u'$ is $p(1,w)^2$. Being a complete intersection of dimension one, finite over $\mathbb{C}\{\rho\}$, it is free of rank $2\mu_v$. The third equation gives $K_m^\natural + \tau\rho^{k-3}\varphi(m)^\natural$, where $\varphi(m)^\natural$ is a unit at v. [16, Lemma 3.14(b)] with $a = 0$ and $c_1 = k - 3$ gives $\lambda_v \leq 2(k-3)\mu_v$ for general τ, and $\sum_v \mu_v = 4$. The general members are linked to shorter schemes, and A is their limit. □

Unlike Proposition 4.4, the cubic equation is not used to eliminate the transverse coordinate, so m may vanish at the roots of p; the price is the factor two in the rank. For $k = 5$ the bound is 40.

5. The map of length twenty-one

This section reduces Theorem 1.1 to 58 Hilbert functions and sorts them by method.

5.1. There are 254 Hilbert functions $(1, 3, H(2), \dots)$ of length twenty-one: the sequences that start with $(1, 3)$, sum to 21 and satisfy Macaulay's growth bound $H(k+1) \leq H(k)^{\langle k \rangle}$ [15, Section 2.4] (script `l21_classify`). Table 1 assigns 196 of them to general results. Each row of [14] and [15] holds verbatim at length twenty-one: its proof uses only that schemes of length at most twenty are smoothable [16], and, for the rows proved by linkage, a complete intersection of local length less than 42. The two rows with a numerical bound have room to spare: two points of the plane give $2s+21 \leq 39$ for the socle degrees $s = 8, 9$ that occur [15, Corollary 3.14], and three points give $27 + 4(s-3) = 39$ for $s = 6$ [15, Corollary 3.15]. The algebras with Hilbert function $(1, 3, 6, 10, 1)$ satisfy $\mathfrak{m}^5 \subseteq I \subseteq \mathfrak{m}^4$; they are very compressed, and those of length at most 95 are smoothable [3, Proposition 1.2].

TABLE 1. General results and the number of Hilbert functions of length twenty-one that they cover. Here $H = H_A$, s is the socle degree, and each Hilbert function is counted in the first row that applies.

Hilbert functions	result	$d = 21$
$H(2) \le 4$, or $H(2) = 5$ and $H(3) \le 2$	[15, Thm. 2.9(1)]	134
$H(2) = 5$, $H(3) = 3$	[15, Thm. 4.12]	12
$H(2) = 5$, $s \le 4$	[15, Cor. 4.7]	2
$H(2) = 5$, $s = 5$, $H(5) = 1$	[15, Cor. 4.9]	3
$H(2) = 5$, $s = 5$, $H(4) = H(5) \le 3$	[15, Cor. 4.10]	1
$H(2) = 5$, and either $s = 6$ and $(H(5) = H(6) = 1$ or $H(4) = H(5) = 2)$, or $s = 7$, $H(6) = H(7) = 1$ and $(H(5) = 1$ or $H(4) = H(5) = 2)$	[15, Prop. 4.11]	10
$H(2) = 6$, $H(4) = \cdots = H(s) = 1$, $s \ge 5$	[15, Prop. 2.12(1)]	9
$s \ge 2k_0$, where k_0 is the largest k with $H(k) > 1$	[15, Prop. 2.12(2)]	17
$H(2) = 6$, $s = 7$, $H(5) = H(6) = H(7) = 1$	[15, Lemma 5.14]	3
$H(2) = 6$, $H(3) = H(4) = H(5) = 2$	[15, Cor. 3.14]	2
$H(2) = 6$, $H(3) = H(4) = 3$, $s = 6$	[15, Cor. 3.15]	1
$(1, 3, 6, 10, 1)$, very compressed	[3, Prop. 1.2]	1
$(1, 3, 6, c, 2, 1, 1)$, $c \ge 2$	[15, Thm. 5.21]	1
individual	Table 2	58
total		254

5.2. The enumeration is now complete: Table 1 treats 196 Hilbert functions, and the remaining 58 fall into three groups, treated in Sections 6, 7 and 8: 15 with $H(2) = 6$ and socle degree at most five, 19 with $H(2) = 6$ and socle degree at least six, and 24 with $H(2) = 5$. Each row of Table 2 names the result that proves it, with its methods; the rest of the paper supplies these results.

Table 2: The 58 Hilbert functions not in Table 1, by section, with the result that treats them, its main method and at most one secondary method. Numbers in parentheses are local lengths of the complete intersections used.

Hilbert function	result	methods
Section 6: $H(2) = 6$, socle degree at most five		
$(1, 3, 6, 9, 2)$	Thm. A.5	rays
$(1, 3, 6, 8, 3)$	Thm. A.5	anchor (36); expected dimension
$(1, 3, 6, 7, 4)$	Thm. A.5	global linkage; expected dimension
$(1, 3, 6, 6, 5)$	Thm. A.5	global linkage; rays
$(1, 3, 6, 5, 6)$	Thm. A.5	rays; limits
$(1, 3, 6, 4, 5, 2)$	Thm. 6.4	prescribed equations (40)
$(1, 3, 6, 4, 4, 3)$	Thm. 6.5	prescribed equations (40); excess
$(1, 3, 6, 5, 5, 1)$	Thm. A.7	line plus point; prescribed equations (38)
$(1, 3, 6, 5, 4, 2)$	Thm. A.8	line plus point; prescribed equations (40)
$(1, 3, 6, 5, 3, 3)$	Thm. A.9	line plus point; prescribed equations (39)
$(1, 3, 6, 6, 4, 1)$	Thm. A.11	prescribed quartics; expected dimension

Table 2, continued

Hilbert function	result	methods
$(1,3,6,6,3,2)$	Thm. A.12	prescribed quartics; anchors (36–41)
$(1,3,6,8,2,1)$	Thm. A.15	anchors (36, 40); cubic reduction
$(1,3,6,7,3,1)$	Thm. A.13	prescribed equations (27–33); expected dimension
$(1,3,6,7,2,2)$	Thm. A.14	anchor (32); expected dimension
Section 7: $H(2)=6$*, socle degree at least six*		
$(1,3,6,3,3,3,1,1)$	Thm. B.7	excess; free quartic tails (39)
$(1,3,6,3,3,2,2,1)$	Thm. B.8	excess; free quartic tails (39)
$(1,3,6,3,2,2,2,2)$	Thm. B.9	excess; linkage at infinity (39)
$(1,3,6,3,2,2,2,1,1)$	Thm. B.10	excess; linkage at infinity (40, 41)
$(1,3,6,3,2,2,1,1,1,1)$	Thm. B.11	rays
$(1,3,6,4,5,1,1)$	Thm. B.13	rays
$(1,3,6,4,2,2,1,1,1)$	Thm. B.12	rays; excess at one member (39)
$(1,3,6,6,3,1,1)$	Thm. B.14	rays; anchor (30)
$(1,3,6,6,2,2,1)$	Thm. B.15	rays; few bad flags
$(1,3,6,5,2,2,2)$	Thm. B.16	rays; few bad flags
$(1,3,6,5,2,2,1,1)$	Thm. B.17	rays; anchors (32, 33)
$(1,3,6,4,2,2,2,1)$	Thm. B.18	excess; anchors (36–41)
$(1,3,6,4,3,3,1)$	Thm. B.22	excess; multiplicity (41)
$(1,3,6,4,4,2,1)$	Thm. B.23	excess; anchors (33–40)
$(1,3,6,4,3,2,1,1)$	Thm. B.24	excess; anchors (34–41)
$(1,3,6,3,3,2,1,1,1)$	Thm. 7.1	linkage at infinity (41); weighted contraction (39)
$(1,3,6,4,3,2,2)$	Thm. B.21	linkage at infinity (38–41); excess
$(1,3,6,5,4,1,1)$	Thm. B.25	linkage at infinity (41); rays
$(1,3,6,5,3,2,1)$	Thm. B.28	linkage at infinity; excess
Section 8: $H(2)=5$		
$(1,3,5,4,4,3,1)$	Thm. 8.1	turning power (36)
$(1,3,5,4,4,2,2)$	Thm. 8.1	turning power (36)
$(1,3,5,4,3,3,2)$	Thm. 8.1	turning power (36)
$(1,3,5,4,4,2,1,1)$	Thm. C.27	turning power (40); turning
$(1,3,5,4,3,3,1,1)$	Thm. C.28	turning power (40); turning
$(1,3,5,4,3,2,2,1)$	Thm. C.29	turning power (40); turning
$(1,3,5,4,3,2,1,1,1)$	Thm. C.32	turning power (41); turning
$(1,3,5,4,2,2,2,2)$	Thm. C.30	turning power (40); turning
$(1,3,5,4,2,2,2,1,1)$	Thm. C.31	turning power (38); turning
$(1,3,5,5,4,2,1)$	Thm. C.19	turning power (36); sliding (40)
$(1,3,5,5,3,3,1)$	Thm. C.20	turning power (36); sliding (40)
$(1,3,5,5,3,2,2)$	Thm. 8.3	sliding (40); turning
$(1,3,5,6,3,2,1)$	Thm. C.18	sliding (40); turning
$(1,3,5,5,5,2)$	Thm. C.13	weighted linkage (40); sliding
$(1,3,5,5,3,2,1,1)$	Thm. C.23	sliding (40); turning past a defect
$(1,3,5,4,4,4)$	Thm. C.16	weighted linkage (32)
$(1,3,5,5,4,3)$	Thm. C.14	weighted linkage, bad roots (40)
$(1,3,5,4,5,3)$	Thm. C.15	weighted linkage (40)
$(1,3,5,6,4,2)$	Thm. C.12	weighted linkage, bad roots (40)
$(1,3,5,7,3,2)$	Thm. C.11	two quartics (40); turning
$(1,3,5,4,5,2,1)$	Thm. C.21	quintic radial costs (40, 41); turning
$(1,3,5,5,2,2,2,1)$	Thm. C.24	turning (no induction)
$(1,3,5,5,2,2,1,1,1)$	Thm. C.24	turning (no induction)

$(1,3,5,4,2,2,1,1,1,1)$ Thm. C.33 turning (no induction)

5.3. The methods in Table 2 are of four kinds.

A short complete intersection, through the algebra, through a limit of it, or through the general member of a flat family through it: *excess*, Theorem 2.6; *free quartic tails*, [15, Lemma D.3] (Section 2.7); *global linkage*, Section 2.7; *multiplicity*, Proposition 2.16; *weighted linkage*, Proposition 2.12 and its variants in Appendix C, among them *bad roots* (Lemma C.6); *two quartics* and *quintic radial costs*, Appendix C; *prescribed equations*, Lemma 2.14 and Proposition 6.2; *prescribed quartics*, Lemma A.4; *linkage at infinity*, Section 3; *weighted contraction*, a rescaling of the tails compatible with closedness, with a linkage anchor at $t = 0$ (Theorem 7.1); *turning power*, Proposition 4.4; *sliding*, Lemma 4.6 and Proposition 4.7.

A ray: *rays*, a ray of the algebra (Section 2.3), or of the members of a dense subset of an irreducible family.

A surface with a rational double point: *turning*, the turning operators of Section 4.1; *turning past a defect*, Lemma C.3.

Transport of an open condition through an irreducible family: an *anchor* is a member in a complete intersection of the stated local length (Section 2.4); *excess at one member*, excess linkage at a general member; *limits*, smoothable members are dense; *expected dimension*, Section 2.2; *few bad flags*, [16, Lemma C.6]; *cubic reduction*, [15, Lemmas A.14 and A.15]; *line plus point*, [16, Section A.8].

"No induction" marks proofs that do not use shorter schemes.

5.4. *Proof of Theorem 1.1.* By [16] and [15, Proposition 2.2(3)], it suffices to treat local algebras of length twenty-one with $I \subseteq \mathfrak{m}^2$. Their Hilbert functions are covered by Tables 1 and 2. □

6. Socle degree at most five

In this section $H_A(2) = 6$ and the socle degree is at most five. Fifteen Hilbert functions of length twenty-one remain. We prove two of them here, by the extension of [16, Proposition A.9] that length twenty-one requires, and the other thirteen in Appendix A.

The five Hilbert functions of socle degree four, $(1,3,6,c,11-c)$ with $5 \leq c \leq 9$, behave as at length twenty [16, Theorem 5.1]. A single cubic lets a point leave along a line, and a pencil of cubics gives one irreducible family with an anchor. More cubics without a common factor are linked globally. Some strata with a common factor are too small to be components; the others describe a line with residual points, and moving the points off the line gives rays. In socle degree five every weight $k(p) - 3$ of excess linkage is at most two, so a finite cubic base scheme of length at most five costs at most $27 + 2 \cdot 6 = 39$ [16, Corollary 3.3]. Two things are new at length twenty-one. When $H_A(3) = 6$, the base scheme can have length six, and then a fat point $V(\mathfrak{m}_p^3)$, with multiplicity nine, would cost $27 + 2 \cdot 9 = 45$;

a prescribed quartic tail lowers the weights (Lemma A.4). And when the cubics share a line, excess linkage fails, and we prescribe three equations through the tails (Lemma 2.14). This is the method of [16, Proposition A.9], which needs one more numerical input at length twenty-one.

6.1. Suppose that, in suitable coordinates, $I_3^* = xQ$ for a space $Q \subseteq S_2$ of quadrics with $S_1Q = S_3$; for instance $Q = S_2$, or Q the hyperplane orthogonal to $x \circ G$ for a cubic G such that $x \circ G$ is not a square [16, proof of Proposition A.9]. Then the cubic base scheme is the line $x = 0$, and the top forms of degree four and five are binary. Let $K \subseteq \bar{S}_4$ be the space of binary quartics that annihilate D_5, and δ the degree of its greatest common divisor.

Proposition 6.2. *Let A be a local algebra of length d with $H_A(2) = 6$ and socle degree five, and suppose that every finite subscheme of $\mathbb{A}^3$ of length less than d is smoothable. Suppose that $I_3^* = xQ$ with $S_1Q = S_3$, and that $K \neq 0$. If $36 + \delta < 2d$, then some algebra with the graded data of A lies in a complete intersection of local length at most $36 + \delta$, and every algebra with the graded data of A is smoothable.*

At $d = 21$ the bound $36 + \delta \leq 40$ always suffices. Not every algebra with these graded data lies in such a complete intersection: for $H_A = (1, 3, 6, 4, 5, 2)$, $D_5 = \langle Y^5, Z^5 \rangle$ and all tails zero, $\delta = 2$, but $e(I) = e(I^*) = 49$.

Proof. We bound the multiplicity of the ideal of the parts that Lemma 2.14 lets us prescribe. As $S_1Q = S_3$, the quartics D_3' whose first derivatives lie in D_3 form $(xS_1Q)^\perp = (xS_3)^\perp = \bar{T}_4$, so $D_4 \subseteq \bar{T}_4$. For $F \in D_5$ all first derivatives are binary, so $x \circ F$ is a binary quartic all of whose derivatives vanish; hence $x \circ F = 0$ and $D_5 \subseteq \bar{T}_5$. A binary quartic p annihilates F if and only if $\langle pv, F \rangle = 0$ for all $v \in S_1$, so $K = (S_1 \circ D_5)^\perp \cap \bar{S}_4$.

Write $K = gH$ with g the greatest common divisor, of degree δ, and $H \subseteq \bar{S}_{4-\delta}$ without common zero ($H = \mathbb{C}$ if $\delta = 4$). The chain of ideals

$$N = xQ + K + \mathfrak{m}^5 \subseteq L = x\mathfrak{m}^2 + K + \mathfrak{m}^5 \subseteq L' = x\mathfrak{m}^2 + g \cdot (y, z)^{4-\delta} + \mathfrak{m}^5$$

satisfies $L^2 = NL$ and $L'^2 = LL'$, as in the proof of [16, Proposition A.9]: $(x\mathfrak{m}^2)^2 = (xQ)(x\mathfrak{m}^2)$ because $Q\mathfrak{m}^2 = \mathfrak{m}^4$, as $S_2Q = S_1(S_1Q) = S_4$, and $(g \cdot (y, z)^{4-\delta})^2 = K \cdot g \cdot (y, z)^{4-\delta}$ because $H\bar{S}_n = \bar{S}_{2n}$ for $n = 4 - \delta$: this is clear for $n = 0$, and for $n > 0$ two general members of H have no common root, so they form a regular sequence of degrees n, n in $\mathbb{C}[y, z]$, and the ideal they generate contains all forms of degree $2n$. So N is a reduction of L'. After a linear change of y and z we may assume $g(0, 1) = 1$; then the substitution $y \mapsto ty$ carries L' to ideals that tend, as $t \to 0$, to the monomial ideal $L_\delta^0 = x\mathfrak{m}^2 + z^\delta(y, z)^{4-\delta} + \mathfrak{m}^5$. For $\delta \leq 3$, the proof of [16, Proposition A.9] gives triples in L_δ^0 of local length $36 + \delta$. For $\delta = 4$,

$$x^3 - z^4, \qquad xy^2 + y^4z, \qquad xz^2 - y^5$$

lie in $L_4^0 = x\mathfrak{m}^2 + (z^4) + \mathfrak{m}^5$ and have local length 40 (script `l21_general_LD`, part (A); certified by a local standard basis and by stabilization). By openness

of the local length, the same combinations of the generators of the deformed ideals, for small $t \neq 0$, have local length at most $36+\delta$, and pulling back gives three such elements of L'. By Proposition 2.16(a), three general elements of N generate an ideal of colength at most $36+\delta$. They have the form $w_i + p_i + r_i$ with $w_i \in xQ = I_3^*$ linearly independent, $p_i \in K$ and $r_i \in S_5$, so Lemma 2.14 places them in the ideal of an algebra with the graded data of A, and concludes. □

Special triples of L_4^0 can be longer: $(x^3-z^4+y^5,\ xy^2+z^4+y^4z,\ xz^2-y^5+z^4)$ has local length 42 (script `121_map_LD`, part (A), the row $\delta = 4$), while twelve random triples have local length 40 (part (B)).

6.3. We apply the proposition to the two Hilbert functions with $H_A(3) = 4$ that remain.

Theorem 6.4. *Every local algebra with Hilbert function* $(1,3,6,4,5,2)$ *is smoothable.*

Proof. The ideal generated by $D_3^\perp$ has Hilbert function 4 in degree three and $\dim D_3' \geq \dim D_4 = 5 = 4^{\langle 3\rangle}$ in degree four, where $D_3' \subseteq T_4$ is the space of quartics whose first derivatives lie in D_3; so by the proof of [15, Lemma A.11], which uses Gotzmann's persistence theorem, $D_3 = \bar{T}_3$ in suitable coordinates, and $I_3^* = xS_2$. The binary quartics killing the pencil $D_5 \subseteq \bar{T}_5$ are cut out by $2 \cdot 2 = 4$ linear conditions in $\bar{S}_4$, of dimension five, so $K \neq 0$. Proposition 6.2 with $Q = S_2$ applies, with $36 + \delta \leq 40 < 42$ (script `121_136452`, class S, checks the proposition on algebras with six pencils D_5, with $\delta = 2, 3, 4$). □

Theorem 6.5. *Every local algebra with Hilbert function* $(1,3,6,4,4,3)$ *is smoothable.*

Proof. By [15, Lemma A.11], either the cubic base scheme is finite of length at most four, or $D_3 = \bar{T}_3$ in suitable coordinates. In the first case $\sum_p e_p \leq 5$ [16, Lemma 3.2], every weight is at most two, and $\dim I_3^* = 6 \geq 3$, so Theorem 2.6 gives $27 + 10 = 37 < 42$. In the second, $D_4 \subseteq \bar{T}_4$ and $D_5 \subseteq \bar{T}_5$ as above, and $S_1 \circ D_5 \subseteq D_4$ has dimension at most four, less than $\dim \bar{T}_4 = 5$; so $K \neq 0$, and Proposition 6.2 gives at most 40. The case $\delta = 4$, where K is a line, occurs, for example for $D_5 = \langle Y^5, Z^5, Y^3Z^2\rangle$. □

7. Socle degree at least six

In this section $H_A(2) = 6$ and the socle degree is at least six. Nineteen Hilbert functions of length twenty-one remain. In most of them a plateau of the Hilbert function makes the graded data those of a scheme Γ of length two or three in the plane, and the top form is a power X^s. Excess linkage then has weight up to $s-3$ at the point $[X]$, and a non-reduced base point there costs too much. The devices of [15] and [16] reduce this cost: free quartic tails, rays along lines and curves, and irreducible families whose general member has a lighter base scheme. We prove $(1,3,6,3,3,2,1,1,1)$ here, where both

kinds of new argument, a limit at infinity and a contraction of the tails, appear next to three old ones, and the other eighteen in Appendix B.

Theorem 7.1. *Every local algebra with Hilbert function* $(1,3,6,3,3,2,1,1,1)$ *is smoothable.*

Proof. We normalize the lift of the top form as in [16, Theorem 7.1]. Excess linkage handles the base schemes that are not concentrated at $[X]$, and a straight ray handles one coefficient $\delta = 0$. Three strata remain: the tangent cone alone suffices for one, cubic tails at infinity for another, and a weighted contraction of the tails for the last.

Setup. As $H_A(3) = H_A(4) = 3$, [15, Lemma 2.16] gives a subscheme $\Gamma \subset \mathbb{P}^2$ of length three with $D_3 = \nu_3(\Gamma)$, $D_4 = \nu_4(\Gamma)$ and $D_k \subseteq \nu_k(\Gamma)$ for $k \geq 5$; as I_Γ is generated in degrees at most three, the cubic base scheme is $\mathfrak{B} = \Gamma$. As $H_A(7) = H_A(8) = 1$, the top form is an eighth power, $D_8 = \mathbb{C}X^8$, so $D_k = \mathbb{C}X^k$ for $k = 6, 7$, and $X^5 \in D_5$; hence $[X] \in \Gamma$. Let $\Phi = X^8 + A_7 + B_6 + C_5 + \cdots$ be the lift of X^8. As in [16, Theorem 7.1], replacing Φ by $\Phi - a\,x^j \circ \Phi$ and the coordinate changes $y \mapsto y + \lambda x^j$, $z \mapsto z + \lambda' x^j$ of [15, Section 5.17] make $A_7 = 0$ and remove X^6, X^5Y, X^5Z from B_6; they do not change the D_k. The space D_5 has dimension two. If $\dim S_1 \circ D_5 = 2$, then $D_5 = \nu_5(\Gamma')$ for a subscheme $\Gamma' \subseteq \Gamma$ of length two through $[X]$, by [15, Lemma 2.16] with $k = 4$ and $n = 2$, applied to the derivatives of D_5; and we call D_5 *tangent* if Γ' is a double point; if $\dim S_1 \circ D_5 = 3$ we call D_5 *active*.

Excess, when Γ *is not concentrated at* $[X]$. At $[X]$ we have $X^k \in D_k$ for all $k \leq 8$, so $k([X]) = 8$ and the weight is five. At another point $p = [P]$ of Γ we have $P^4 \in D_4$ and $P^6 \notin D_6$, so the weight is three if $P^5 \in D_5$ and two otherwise; and at most one such point has $P^5 \in D_5$, as $\dim D_5 = 2$. So Theorem 2.6 gives $27 + 5 + 3 + 2 = 37$ if Γ is reduced, $27 + 5 \cdot 2 + 3 = 40$ if Γ is a double point at $[X]$ and a point, and $27 + 5 + 3 \cdot 2 = 38$ if Γ is $[X]$ and a double point. From now on Γ is concentrated at $[X]$: curvilinear, either collinear, $\Gamma = V(z, y^3)$, or on a smooth conic, $\Gamma = V(xz - y^2, yz, z^2)$, or the fat point $V(\mathfrak{m}_X^2)$. Excess linkage now gives $27 + 5 \cdot 3 = 42$ and $27 + 5 \cdot 4 = 47$.

A straight ray. Suppose that D_5 is tangent, with $\Gamma' = V(z, y^2)$, so $D_5 = \langle X^5, X^4Y \rangle$. The conditions $y \circ B_6$, $z \circ B_6 \in D_5$, which hold because $y \circ \Phi$ and $z \circ \Phi$ have degree at most five, leave $B_6 = \delta X^4Y^2$ after the normalization [16, proof of Theorem 7.1]. Then $x^3 \circ \Phi / 336 \equiv X^5 + \frac{\delta}{14} XY^2$ modulo $T_{\leq 2}$. If $\delta = 0$, or if $XY^2 \in D_3$, which is the collinear case, then $X^5 \in M$; the operators x^5y and x^5z kill $M \cap T_{\leq 5}$ by order and kill Φ, $x \circ \Phi$, $x^2 \circ \Phi$ by the normalization, so [15, Lemma 5.4] with $k = 5$ makes the line of X a ray.

Active D_5*: the tangent cone suffices.* Let Γ be curvilinear and D_5 active. Then $D_5 = \langle X^5, V_0 + cX^4Y \rangle$ with $V_0 = X^3Y^2$ in the collinear case and $V_0 = 2X^3Y^2 + X^4Z$ on the conic. The substitutions $Y \mapsto Y + \mu X$, respectively $Y \mapsto Y + 2\mu X$, $Z \mapsto Z + \mu Y + \mu^2 X$, fix X and Γ and replace c by $c + 2\mu$, respectively $c + 9\mu$ (script `l21_136332111`, part (U2)). So $c = 0$, and the

graded data are fixed: $D_k = \nu_k(\Gamma)$ for $k = 3, 4$, $D_5 = \langle X^5, V_0\rangle$ and $D_k = \mathbb{C}X^k$ for $k \geq 6$. The tangent cone ideal I^* contains

$$J_{\mathrm{col}} = \big(x^2z,\ y^3{+}x^9,\ z^3{+}x^4y\big), \quad \text{resp.} \quad J_{\mathrm{con}} = \big(x^2z{-}xy^2,\ y^3{+}x^9,\ z^3{+}x^4y\big);$$

the cubic parts lie in $(I_\Gamma)_3 = I_3^*$, x^4y is orthogonal to D_5, and $\mathfrak{m}^9 \subseteq I^*$. The cubic cones meet only at $[X]$, and in the chart $x = \rho$, $y = \rho\eta$, $z = \rho\zeta$ the proper transforms are ζ, resp. $\zeta - \eta^2$, and $\eta^3 + \rho^6$, $\zeta^3 + \rho^2\eta$. Substituting $\zeta = 0$, resp. $\zeta = \eta^2$, leaves $\eta^3 + \rho^6$ and $\rho^2\eta$, resp. $\eta(\eta^5 + \rho^2)$, which meet in length $6 + 6 = 12$. So the local length is $27 + 12 = 39$ [16, Lemma 3.14(a)] (script `l21_explicit_triples`, part (C)), and Proposition 2.16(b) with $L = I^*$ makes every algebra with these graded data smoothable.

Γ *on the conic,* D_5 *tangent: cubic tails at infinity.* Now $\Gamma = V(xz - y^2, yz, z^2)$ and $D_5 = \nu_5(\Gamma') = \langle X^5, X^4Y\rangle$, so the graded data are fixed. Excess linkage and the tangent cone both give 42. Take $Q(X^8) = -6720\,Y^3$ in Corollary 3.4. The three elements

$$x^2z - xy^2, \qquad y^3 + x^8, \qquad z^3 + x^5y$$

have the form (5): their cubic parts lie in $(I_\Gamma)_3 = I_3^*$, $x^5y \in I_6^*$, and $\langle y^3 + x^8, X^8 - 6720Y^3\rangle = 8! - 6 \cdot 6720 = 0$. The cubic cones meet only at $[X]$, and in the chart $x = \rho$, $y = \rho\eta$, $z = \rho\zeta$ the proper transforms are $\zeta - \eta^2$, $\eta^3 + \rho^5$ and $\zeta^3 + \rho^3\eta$. Substituting $\zeta = \eta^2$ leaves $\eta^3 + \rho^5$ and $\eta(\eta^5 + \rho^3)$, which meet in length $5 + 9 = 14$. So the local length is $27 + 14 = 41$ [16, Lemma 3.14(a)] (part (W)), and Corollary 3.4 settles this stratum for all tails.

Γ *fat,* $\delta \neq 0$*: a weighted contraction.* Here the coefficient δ of the sextic tail is the tail we must keep, and the closedness conditions below are weighted homogeneous in the other tails; so a weighted rescaling of these tails, compatible with closedness, contracts the algebra to one with the single tail δX^4Y^2, which lies in a short complete intersection. Now $D_3 = X^2T_1$, $D_4 = X^3T_1$, and $D_5 = \langle X^5, X^4Y\rangle$ after a rotation of Y, Z fixing X. Excess linkage gives 47, the tangent cone 47, Proposition 3.7(c) 45, and free quartic tails 43. Subtracting a multiple of $x^3 \circ \Phi$ and the coordinate changes $y \mapsto y + \lambda x^4$, $z \mapsto z + \lambda' x^4$ remove X^5, X^4Y, X^4Z from the quintic tail, so

$$\Phi = X^8 + \delta X^4Y^2 + \Phi_5 + \Phi_4 + \Phi_3, \qquad W = X^3Z + W_3,$$

where $W \in M$ is the lift of X^3Z and Φ_k, W_3 are forms of degree k, 3. As $\delta \neq 0$, the elements $y \circ \Phi$ and $xy \circ \Phi$ have top forms $2\delta X^4Y$ and $8\delta X^3Y$; together with $T_{\leq 2}$ and X^2T_1, the elements Φ, $x^j \circ \Phi$ for $1 \leq j \leq 4$, $y \circ \Phi$, $xy \circ \Phi$ and W have independent top forms, so they span M. Reducing every first derivative against this basis, with all coefficients of Φ_5, Φ_4, Φ_3, W_3 as indeterminates over $\mathbb{Q}(\delta)$, uses only constant pivots (part (F)). The remainders of degree at least four vanish exactly when $z \circ \Phi_5 = pX^3Y + qX^3Z$ for some $p, q \in \mathbb{C}$, and then only two remainders are left, those of $y^2 \circ \Phi$ and

of $z \circ \Phi$:

$$\rho_1 = y^2 \circ \Phi_5, \qquad \rho_2 = z \circ \Phi_4 - \frac{p}{8\delta}\, xy \circ \Phi_5 - qW_3 \qquad \text{modulo } X^2T_1.$$

The equations $\rho_2 = 0$ solve seven coefficients of Φ_4 with an invertible constant matrix, and ρ_1 involves none of these coefficients; so M is closed if and only if $z \circ \Phi_5 = pX^3Y + qX^3Z$ and $\rho_1 = \rho_2 = 0$, and no further exceptional locus occurs. For the rescaling

$$(\Phi_5, \Phi_4, \Phi_3, W_3, p, q) \mapsto (t\Phi_5, t^2\Phi_4, t\Phi_3, tW_3, tp, tq)$$

these three conditions are homogeneous of weights 1, 1, 2. So the spaces M_t spanned in the same way by $T_{\le 2}$, X^2T_1, the derivatives of $\Phi_t = X^8 + \delta X^4Y^2 + t\Phi_5 + t^2\Phi_4 + t\Phi_3$ and by $W_t = X^3Z + tW_3$ are closed for every $t \in \mathbb{C}$, with constant top forms; by [15, Lemma 2.3(2)] they form a family with the normalized A at $t = 1$. At $t = 0$ only the tail δX^4Y^2 survives, $M_0 = T_{\le 2} + X^2T_1 + S \circ (X^8 + \delta X^4Y^2) + \mathbb{C}X^3Z$, and its ideal contains

$$J = \Big(y^3, \quad xz^2, \quad z^3 + xy^2 - \frac{\delta}{840}\, x^5 \Big),$$

since y^3 and xz^2 kill $X^8 + \delta X^4Y^2$, X^3Z and X^2T_1, and $xy^2 \circ (X^8 + \delta X^4Y^2) = 8\delta X^3 = \frac{\delta}{840}\, x^5 \circ X^8$. The ring $R = \mathbb{C}[[x, y, z]]/(y^3,\, z^3 + xy^2 - \frac{\delta}{840}x^5)$ is a complete intersection of dimension one, on which x and z are parameters. Here $\ell(R/xR) = \ell(\mathbb{C}[[y, z]]/(y^3, z^3)) = 9$, and $\ell(R/zR) = 15$, because multiplication by $xy^2 - \frac{\delta}{840}x^5$ on $\mathbb{C}[[x]][y]/(y^3)$ has determinant $(-\frac{\delta}{840})^3 x^{15}$. So

$$\ell(S_{\mathfrak{m}}/J) = \ell(R/xz^2R) = 9 + 2 \cdot 15 = 39 < 42$$

(part (A1): 39 by both certificates for $\delta = 1$). The member M_0 is a linkage anchor of the t-line, and Section 2.4 makes A smoothable.

These are all cases: a tangent D_5 with $\delta = 0$ or a collinear Γ has a ray; an active D_5 requires a curvilinear Γ; and when Γ is fat, $D_5 \subseteq \nu_5(\Gamma) = X^4T_1$ is always tangent. □

The limits of Section 3 keep only cubic and quartic tails of the top form, so they lose the coefficient δ of the sextic tail X^4Y^2; on the fat stratum Proposition 3.7(c) gives 45, and free quartic tails give 43. The weighted contraction keeps δ and discards all other tails.

8. One quadric in the tangent cone

In this section $H_A(2) = 5$. Twenty-four Hilbert functions of length twenty-one remain; we prove four of them here and the other twenty in Appendix C. By Section 4.1 we may assume $I_2^* = \mathbb{C}x^2$, and when $\mathfrak{m}^7 \subseteq I$, that is, in socle degree at most six, also the binary case $I_3^* = x^2S_1 \oplus xW_2$, $\gamma_f = 0$, and f with cubic part in x^2S_1. The natural candidates for a complete intersection are then f, an element K_m with cubic part xm, and an element P whose leading form restricts to a nonzero binary quartic p. They cost 24, plus a contribution of each root of p on the line $x = 0$. At length twenty these contributions were measured by radial costs and avoided by weighted linkage.

At length twenty-one the radial costs reach 42 or more for several Hilbert functions, and the moves of Section 4 cap them. The first theorem caps them with a turning power; the second slides a cubic equation, and where the top forms leave the line $x = 0$ it finds a turning operator instead.

Theorem 8.1. *Every local algebra with Hilbert function* $(1, 3, 5, 4, 4, 3, 1)$, $(1, 3, 5, 4, 4, 2, 2)$ *or* $(1, 3, 5, 4, 3, 3, 2)$ *is smoothable.*

These are exactly the Hilbert functions $(1, 3, 5, 4, h_4, h_5, h_6)$ of length twenty-one with socle degree six and $h_4 \leq 4$: then $h_5 \leq h_4^{\langle 4 \rangle} = h_4$ and $h_6 \leq h_5$, and $h_4 + h_5 + h_6 = 8$.

Proof. We show that all three Hilbert functions satisfy the hypotheses of Proposition 4.4 with $s = 6$, which gives $24+4\cdot 3 = 36$. As $\mathfrak{m}^7 \subseteq I$, an element of I with a cubic leading form c such that $\pi(c) \neq 0$ would be a turning operator of type (b); so $I_3^* \subseteq xS_2$, and as $\dim I_3^* = 10 - 4 = 6 = \dim xS_2$, we have $I_3^* = xS_2$ and $W_2 = \bar{S}_2$, so $D_k \subseteq \bar{T}_k$ for $k \geq 3$ [15, remark after Corollary 4.8]. If $\gamma_f \neq 0$, [16, Lemma 3.9] applies, so $\gamma_f = 0$. As $\dim D_4 \leq 4 < 5$, $J_4 = D_4^{\perp} \cap \bar{S}_4$ is not zero. Choose $0 \neq p \in J_4$, and $r \in \bar{S}_2$ and $b \in \bar{S}_1$ vanishing at no root of p. Proposition 4.4 gives a complete intersection of local length at most $36 < 42$ for the general member of the turning family of b^5, and A is its limit. □

Radial costs alone [16, Theorem 3.15] do not always suffice. For $(1, 3, 5, 4, 4, 2, 2)$ the top forms are $D_k = \nu_k(\Gamma)$, $k = 5, 6$, for a scheme Γ of length two on the line $x = 0$ [15, Lemma 2.16], and at a point of Γ every graded equation of degree at most six vanishes, so $\kappa = 7$ and each unit of multiplicity costs five. If $\Gamma = \{[Y], [Z]\}$ and $J_4 = \mathbb{C}\, y^2 z(y + z)$, the roots in Γ have total multiplicity three, and Theorem 2.10 gives $24 + 5 \cdot 3 + 3 = 42$; the turning power lowers the cost of each unit to three (script `121_1354422`, class S: 42 on the graded algebras, 36 after turning).

8.2. When $H_A(5) = H_A(6) = 2$, the top forms are those of a scheme of length two, and whether this scheme lies on the line $x = 0$ decides the method.

Theorem 8.3. *Every local algebra with Hilbert function* $(1, 3, 5, 5, 3, 2, 2)$ *is smoothable.*

Proof. If the top forms are binary, we slide a cubic equation and apply Proposition 4.7, with the bound 40. Otherwise two identities between derivatives of the lifts of the top forms produce a turning operator.

Setup. As $\mathfrak{m}^7 \subseteq I$, we are in the binary case with $\gamma_f = 0$ and $f_3 \in x^2 S_1$, and W_2 is a pencil, as $\dim I_3^* = 10 - 5 = 5$. So $D_3 = \bar{T}_3 \oplus \mathbb{C}\, XQ_D$, where $Q_D \in \bar{T}_2$ spans the orthogonal complement of W_2 in $\bar{T}_2$. As $\dim I_4^* = 15 - 3 = 12$ and $\dim(I_4^* \cap xS_3) \leq 10$, some $P \in I$ has a quartic leading form p_4 with $p = \pi(p_4) \neq 0$. By [15, Lemma 2.16], $D_k = \nu_k(\Gamma)$ for $k = 5, 6$ and a scheme Γ of length two, and as $x^2 S_4 \subseteq I_6^* = (I_\Gamma)_6$, we have $x^2 \in I_\Gamma$. So either Γ lies on

the line $x = 0$, or Γ is a double point at a point of the line with a transverse tangent; in the second case, after a linear change of coordinates that keeps x^2, the plane $x = 0$, the binary case and $\gamma_f = 0$, we have $D_6 = \langle Y^6, XY^5\rangle$ and $D_5 = \langle Y^5, XY^4\rangle$.

Γ *on the line: sliding.* Now D_5 and D_6 are binary, so $x \circ D_6 = 0$. Every linear map $\varphi\colon W_2 \to S_5$ satisfies condition (i) of Lemma 4.6: $x\varphi(m)$ is a sextic orthogonal to D_6, and $\mathfrak{m}^2\varphi(m) \subseteq \mathfrak{m}^7 \subseteq I$. A pencil of binary quadrics either has no common zero or has a common linear factor. If $W_2 = \langle m_1, m_2\rangle$ has no common zero, there is no linear syzygy, so (ii) is empty, and we take $\varphi(m_1) = \ell^5$ and $\varphi(m_2) = 0$, with $\ell \in \bar{S}_1$ vanishing at no root of p. If $W_2 = l_0\bar{S}_1$, we take $\varphi(l_0 l) = l\,\ell_4$ with a binary quartic ℓ_4 vanishing at no root of p, which satisfies (ii), and $m = l_0 l_1$ with l_1 vanishing at no root of p. In both cases $\varphi(m)$ has order five and does not vanish at the roots of p, and Proposition 4.7 with $E = \mathbb{C}p_4$ gives $24 + 8 \cdot 2 = 40 < 42$. Triples of local length exactly 40 occur (script `121_1355322`, part (B)).

Γ *transverse: a turning operator.* Now $D_6 = \langle Y^6, XY^5\rangle$. For $m \in W_2$ the cubic xm lies in I_3^*, so $0 = (xm)\circ XY^5 = m\circ Y^5$, and m vanishes at $[Y]$; hence $W_2 = z\bar{S}_1$, $Q_D = Y^2$ and $D_3 = \bar{T}_3 \oplus \mathbb{C}XY^2$. Next, $D_4 \supseteq S_1 \circ D_5 = \langle Y^4, XY^3\rangle$, and $D_4 = \langle Y^4, XY^3, E_4\rangle$, where the first derivatives of E_4 lie in D_3; writing E_4 by powers of X, this forces $E_4 \equiv E_0$ modulo XY^3 for a binary quartic E_0. Choose $p_3 = z\,k_2 \neq 0$ with $k_2 \in \bar{S}_2$ and $k_2 \circ (z \circ E_0) = 0$, two linear conditions on $\bar{S}_2$. Then p_3 kills Y^4, XY^3 and E_0, so $p_3 \circ D_4 = 0$, and hence $p_3 \circ D_5 = p_3 \circ D_6 = 0$.

Let $\Phi = Y^6 + P_5 + \cdots$ and $\Psi = XY^5 + Q_5 + \cdots$ be the lifts of the top forms. The elements $x \circ \Phi$, $z \circ \Phi$, $z \circ \Psi$ and $y \circ \Phi - 6\,x \circ \Psi$ of M have degree at most four, as their parts of degree five cancel; so their quartic parts $x \circ P_5$, $z \circ P_5$, $z \circ Q_5$ and $y \circ P_5 - 6\,x \circ Q_5$ lie in D_4, which p_3 kills. Hence $x \circ (p_3 \circ P_5) = z \circ (p_3 \circ P_5) = 0$, so $p_3 \circ P_5 = cY^2$ for some $c \in \mathbb{C}$; and $z \circ (p_3 \circ Q_5) = 0$, while $x \circ (p_3 \circ Q_5) = \frac{1}{6}\,y \circ (p_3 \circ P_5) = \frac{c}{3}Y$, so $p_3 \circ Q_5 = \frac{c}{3}XY + c'Y^2$. Put

$$u = p_3 - \frac{c'}{60}\,xy^3 - \frac{c}{360}\,y^4.$$

As $y^4 \circ Y^6 = 360Y^2$, $xy^3 \circ Y^6 = 0$, $xy^3 \circ XY^5 = 60Y^2$ and $y^4 \circ XY^5 = 120XY$, the parts of degree at least two of $u \circ \Phi$ and $u \circ \Psi$ vanish. The elements of $M \cap T_{\leq 5}$, which together with Φ and Ψ span M, have quintic parts in D_5, which p_3 kills, so u maps them into $T_{\leq 1}$. Hence $u \in (I : \mathfrak{m}^2) \cap \mathfrak{m}^3$, and its cubic part p_3 is a nonzero binary cubic: a turning operator of type (b). (Script `121_1355322`, part (N), checks the four differentiation constants, and the shapes and the corrected operator on seeded actual algebras and on an explicit example in which p_3 alone is not in $(I : \mathfrak{m}^2)$.) $\square$

In the second case no induction is used. The two derivative identities are the whole content: they say that the tails of the two top forms are tied

together by the relation $y \circ Y^6 = 6\, x \circ XY^5$, and the quartic corrections of u absorb exactly this tie.

Appendix A. Socle degree at most five: the remaining cases

A.1. This appendix proves the thirteen Hilbert functions with $H_A(2) = 6$ and socle degree at most five left by Section 6. Throughout, $W = I_3^*$ and $D_3 = W^\perp$. In socle degree five, M is written as in (2), $r = \dim S_1 \circ D_5$, and Syz is the kernel of $S_1 \otimes D_5 \to T_4$, $v \otimes F \mapsto v \circ F$.

A.2. By (3), τ is determined on $S_1 \circ D_5$ by R, and R must satisfy $\sum_j v_j \circ R(F_j) \in D_3$ for all $\sum_j v_j \otimes F_j \in \mathrm{Syz}$. Dually, these conditions have rank $\dim(W \cdot \mathrm{Syz})$, where $W \cdot \mathrm{Syz} \subseteq S_4^{H_A(5)}$ is spanned by the vectors $(v_j w)_j$ with $w \in W$ [16, proof of Theorem A.11]. So for fixed graded data the tails (τ, R), with R modulo D_4, form a space of dimension

$$(H_A(4) - r)(10 - H_A(3)) + (15 - H_A(4))H_A(5) - \dim(W \cdot \mathrm{Syz}), \tag{6}$$

and E is free. Over an irreducible variety of graded data forming vector bundles, with r and $\dim(W \cdot \mathrm{Syz})$ constant, the algebras form one irreducible family [15, Lemmas 2.3(2) and 5.16]. In (6) R is counted modulo D_4. Some counts below take R *raw*, as a linear map $D_5 \to T_4$; this adds $H_A(4)H_A(5)$ both to the dimension of the space of all data and to that of every stratum, so it changes no comparison. We say in each count which convention it uses.

Lemma A.3. *Let $H_A(3) = 6$, and suppose that the cubics of W have no common factor. Then $\sum_{p \in \mathfrak{B}} e_p \le 7$, unless $\mathfrak{B} = V(\mathfrak{m}_p^3)$, where $e_p = 9$.*

Proof. By [16, proof of Theorem A.14, case $F = X^5$], $\mathfrak{B}$ is finite of length at most six, and $W = (I_{\mathfrak{B}})_3$ if the length is six. If $\ell(\mathfrak{B}) \le 5$, then $\sum_p e_p \le \ell(\mathfrak{B}) + 1 \le 6$ [16, Lemma 3.2]. Let $\ell(\mathfrak{B}) = 6$. By the proof of [16, Lemma 3.2], $e_p = \ell_p$ at a curvilinear point p of length ℓ_p, and $e_p \le \ell_p + 1$ at a non-curvilinear point of length at most five. A non-curvilinear point has length at least three, so two of them would form $V(\mathfrak{m}_p^2) \cup V(\mathfrak{m}_q^2)$; the line through p and q meets this scheme in length four, so it would divide every cubic of $W = (I_{\mathfrak{B}})_3$. Hence $\sum_p e_p \le 7$ unless $\mathfrak{B}$ is a single non-curvilinear point p of length six, whose local ideal $\mathfrak{b}$ lies in $\mathfrak{m}_p^2$; let u, v be local coordinates at p.

The Hilbert function of $\mathcal{O}_p/\mathfrak{b}$ is $(1,2,3)$, $(1,2,1,1,1)$ or $(1,2,2,1)$, and $e_p \le e(\mathfrak{b}^*)$ for the leading ideal $\mathfrak{b}^*$; the multiplicity of an ideal is at most the colength of a parameter ideal inside it. For $(1,2,3)$, $\mathfrak{b} = \mathfrak{m}_p^3$ and $e_p = 9$. For $(1,2,1,1,1)$, $\mathfrak{b}^* = (u^2, uv, v^5) \supseteq (uv, u^2 + v^5)$, of colength 7. For $(1,2,2,1)$, up to a linear change of u, v, $\mathfrak{b}^*$ is one of the complete intersections (u^2, v^3), $(u^2, uv^2 + \beta v^3)$, $(uv, u^3 + \beta v^3)$ with $\beta \neq 0$, of colength 6; or $(uv, u^3, v^4) \supseteq (uv, u^3 + v^4)$, of colength 7; or (u^2, uv^2, v^4). In the last case $\mathfrak{b} \supseteq \mathfrak{m}_p^4$ contains uv^2 and $u^2 + \gamma v^3$ for some γ. If $\gamma \neq 0$, $(u^2 + \gamma v^3, uv^2)$ has colength $3 + 2 \cdot 2 = 7$. If $\gamma = 0$, then $\mathfrak{b} = (u^2, uv^2, v^4)$, and for $p = [Z]$, $u = x/z$, $v = y/z$ the cubics of $I_{\mathfrak{B}}$ are x^3, x^2y, x^2z, xy^2, all divisible by x; as $W = (I_{\mathfrak{B}})_3$, this case is excluded by the hypothesis. □

In socle degree five a point $p = [P]$ of $\mathfrak{B}$ costs $2e_p$ in Theorem 2.6 if $P^4 \in D_4$; a quartic part prescribed by Lemma 2.14 halves this unless P^4 is a derivative of a top form.

Lemma A.4. *Let A be a local algebra of length d with $H_A(2) = 6$, socle degree five, $\dim I_3^* \geq 3$ and $\mathfrak{B}$ finite. Let $\omega(p) = 1$ if $P^4 \notin S_1 \circ D_5$, and $\omega(p) = 2$ otherwise. If $27 + \sum_{p\in\mathfrak{B}} \omega(p)\, e_p < 2d$, then every algebra with the graded data of A is smoothable.*

Proof. Choose general $w_1, w_2, w_3 \in I_3^*$, so that their common zero set is the support of $\mathfrak{B}$ and, at each $v \in \mathfrak{B}$, the germs of w_1, w_2 generate a reduction of the local ideal $\mathfrak{b}_v$ of $\mathfrak{B}$; and take $L \in S_1$, a quartic p_3 killing D_5 (the quartic part of the third equation in Lemma 2.14), and σ_1, σ_2, all general. By Lemma 2.14 some algebra with the graded data of A contains w_1, w_2, $w_3 + \sigma_1 p_3 + \sigma_2 L^5$. At $v \in \mathfrak{B}$, in the chart $x = \rho$, $y = \rho\eta$, $z = \rho\zeta$ with $v = [1 : 0 : 0]$, the cones $V(w_1), V(w_2)$ meet in a scheme free of rank e_v over $\mathbb{C}\{\rho\}$, as w_1, w_2 generate a reduction of $\mathfrak{b}_v$. On it the proper transform of the third surface is cut out by $\tilde{w}_3 + \sigma_1\rho\,\tilde{p}_3 + \sigma_2\rho^2\tilde{L}^5$, where $\tilde{g} = g(1, \eta, \zeta)$ for a form g. As the quartics killing D_5 form $(S_1 \circ D_5)^\perp \cap S_4$, $\tilde{p}_3$ is a unit if $\omega(v) = 1$, and [16, Lemma 3.14(b)], with $h = \tilde{w}_3$, $a = 0$ and $c_1 = 1$, bounds the length over v by e_v; otherwise the unit $\tilde{L}^5$, with $c_1 = 2$, bounds it by $2e_v$. [16, Lemma 3.14(a)] adds 27. □

Theorem A.5. *Every local algebra of length twenty-one with $H_A(2) = 6$ and socle degree four, other than the very compressed ones with Hilbert function $(1, 3, 6, 10, 1)$ (Table 1), is smoothable. These are the Hilbert functions $(1, 3, 6, c, 11 - c)$, $5 \leq c \leq 9$.*

Proof. We transfer [16, Theorem 5.1]; $c \geq 5$ as $4^{\langle 3\rangle} = 5 < 7$. By [15, Lemma 2.7], $M = T_{\leq 2} \oplus D_3 \oplus \{F + \tau(F) : F \in D_4\}$ with $D_4 \subseteq D_3' = (S_1 W)^\perp$ and τ free; counts use [15, Lemmas 2.17 and 2.18] on the pairs (D_3, D_4).

$c = 9$: rays. The data (w, p, D_4, τ), $W = \mathbb{C}w$, $w(p) = 0$, $D_4 \in \mathrm{Gr}(2, D_3')$, $\dim D_3' = 12$, form an irreducible family containing every algebra (plane cubics have zeros). Injectivity of $(N_P \otimes D_4) \oplus \mathbb{C} \to T_3$, $(v \otimes F, \lambda) \mapsto v \circ F - \lambda P^3$, is open and gives a ray [15, Lemma 5.2]; it holds for $w = x^3$, $D_4 = \langle Y^4 + Z^4 + X^2YZ, X^2Z^2\rangle$, $p = [Y]$, as $2XYZ$, $4Z^3 + X^2Y$, $2XZ^2$, $2X^2Z$, Y^3 are independent.

$c = 8$: an anchor. Here W is a pencil. The pairs $(D_3, D_4) \in \mathrm{Gr}(8, T_3) \times \mathrm{Gr}(3, T_4)$ with $S_1 \circ D_4 \subseteq D_3$ are cut out by $3 \cdot 3 \cdot 2$ equations, so every component has dimension at least $52 - 18 = 34$. A coprime pencil has no linear syzygy, so $\dim S_1 W = 6$ and $D_4 \in \mathrm{Gr}(3, 9)$: these pairs form an irreducible open set of dimension $16 + 18 = 34$. The pencils with a common factor are $l\langle a, b\rangle$ with coprime quadrics a, b, and $q\langle l_1, l_2\rangle$; they give at most $2 + 8 + 18 = 28$ and $5 + 2 + 21 = 28$. So the pairs form one irreducible variety, the closure of the coprime stratum [15, Lemma 2.18], and as τ is free, the algebras form one irreducible family. Its member $T_{\leq 2} \oplus \langle$cubic monomials

other than $X^3, Y^3\rangle \oplus \langle X^2Y^2, X^2YZ, XY^2Z\rangle$ is killed by x^3, y^3, z^4, of local length $3 \cdot 3 \cdot 4 = 36 < 42$, so it is a linkage anchor of the family.

$c = 7, 6$. Without a common factor, [15, Proposition 3.18] applies. For $c = 7$ the bound is $65 - 36 = 29$, and a common factor gives $W = qS_1$ $(5+20)$ or $W = lV$ with $k = \dim S_1V \geq 10 - 3^{\langle 2\rangle} = 6$: $2 + 9 + 4(11 - k) \leq 27$ for $k \geq 7$, and $2 + 2 + 20$ for $k = 6$, where $V = l'S_1$ (Gotzmann). For $c = 6$ the bound is $74 - 60 = 14$, $W = lV$ (a quadric factor allows three cubics only) with $k \in \{8, 9, 10\}$, and $k = 10, 9$ give $2 + 8$ and $2 + 6 + 5 = 13$ as in [16, Theorem 5.1]. The stratum $k = 8$, of dimension $2+4+\dim \mathrm{Gr}(5, 7) = 16$, has $V = (I_\Gamma)_2$ with $\ell(\Gamma) = 2$ [15, Lemma 2.16], so $W = l(I_\Gamma)_2$. Let $\mathfrak{B}_{l,\Gamma} = V(lI_\Gamma)$. It is the line $V(l)$ with Γ attached; where Γ meets the line, $\mathfrak{B}_{l,\Gamma}$ has embedded points, so it equals the union $V(l) \cup \Gamma$ only when Γ misses the line. The exact sequence

$$0 \to (S/I_\Gamma)(-1) \xrightarrow{\cdot l} S/(lI_\Gamma) \to S/(l) \to 0$$

gives $h_{\mathfrak{B}_{l,\Gamma}}(k) = (k + 1) + 2$ for $k \geq 2$. As $(I_\Gamma)_3 = S_1(I_\Gamma)_2$, D_4 is orthogonal to $S_1W = (lI_\Gamma)_4$, so $D_4 \subseteq \nu_4(\mathfrak{B}_{l,\Gamma}) = (lI_\Gamma)_4^\perp$, of dimension 7 for all (l, Γ): the stratum is a $\mathrm{Gr}(5, 7)$-bundle over $\mathbb{P}^2 \times \mathrm{Hilb}_2(\mathbb{P}^2)$. For dense $\Gamma = \{X, Q\}$ off the line, $\mathfrak{B}_{l,\Gamma} = V(l) \cup \Gamma$ and $N_X \circ \nu_4(\mathfrak{B}_{l,\Gamma}) \subseteq \nu_3(V(l)) + \mathbb{C}Q^3 \not\ni X^3$: a ray.

$c = 5$. As $\dim D_3' \geq 6 = 5^{\langle 3\rangle}$, [15, Lemma A.12](c) gives $D_k = \nu_k(\mathfrak{B})$, $k = 3, 4$, for $\mathfrak{B} = V(l\mathfrak{m}_q)$. As $h_{\mathfrak{B}}(k) = k + 2$ for all (l, q), the algebras form a bundle of rank $6 \cdot 5 = 30$ over $\mathbb{P}^2 \times \mathbb{P}^2$, with rays when $q = [X] \notin V(l)$, $l = x$: $N_X \circ D_4 \subseteq \bar{T}_3 \not\ni X^3$. [15, Proposition 2.2(1)] concludes. □

A.6. For $H_A(3) = 5$ we apply [15, Lemma A.12]. In cases (a), (b), [16, Corollary 3.3] gives at most 39. In case (d), Proposition 6.2 applies, as $S_1 \circ D_5 \subseteq \bar{T}_4$ has dimension at most $\min(H_A(4), 2H_A(5)) < 5$. In case (c), $\mathfrak{B} = V(x\mathfrak{m}_q)$, $W = x(\mathfrak{m}_q)_2$ and $D_k \subseteq \nu_k(\mathfrak{B})$ for $k = 4, 5$; if $q \notin V(x)$, [16, Lemma A.3] gives a ray. As $\nu_k(\mathfrak{B})$ form bundles over the pairs (line, point) [16, Section A.8], *an algebra with $q \in V(x)$ is smoothable once its graded data lie in an irreducible family of such data, with $q \notin V(x)$ dense, on which r and* $\dim(W \cdot \mathrm{Syz})$ *are constant* (Section A.2). For a point $N \neq q$ of the line, $\dim N_NW = 10 - 2 = 8$, and $\dim N_qW = 10 - 3 = 7$, as $(x\mathfrak{m}_q : \mathfrak{m}_N) = x\mathfrak{m}_q$ and $(x\mathfrak{m}_q : \mathfrak{m}_q) = (x)$.

Theorem A.7. *Every local algebra with Hilbert function* $(1, 3, 6, 5, 5, 1)$ *is smoothable.*

Proof. We transfer [16, Theorem A.10], whose five types of $D_5 = \mathbb{C}F$ for embedded q do not involve $D_4 \in \mathrm{Gr}(5 - r, 6 - r)$. By (6), the tails have dimension $(5 - r)5 + 10 - \dim(W \cdot \mathrm{Syz})$. For F concise, $r = 3$ and $\mathrm{Syz} = 0$, so the tails are 20. For F binary on the line, $r = 2$ and $\mathrm{Syz} = \mathbb{C}x \otimes F$, so $W \cdot \mathrm{Syz} \cong xW$ and the tails are $15 + 10 - 5 = 20$. For $F = N^5$ with $N \neq q$ on the line, and for $F = q^5$, $r = 1$ and $\mathrm{Syz} = N_N \otimes F$, respectively $N_q \otimes F$, so $W \cdot \mathrm{Syz} \cong N_NW$, respectively N_qW, and the tails are $20 + 10 - 8 = 22$,

respectively $20+10-7=23$ (Section A.6). So these four types are families as in Section A.6. The fifth type, $F = Z^4(c(X+\alpha Y)+bZ)$ with $c \neq 0$ and $q = [Z]$, is a limit of the concise type along the curve $F + tG'$ of [16, Theorem A.10], with one more quartic in $D_4(t)$; the tails are 20 along it (at $t = 0$, $\mathrm{Syz} = \mathbb{C}(y - \alpha x) \otimes F$) (script `121_136551`, parts (T), (D)). □

Theorem A.8. *Every local algebra with Hilbert function* $(1,3,6,5,4,2)$ *is smoothable.*

Proof. Here $r \in \{2,3,4\}$, and the tails are $(4-r)5 + 22 - \dim(W \cdot \mathrm{Syz})$. For $r \leq 3$, the six families of pencils of [16, Theorem A.11], with $\dim(W \cdot \mathrm{Syz}) = 16, 15, 13, 12, 14, 13$, and $D_4/S_1 \circ D_5 \in \mathrm{Gr}(4-r, 6-r)$ give tails $16, 17, 14, 15, 13, 14$ (script `121_136542`, parts (R2), (R3)). Let $r = 4$, so $D_4 = S_1 \circ D_5$, $q^5 \notin D_5$ (else $r \leq 3$), and D_5 has at most one fifth power. If $L^5 \in D_5$, then $\mathrm{Syz} = N_L \otimes L^5$ and the tails are $22 - 8 = 14$, over an open set of a $\mathbb{P}^5$-bundle over the triples (line, q, L). Otherwise $\dim(W \cdot \mathrm{Syz}) = 10$, the maximum ($x \otimes F_2 \in \mathrm{Syz}$ for a binary $F_2 \in D_5$, and a second relation $a_1 \circ F_1 + a_2 \circ F_2 = 0$ has $a_1 \neq 0$, else F_2 is a fifth power), and we move q off the line. For binary D_5 this is clear. Otherwise $D_5 = \langle G_1 + XZ^4, G_2 \rangle$, G_i binary, with a relation $\alpha Z^4 + \beta y \circ G_1 + (\gamma y + \delta z) \circ G_2 = 0$. Let $q_t = [Z + tX]$, $h_t = ((Z+tX)^5 - Z^5)/5t$. As $(G_1', G_2') \mapsto (\beta y - \alpha t z) \circ G_1' + (\gamma y + \delta z) \circ G_2'$ maps onto $\bar{T}_4$ at $t = 0$, regular binary $G_i(t)$ with $G_i(0) = G_i$ are sent to $-\alpha Z^4$; then the derivatives of $\langle G_1(t) + h_t, G_2(t) \rangle \subseteq \nu_5(V(x\mathfrak{m}_{q_t}))$ keep a relation, and $r = 4$ near $t = 0$ (part (CV)). □

Theorem A.9. *Every local algebra with Hilbert function* $(1,3,6,5,3,3)$ *is smoothable.*

Proof. By [15, Lemma 2.16], $D_k = \nu_k(\Gamma)$ for $k = 4, 5$ with $\ell(\Gamma) = 3$; so $r = 3$, the tails are $36 - \dim(W \cdot \mathrm{Syz})$, and $\Gamma \subseteq \mathfrak{B}$. Let $q = [Z]$ and $V = (\mathfrak{m}_q)_2$. If $\Gamma \subset V(x)$, then $\mathrm{Syz} = x \otimes D_5 \oplus \mathrm{Syz}_{\mathrm{bin}}$, and $W \cdot \mathrm{Syz}$ is the direct sum of $(x^2V)^3$, $xy\,\bar{S}_1\mathrm{Syz}_{\mathrm{bin}}$ and x^2z^2 times the values of $\mathrm{Syz}_{\mathrm{bin}}$ at q, or, off the line, of $(x^2V)^3$ and $x\bar{S}_2\mathrm{Syz}_{\mathrm{bin}}$; these have dimensions 15, 6, 9 (script `121_136533`, part (M)), and the values span 3 dimensions if $q \notin \Gamma$, 2 if $q \in \Gamma$. So for $q \notin \Gamma$, $\dim(W \cdot \mathrm{Syz})$ is $15 + 6 + 3 = 24$, or $15 + 9 = 24$ off the line, and the tails are $36 - 24$. If $q \in \Gamma$, the base is the closure of $\{(\ell, q, \Gamma_2 \sqcup q) : q \notin \ell, \Gamma_2 \subset \ell\}$, type (5) of [16, proof of Theorem A.11]; it contains every embedded $\Gamma \ni q$: those in the line (when reduced, limits of $\{N_1, N_2, q_t\}$), N with a transverse double point at q, and $V(\mathfrak{m}_q^2)$ and $V(x - cy^2, y^3)$ (chart $z = 1$). The tails are $36 - 23$: off the line $16 + 7$, in the line $15 + 6 + 2$, and 23 at the three other orbits (part (T2)). □

A.10. For $H_A(3) = 6$, either W has no common factor, and then Lemmas A.3 and A.4 settle every algebra except those over $V(\mathfrak{m}_p^3)$ with $\omega(p) = 2$, as $27 + 2 \cdot 7 = 41$ and $27 + 9 = 36$; or $W = xQ$, $Q \in \mathrm{Gr}(4, S_2)$, as a quadric factor leaves room for three cubics only. Let $k = 10 - \dim S_1 Q \leq 2$ ($2^{\langle 2 \rangle} = 2$). If $k = 0$, then $S_1 Q = S_3$, and Proposition 6.2 applies: $K \neq 0$, as $D_5 \subseteq \bar{T}_5$ (proof

of Proposition 6.2), so $S_1 \circ D_5 \subseteq \bar{T}_4$ has dimension at most $2 \dim D_5 \leq 4 < 5$. If $k = 1$, a cubic G spans $(S_1 Q)^\perp$, with $S_1 \circ G \subseteq Q^\perp$; two essential variables would give $Q \supseteq mS_1$ and $\dim S_1 Q \leq 8$, so $G = N^3$, Q is a hyperplane of the quadrics $I(N)_2$ through N, and $S_1 Q = I(N)_3$; we call N the *square point*. If $k = 2$, $Q = (I_\Gamma)_2$ with $\ell(\Gamma) = 2$ [15, Lemma 2.16].

Theorem A.11. *Every local algebra with Hilbert function* $(1, 3, 6, 6, 4, 1)$ *is smoothable.*

Proof. We count all data for each type of top form; strata of exactly the expected dimension are settled directly. Let $T_4(W) = (S_1 W)^\perp$.

F concise. Then $D_4 = S_1 \circ F \oplus \mathbb{C}u$, and with R raw the tails are $15+4+4 = 23$ ($R \in T_4$, $\tau(u)$ and E in T_3/D_3; with R modulo D_4 they are 19, and all dimensions below drop by four). The pairs $([F], D_3)$ with $S_2 \circ F \subseteq D_3$ have dimension at least $20 + 24 - 24$, and $[u] \in \mathbb{P}^{11}$ meets 12 conditions: the bound is $20 + 11 - 12 + 23 = 42$. Over the fat point $\mathfrak{B} = V(\mathfrak{m}_p^3)$, $W = (I_\mathfrak{B})_3$ and $I_\mathfrak{B} = \mathfrak{m}_p^3$ give $D_4 \subseteq \nu_4(\mathfrak{B})$ and $F \in \nu_5(\mathfrak{B})$; with p, $[F] \in \mathbb{P}(\nu_5(\mathfrak{B}))$, $[u] \in \mathbb{P}(\nu_4(\mathfrak{B})/S_1 \circ F)$ and the tails, these algebras have dimension at most $2 + 5 + 2 + 23 = 32$. For $W = lQ$, Q kills $l \circ F \neq 0$; if $l \circ F = N^4$, then l, N, $[F] \in \mathbb{P}^6$, $Q \subseteq I(N)_2$, $[u] \in \mathbb{P}^3$ give $2+2+6+4+3+23 = 40$, and otherwise $Q = (I_{\Gamma'})_2$ with $l \circ F \in \nu_4(\Gamma')$ [15, Lemma 2.16], giving $2+4+7+3+23 = 39$.

$F = X^5$. The data p, $D_3 \ni X^3$, $D_4 \ni X^4$, τ, R modulo D_4, E have dimension $2+20+33+16+11+4 = 86$, with $36+12$ equations: the bound is 38. The tails, with R modulo D_4, are $19 + r'$, $r' = \dim\{q : yq, zq \in W\} \leq 3$. Counting W with p, D_4/X^4 and the tails, the fat point gives $2 + 6 + 22$, and $W = lQ \subseteq I(p)_3$ gives $11 + 3 + 22$ if $\delta' = 10 - \dim S_1 Q = 0$, $9 + 6 + 22$ if $\delta' = 1$, and, for $\delta' = 2$, where $Q = (I_{\Gamma''})_2$ and $r' = \dim(I_{\Gamma''} : I_p)_1$, $7 + 9 + 20$ if $l(p) = 0$, $p \notin \Gamma''$; $5 + 9 + 21$ if $l(p) = 0$, $p \in \Gamma''$; $6 + 9 + 21$ if $l(p) \neq 0$, $p \in \Gamma''$: at most 37.

F binary. Let $F \in \mathbb{C}[X, Y]_5$, $c = \dim S_2 \circ F \in \{2, 3\}$, and $Z \subset V(z)$ the scheme of the binary apolar generator of degree c, so $W \subseteq \operatorname{Ann}(F)_3 = (I_Z)_3$. With $D_4 = S_1 \circ F \oplus U'$, closedness says $z \circ R \in D_3$, the tails with R raw are $11 + 8 + 4 = 23$ (the same for all W and U'), and the data (W, U') have dimension at least $12 + 22 - 24 = 10$ if $c = 3$, and $16 + 22 - 24 = 14$ if $c = 2$. The fat point gives 4. A factor $l \neq z$ gives at most $2 + 6 = 8$ if $c = 3$, as no derivative of F is a fourth power (proof of Theorem A.15), so l determines $Q = (I_{\Gamma'})_2$; and at most 12 if $c = 2$. For $W = zQ$, $T_4(W)$ is the preimage of $(S_1 Q)^\perp$ under $z\circ$, so $U' \in \operatorname{Gr}(2, 3 + \delta')$, and $\delta' = 0, 1, 2$ give $8+2$, $6+4$, $4+6$, all $10 < 14$. For $c = 3$ we settle them directly. If $\delta' = 0$, the binary quartics killing F form a space of dimension 3, so $\delta \leq 2$, and Proposition 6.2 gives 38. If $Q \subseteq I(N)_2$ or $Q = (I_{\Gamma''})_2$, the stratum is irreducible with constant tails, and N, respectively Γ'', may be taken off the line. Then $N^3 \in D_3$, $v \mapsto v \circ F$ is injective on N_N, and the cubic parts of the elements of $N_N \circ M$ of degree at most three lie in $N_N \circ D_4 \subseteq \mathbb{C}[X, Y]_3$, respectively $\mathbb{C}[X, Y]_3 + \mathbb{C}N'^3$, which does not contain N^3: a ray [15, Theorem 2.11(2)].

[15, Proposition 2.2(1)] concludes (script `121_136641_0`). □

Theorem A.12. *Every local algebra with Hilbert function* $(1,3,6,6,3,2)$ *is smoothable.*

Proof. Let $V = D_5$; then $r \in \{2,3\}$, and $D_4 = S_1 \circ V$ if $r = 3$. By Section A.10 every algebra falls into one of the following cases.

case	method	local length or count
W without common factor, $\mathfrak{B} \neq V(\mathfrak{m}_p^3)$	excess with weights (Lemmas A.3 and A.4)	at most $27 + 2 \cdot 7 = 41$
$\mathfrak{B} = V(\mathfrak{m}_p^3)$	four irreducible families, each with an anchor	37, 36, 39, 41
$W = xQ$, $k = 0$	Proposition 6.2	at most $36 + \delta \leq 40$
$k = 1$, N off the line	the proof of Proposition 6.2 with $x(y,z)\mathfrak{m}$ in place of $x\mathfrak{m}^2$	at most $38 + \delta \leq 41$
$k = 1$, N on the line	the same with $x(x,z)\mathfrak{m}$; limits with constant tails	37 to 41; ρ constant
$k = 2$	a count with V fixed	33, $32 < 34$; $38 < 42$

The first and third rows are settled in Section A.10. We treat the others in turn.

The fat point. Let $\mathfrak{B} = V(\mathfrak{m}_p^3)$, $p = [Z]$; then $D_3 = ZT_2$, $D_4 \subseteq Z^2T_2$ and $V = Z^3\bar{V}$. Write $g \in \bar{V}$ as $g_0 + Zg_1 + cZ^2$ with g_0, g_1 free of Z. If the g_0 span two dimensions, then $r \geq 4$; if one, $r \leq 3$ forces g_0 to be a multiple of the square of the g_1 of the other generator, or of X^2 if that g_1 is zero; if none, $V = Z^4\Lambda$. So up to the stabilizer of p, V is $\langle Z^5, XZ^4\rangle$ ($r = 2$, $[u] \in \mathbb{P}^3$), $\langle Z^5, X^2Z^3 + bYZ^4\rangle$, $\langle XZ^4, X^2Z^3 + bYZ^4 + cZ^5\rangle$ or $\langle XZ^4, YZ^4\rangle$. Part (C) of script `l21_136632_fat` checks the seven orbits in these families: (a); (b1), (b2) for $b = 0$, $b \neq 0$; (c1), (c2), (e) for $b = c = 0$, $b = 0 \neq c$, $b \neq 0$; and (d). As $Z^4 \in S_1 \circ V$, Lemma A.4 gives only 45. These four families are irreducible, with tails (τ, R), R modulo D_4, of dimensions 15, 15, 12, 14 (part (T)) and E free, and have members of local length 37, 36, 39, 41 (script `l21_136632`, parts (A7)–(Ad)). The members are spanned by $T_{\leq 2}$, ZT_2 and the following lifts, and they are killed by the triples shown:

- $V = \langle Z^5, XZ^4\rangle$: $Z^4 + 4X^3$, $XZ^3 + \frac{1}{4}Y^3$, $YZ^3 - 3X^2Y$, $XZ^4 + X^4 + Y^3Z$, $Z^5 + 20X^3Z$; killed by $x^3 - z^4$, $y^3 - \frac{1}{4}xz^3$, $x^2y + yz^3$, of local length 37;
- $V = \langle Z^5, X^2Z^3\rangle$: $Z^4 + 4X^3 + 4Y^3$, XZ^3, $X^2Z^2 - 2X^2Y$, $Z^5 + 20(X^3 + Y^3)Z$, $X^2Z^3 - 6X^2YZ$; killed by $x^3 - z^4$, $y^3 - z^4$, $x^2y + x^2z^2$, of local length 36;
- $V = \langle XZ^4, X^2Z^3\rangle$: $Z^4 + 4X^3$, XZ^3, X^2Z^2, $XZ^4 + X^4$, X^2Z^3; killed by $x^3 - z^4$, $y^3 + z^5$, $x^2y + yz^3$, of local length 39;

- $V = \langle XZ^4, YZ^4\rangle$: Z^4, $XZ^3 - X^3 - 3X^2Y$, $YZ^3 - Y^3 - 6X^2Y$, $XZ^4 - 4X^3Z - 12X^2YZ$, $YZ^4 - 4Y^3Z - 24X^2YZ$; killed by $x^3 + xz^3$, $y^3 + yz^3$, $x^2y + xy^2 + (x + 2y)z^3 + z^5$, of local length 41.

$k = 1$, N off the line. Let $N = X$, so $S_1Q = (y, z)S_2$, $D_3' = \bar{T}_4 \oplus \mathbb{C}X^4$, $V \subseteq \bar{T}_5 \oplus \mathbb{C}X^5$, and the quartics killing V are $x(y, z)S_2 \oplus K''$, with K'' the binary quartics killing the projection of V to $\bar{T}_5$, and x^4 if $V \subseteq \bar{T}_5$. Let δ be the degree of the greatest common divisor of K''. As K'' is the annihilator in $\bar{S}_4$ of the projection of $S_1 \circ V$ to $\bar{T}_4$, of dimension at most $r \leq 3$, $\dim K'' \geq 2$ and $\delta \leq 3$. The proof of Proposition 6.2 with $x(y, z)\mathfrak{m}$ in place of $x\mathfrak{m}^2$, as $x^2(y, z)^2S_2 = x^2\langle y, z\rangle S_1Q$, gives at most $38 + \delta \leq 41$ (script `l21_136632_line`, part (K)).

$k = 1$, N on the line. Let $N = Y$. Then Q is a hyperplane of $I(Y)_2 = \langle x^2, xy, xz, yz, z^2\rangle$ with $S_1Q = I(Y)_3$, and $xS_1Q = (x\mathfrak{m}_Y)_4$ is the part of degree four of the ideal of $V(x\mathfrak{m}_Y)$, the line with an embedded point at Y. Hence $D_3' = \bar{T}_4 \oplus \mathbb{C}XY^3$, $D_4 \subseteq D_3'$ and $V \subseteq \bar{T}_5 \oplus \mathbb{C}XY^4$, the spaces orthogonal to $(x\mathfrak{m}_Y)_4$ and $(x\mathfrak{m}_Y)_5$. So (D_4, V) are the data in degrees four and five of an algebra with Hilbert function $(1, 3, 6, 5, 3, 2)$ whose cubic base scheme is the line with an embedded point, and they fall into the six types (1)–(6) of [16, proof of Theorem A.11]. Two facts hold for every square point N, on the line or off it.

(Q1) Q contains no mS_1 with $m \in N_N$: otherwise $Q = mS_1 \oplus \mathbb{C}q$ with $m \nmid q$, and $S_1Q = mS_2 + qS_1$ has dimension $6 + 3 - 1 = 8$, so $k = 2$.

(Q2) By (6), with R modulo D_4, the tails have dimension $4(3 - r) + 24 - \rho$, where $\rho = \dim(W \cdot \mathrm{Syz})$. As $W \cdot \mathrm{Syz}$ is the image of $W \otimes \mathrm{Syz} \to S_4^2$, $\rho = 4 \dim \mathrm{Syz} - \dim \mathcal{K}_V$ for the kernel $\mathcal{K}_V$ of this map. So, as in Section A.6, an algebra with N on the line is smoothable once its graded data lie in an irreducible family of graded data on which r and ρ are constant and in which the data with N off the line are dense: the algebras then form one irreducible family (Section A.2), its members with N off the line are smoothable by the previous paragraph, and [15, Proposition 2.2(1)] applies.

Multiplicities. Let g' be the greatest common divisor of the binary quartics killing V, δ' its degree and μ its order at Y. These quartics form the annihilator in $\bar{S}_4$ of the projection of $S_1 \circ V$ to $\bar{T}_4$, of dimension at least $5 - 3$, so $\delta' \leq 3$. With $x(x, z)\mathfrak{m}$ in place of $x(y, z)\mathfrak{m}$, as $x^2(x, z)^2S_2 = x^2\langle x, z\rangle S_1Q$, and with $z \mapsto tz$ in place of $y \mapsto ty$, under which $t^{-\mu}g'(y, tz)$ tends to a nonzero multiple of $y^{\delta'-\mu}z^\mu$, the argument for N off the line bounds the local length by that of a triple in the monomial ideal $x(x, z)\mathfrak{m} + y^{\delta'-\mu}z^\mu(y, z)^{4-\delta'} + \mathfrak{m}^5$. For $(\delta', \mu) = (0, 0)$, $(1, 0)$, $(1, 1)$, $(2, 0)$, $(2, 1)$, $(2, 2)$, $(3, 0)$ these lengths are 37, 38, 40, 39, 41, 41, 40 (script `l21_136632_line2`, part (MC)). This settles $r = 2$, that is, types (1) and (2): if $V = \nu_5(\Gamma)$ for Γ in the line, g' is the quadric of Γ and $\delta' = 2 \geq \mu$; if Γ is the double point at Y transverse to the line, $V = \langle Y^5, XY^4\rangle$, $g' = z$ and $(\delta', \mu) = (1, 1)$. It settles type (6), where $V = \langle Z^5, XY^4\rangle$, $g' = yz$ and $(\delta', \mu) = (2, 1)$, and two of the three kinds of type (5): over a point of the line and the transverse double point at Y,

where $V = \langle XY^4 + bZ^5, Y^5 + dZ^5\rangle$ with $d \neq 0$, $g' = yz$ and $(\delta', \mu) = (2, 1)$; and over $V(\mathfrak{m}_Y^2)$, where $V \subseteq Y^4T_1$, the quartics killing V contain $z^2\bar{S}_2$, and $\mu = \delta' \leq 2$. The normal forms are reached by coordinate changes that fix the line and Y (script `l21_136632_line4`, part (MV)).

Limits with constant tails. The remaining types have $r = 3$ and $D_4 = S_1 \circ V$. For a basis F_1, F_2 of V we write an element $v_1 \otimes F_1 + v_2 \otimes F_2$ of Syz as (v_1, v_2). For a basis s_1, s_2, s_3 of Syz, an element $\sum_j w_j \otimes s_j$ of $\mathcal{K}_V$ satisfies two equations in S_4, and as each w_j is a multiple of x, they are solved by divisibility. For instance, if $s_1 = (x, 0)$, $s_2 = (0, x)$, $s_3 = (a, b)$ with $a \notin \mathbb{C}x$, then $xw_1 + aw_3 = xw_2 + bw_3 = 0$ forces $w_3 = x^2l$, $w_1 = -axl$ and $w_2 = -bxl$ for a linear form l, and these lie in $W = xQ$ if and only if $l\langle x, a, b\rangle \subseteq Q$. The table lists the cases.

type	V	Syz	$\mathcal{K}_V$	ρ
(3)	binary, no fifth power	$(x,0)$, $(0,x)$, (a,b)	$\{l : lS_1 \subseteq Q\} = 0$	12
(3)	$\langle Z^5, F\rangle$, F binary	$(x,0)$, $(0,x)$, $(y,0)$	$\{l : l\langle x,y\rangle \subseteq Q\}$	$12 - \dim \mathcal{K}_V$
(4)	$\langle Y^5, F\rangle$, F binary	$(x,0)$, $(z,0)$, $(0,x)$	$\{l : lN_Y \subseteq Q\}$	11
(4)	$\langle Y^5, XY^4 + G\rangle$, G binary	$(x,0)$, $(z,0)$, $(-\frac{1}{5}y, x)$	$\{l : lN_Y \subseteq Q\}$	11
(5)	$\langle Y^4Z, 5cXY^4 + 10Y^3Z^2\rangle$	$(x,0)$, $(-5y,z)$, $(-5cz,x)$	$\{\lambda \in \mathbb{C} : \lambda x^2, \lambda xz, \lambda(xy - cz^2) \in Q\}$	12 or 11

In the first row $\langle a, b\rangle = \langle y, z\rangle$, since dependent a, b would give a fifth power in V; and $\{l : lS_1 \subseteq Q\} = 0$ by (Q1), as $lS_1 \subseteq Q \subseteq I(Y)_2$ forces $l \in N_Y$. In the last row $c \neq 0$, and V lies in $\nu_5(\Gamma_c)$ for the curvilinear scheme Γ_c of length three at Y tangent to the line, with $(I_{\Gamma_c})_2 = \langle x^2, xz, xy - cz^2\rangle$; so $\rho = 11$ if $Q \supseteq (I_{\Gamma_c})_2$, and $\rho = 12$ otherwise. There $g' = z^3$, so the multiplicities above do not apply. We now move N off the line. Each move keeps r and ρ constant, hence the dimension of the solutions of the closedness equations, so the given tails extend to the moving graded data; the general member has its square point off the line, where the case $k = 1$, N off the line, applies, and (Q2) concludes.

(i) *Type (3), no fifth power.* Fix the line $x = 0$ and the binary pencil V, with $r = 3$ and no fifth power; then $D_4 = S_1 \circ V \subseteq \bar{T}_4$ is fixed too. Move the square point: the pairs (N, Q), with $N \in \mathbb{P}^2$ and Q a hyperplane of $I(N)_2$ with $\dim S_1Q = 9$, form a nonempty open subset of a $\mathbb{P}^4$-bundle over $\mathbb{P}^2$, so an irreducible variety in which the pairs with N off the line are dense (its complement, $k = 2$, is another case), and $D_4 \subseteq \bar{T}_4 \subseteq D_3'$ for every pair. Carry the tails: the first row and its proof hold with $I(N)$ and N_N in place of $I(Y)$ and N_Y, so $\rho = 12$ on the whole variety, and (Q2) applies.

(ii) *Type (3), $Z^5 \in V$.* Let σ_t be the substitution $x \mapsto x - ty$, and put $N_t = Y + tX$ and $Q_t = \sigma_t(Q)$, a hyperplane of $\sigma_t(I(Y)_2) = I(N_t)_2$ with $\dim S_1Q_t = 9$; keep V and $D_4 = S_1 \circ V \subseteq \bar{T}_4 \subseteq D_3'$. The second row holds

for Q_t in place of Q, and $\{l : l\langle x,y\rangle \subseteq Q_t\} = \sigma_t(\{l : l\langle x,y\rangle \subseteq Q\})$, as σ_t preserves $\langle x,y\rangle$. So ρ is constant for all $t \in \mathbb{C}$, N_t is off the line for $t \neq 0$, and (Q2) applies.

(iii) *Type (4).* If $V = \langle Y^5, F\rangle$ is binary, we use the data $(N, Q, [F])$, with N and Q as in (i), $F \in \bar{T}_5$ and $V = \langle N^5, F\rangle$ of $r = 3$: an irreducible variety in which N off the line is dense, with $D_4 = S_1 \circ V \subseteq \bar{T}_4 + \mathbb{C}N^4 \subseteq D'_3$. If $V = \langle Y^5, XY^4 + G\rangle$, we use the curve $N_t = Y + tX$, $Q_t = \sigma_t(Q)$, $V_t = \langle N_t^5, (N_t^5 - Y^5)/5t + G\rangle$, which tends to V; here $S_1 \circ V_t \subseteq D'_3(t) = (xI(N_t)_3)^\perp$ for every t (for $t \neq 0$ this is $\bar{T}_4 \oplus \mathbb{C}N_t^4$), and $r = 3$ near $t = 0$. In both cases V contains N^5 for the square point N of Q. We show that then $\mathcal{K}_V \cong \{l : lN_N \subseteq Q\}$, of dimension one, so that $\rho = 11$ on the whole variety, respectively near $t = 0$, and (Q2) applies. Let $N_N = \langle m_1, m_2\rangle$. For $F_1 = N^5$, the elements $(v_1, 0)$ of Syz are those with $v_1 \in N_N$; as $\dim \mathrm{Syz} = 6 - r = 3$, Syz is spanned by $(m_1, 0)$, $(m_2, 0)$ and an element (α, v) with $v \neq 0$. The equations of $\mathcal{K}_V$ are $vw_3 = 0$ and $m_1w_1 + m_2w_2 + \alpha w_3 = 0$; so $w_3 = 0$, and $(w_1, w_2) = (m_2h, -m_1h)$ with $h \in S_2$; as $w_1, w_2 \in (x)$ and x divides at most one of m_1, m_2, $h = xl$ for a linear form l. These lie in W if and only if $lN_N \subseteq Q$. Write $Q = \ker\varphi$ for a linear form φ on $I(N)_2$. As $lm_1, lm_2 \in I(N)_2$, the conditions are the two functionals $l \mapsto \varphi(m_1l)$ and $l \mapsto \varphi(m_2l)$ on S_1, so the space has dimension at least one. If they were dependent, some $m = am_1 + bm_2 \neq 0$ would have $\varphi(mS_1) = 0$, that is, $mS_1 \subseteq Q$, against (Q1). So the dimension is exactly one (script `l21_136632_kernel`, with all parameters symbolic).

(iv) *Type (5) over* Γ_c. We use the curve of [16, proof of Theorem A.11], type (5). Let $P(s) = c(s^2 - t^2)X + Y + sZ$ and $\Gamma(t) = \{P(t), P(-t), P(2t)\}$: the points $P(\pm t) = Y \pm tZ$ lie on the line, $N_t = P(2t) = 3ct^2X + Y + 2tZ$ lies off it for $t \neq 0$, and $\Gamma(t)$ tends to Γ_c. Let V_t be spanned by the divided differences $f[-t,t]$ and $f[-t,t,2t]$ of $f(s) = P(s)^5$. Then $V_0 = \langle 5Y^4Z, 5cXY^4 + 10Y^3Z^2\rangle$, $D_4(t) = S_1 \circ V_t \subseteq \nu_4(\Gamma(t)) \subseteq D'_3(t)$ for every t, and $r = 3$ near $t = 0$. With $F_1 = f[-t,t]$ and $F_2 = f[-t,t,2t]$, a basis of $\mathrm{Syz}(V_t)$, polynomial in t and independent at $t = 0$, is

$$s_1 = (x, 0), \qquad s_2 = (-c(z + 2ty),\, x - 3ct^2y), \qquad s_3 = (-y,\, z - 2ty)$$

(script `l21_136632_line5`, part (K)). So ρ_t is the rank of a matrix polynomial in t, and $\rho_t \geq \rho_0$ near $t = 0$.

- If $Q \not\supseteq (I_{\Gamma_c})_2$, let $Q_t = \{q - q(N_t)y^2 : q \in Q\} \subseteq I(N_t)_2$, where $q(N_t)$ is the value of q at $(3ct^2, 1, 2t)$. Here $\rho_0 = 12 = 4\dim\mathrm{Syz}$ is the maximum, so $\rho_t = 12$ near $t = 0$.
- If $Q = (I_{\Gamma_c})_2 \oplus \mathbb{C}q_4$, let $Q_t = (I_{\Gamma(t)})_2 \oplus \mathbb{C}(q_4 - q_4(N_t)y^2)$, where $(I_{\Gamma(t)})_2 = \langle g_1, g_2, g_3\rangle$ with $g_1 = x(x - 3ct^2y)$, $g_2 = x(z - 2ty)$, $g_3 = xy - cz^2 + ct^2y^2$. These are the 2×2 minors of the matrix with rows s_1, s_2, s_3, on the rows $\{1,2\}$, $\{1,3\}$, $\{2,3\}$, so the two 3×3 determinants with a repeated column give $g_1s_3 - g_2s_2 + g_3s_1 = 0$, identically in t and c. Hence $k(t) = g_1 \otimes s_3 - g_2 \otimes s_2 + g_3 \otimes s_1$ is a

kernel vector of the multiplication map $(I_{\Gamma(t)})_2 \otimes \mathrm{Syz}(V_t) \to S_3^2$ with constant coefficients, and $k(0) \neq 0$. So $xk(t)$ is a nonzero element of $\mathcal{K}_{V_t}$, and near $t = 0$

$$11 = \rho_0 \leq \rho_t \leq 4 \dim \mathrm{Syz} - 1 = 11.$$

In both cases $\dim S_1 Q_t = 9$ near $t = 0$, so N_t is the square point of Q_t, and (Q2) applies along the curve.

These are all cases: type (3) is split by the fifth powers in V, of which there is at most one as $r = 3$, and type (5) by the three schemes of length three in [16, proof of Theorem A.11]. Script `l21_136632_moves`, part (M), checks for all parameters that ρ is constant along moves (i) and (ii); for move (iv) it certifies ρ_0 for every $c \neq 0$, and the syzygy basis and the kernel vector used above; move (iii) is `l21_136632_kernel`. So every algebra with $k = 1$ is smoothable.

$k = 2$: a count with V fixed. Here $W = x_\ell(I_\Gamma)_2$ for a line $\ell = V(x_\ell)$, and $D_3 = \nu_3(\mathfrak{B}_{x_\ell,\Gamma})$, with $\mathfrak{B}_{x_\ell,\Gamma} = V(x_\ell I_\Gamma)$ as in Theorem A.5, determines (ℓ, Γ); and as $(I_\Gamma)_4 = S_2(I_\Gamma)_2$, $S_2 \circ V \subseteq D_3$ means $x_\ell \circ V \subseteq \nu_4(\Gamma)$. These pairs have dimension $\lambda \leq 3$, or 4 if $V \subseteq \nu_5(\ell)$: if $\dim x_\ell \circ V = 2$, Γ is determined; if $x_\ell \circ V = \mathbb{C}g$, the $([F], \ell)$ with $x_\ell \circ F = 0$ have dimension at most one (V has at most two fifth powers), and $\Gamma \ni P$ if $g = P^4$, while otherwise Γ is unique, as $\dim \nu_4(\Gamma_1) \cap \nu_4(\Gamma_2) = \ell(\Gamma_1 \cap \Gamma_2) \leq 1$. Let $r = 3$, $U = S_1 \circ V$, $c = \dim S_1 \circ U$. Then $c \leq 4$: if $F \in V$ is concise and $b \circ G = \alpha(b) \circ F$ for the other generator, the 2×2 minors m_1, m_2, m_3 of $\left(\begin{smallmatrix} x & y & z \\ \alpha x & \alpha y & \alpha z \end{smallmatrix}\right)$ kill F, and they contain at least two independent quadrics: as $\sum_i x_i m_i = 0$ (a determinant with two equal rows), multiples $n_i q$ of one quadric q would give $q \sum_i n_i x_i = 0$, so all m_i would vanish and α would be scalar (script `l21_136632_k2count`, part (Mi)). Hence $c = \dim S_2 \circ F = 6 - \dim \mathrm{Ann}(F)_2 \leq 4$. If V is binary, $S_1 \circ U$ consists of binary cubics, so $c \leq 4$; otherwise two general members are binary with a common derivative L^4, hence $aL^5 + bL^4M$ (as $aL^5 + bM^5$ would give a concise combination), and $V = L^4\Lambda$, $c = 3$. The data $D \supseteq S_1 \circ U$, $\varphi \in \mathrm{Hom}(U, T_3/D)$, $\psi \in \mathrm{Hom}(V, T_4)$, $\chi \in \mathrm{Hom}(V, T_3/D)$ of $M = T_{\leq 2} \oplus D \oplus \{u + \varphi(u)\} \oplus \{v + \psi(v) + \chi(v)\}$ have dimension $4(6 - c) + 12 + 30 + 8$, with ψ raw, and closedness, $a \circ \psi(v) \equiv \varphi(a \circ v)$ modulo D, is 24 conditions: every component has dimension at least $50 - 4c \geq 34$. By duality, over fixed D the solutions have dimension $26 + \xi$, where $\xi = \dim \ker(W \otimes \mathrm{Syz} \to S_4^2) \leq 4$: this kernel lies in that of $W^3 \to S_4$, $(w_i) \mapsto \sum x_i w_i$, of dimension $12 - \dim S_1 W = 12 - 8$, if V has a concise member, and embeds in W otherwise. On the line of a binary $V = \langle F_1, F_2 \rangle$, that is, if $x_\ell \circ V = 0$, Syz is spanned by $x_\ell \otimes F_1$, $x_\ell \otimes F_2$ and some $a \otimes F_1 + b \otimes F_2$ with, say, $a \notin \mathbb{C}x_\ell$. A kernel element (w_1, w_2, w_3) then has $x_\ell w_1 + a w_3 = x_\ell w_2 + b w_3 = 0$ with $w_i \in W = x_\ell(I_\Gamma)_2$, so it is $(-a x_\ell l, -b x_\ell l, x_\ell^2 l)$ with $l\langle x_\ell, a, b \rangle \subseteq (I_\Gamma)_2$. These l form a proper subspace of S_1, as $\dim(x_\ell S_1 + a S_1) = 5 > 4 = \dim(I_\Gamma)_2$; so $\xi \leq 2$ there. So the $k = 2$ points have dimension at most $3 + 26 + 4 = 33$, or $4 + 26 + 2 = 32$ on the line of a binary V; both are less than 34.

For $r = 2$, $V = \nu_5(\Gamma')$ with $\ell(\Gamma') = 2$. The data $D/\nu_3(\Gamma') \in \mathrm{Gr}(4, 8)$, $[u]$ in a chart $\mathbb{A}^{12}$ of $\mathbb{P}(T_4/S_1 \circ V)$, φ, ψ and χ have dimensions 16, 12, 12, 30, 8, and $S_1 \circ u \subseteq D$ and closedness are $12 + 24$ conditions: every component has dimension at least $78 - 36 = 42$. At the $k = 2$ points, $[u] \in \mathbb{P}^4$ and $\xi \leq 4$. Indeed, for a point P of Γ' with $(I_P)_1 = \langle a, b\rangle$, the space $K_P = \{(w, w') \in W^2 : aw + bw' = 0\}$ consists of the pairs $(bx_\ell l, -ax_\ell l)$ with $l\langle a, b\rangle \subseteq (I_\Gamma)_2$, as x_ℓ does not divide both a and b; so $\dim K_P \leq 2$, as above. The kernel is $K_P \oplus K_Q$ if $\Gamma' = \{P, Q\}$, and it has dimension at most $2 \dim K_P$ if Γ' is a double point at P, where $V = \langle P^5, P^4P'\rangle$ for some P'. So these points have dimension at most $4 + 4 + 26 + 4 = 38 < 42$ (script `121_136632_k2count`). So the $k = 2$ points fill no component, and [15, Proposition 2.2(1)] concludes. □

Theorem A.13. *Every local algebra with Hilbert function* $(1, 3, 6, 7, 3, 1)$ *is smoothable.*

Proof. For each type of top form the algebras form one irreducible family with a dense linked subset. Let σ be the dimension of the linear syzygies of the net W: $\sigma \leq 3$, $\sigma = 3$ only for $W = qS_1$ ($7^{\langle 3\rangle} = 9$ and Gotzmann), and $\sigma \geq 2$ forces a common factor. Say W has $(\star)$ if $w_1, w_2, w_3 + L^5$ have local length at most 41 for a basis of W and some L; by Lemma 2.14, this open condition settles every algebra with $D_3 = W^\perp$. Script `121_136731`, part (C), checks the arithmetic of the counts below.

F concise. The tails, with R raw, are $15 + 3$, the same for all D_3; the pairs $([F], D_3)$, $S_2 \circ F \subseteq D_3$, have dimension at least $20 + 21 - 18 = 23$, attained where $h = \dim S_2 \circ F = 6$ $(20 + 3)$, while $h = 5, 4, 3$ give at most $15 + 6 = 21$ (a quadric kills F), $11 + 9 = 20$ (a pencil), $8 + 12 = 20$ ($F \in \nu_5(\Gamma)$). The member $F = XYZ(XY + XZ + YZ)$, $W = \langle x^3, y^3, z^3\rangle$, zero tails lies in $V(x^3, y^3, z^3)$, of length 27.

F binary. With Z as in Theorem A.11, $W \subseteq (I_Z)_3$, $D_4 = S_1 \circ F \oplus \mathbb{C}u$, and the tails, with R raw, are 18 for all W and u; the bound $12 + 12 - 9 = 15$, respectively $15 + 12 - 9 = 18$, is attained over $\sigma = 0$. Over $\sigma \geq 1$ the fibre $\mathbb{P}(T_4(W)/S_1 \circ F)$ has dimension $3 + \sigma \leq 6$, and the nets have dimension at most 8, respectively 10, by the rank of the coefficients of a syzygy; put $H = (I_Z)_3$, of dimension 7, respectively 8. If they span S_1, a change of basis of W makes the syzygy $xw_1 + yw_2 + zw_3 = 0$, so (w_1, w_2, w_3) lies in the kernel of $H^3 \to (I_Z)_4$, of dimension $3 \cdot 7 - 12 = 9$, respectively $3 \cdot 8 - 13 = 11$ (the map is onto, as I_Z is generated in degree at most three): projectively 8, respectively 10. If they span a plane A, then $qA \subseteq W$ for a quadric q with $A \circ (q \circ F) = 0$. For $q \circ F = 0$, the point $[q] \in \mathbb{P}(\mathrm{Ann}(F)_2)$, the plane $A \in \mathrm{Gr}(2, S_1)$ and the third element of W in $\mathbb{P}(H/qA)$ give $2 + 2 + 4 = 8$, respectively $3 + 2 + 5 = 10$. For $q \circ F \neq 0$, $q \circ F$ is one of the finitely many cubes in $S_2 \circ F$ (a line or a plane meets the twisted cubic in finitely many points), which fixes A and confines $[q]$ to a space of dimension 3, respectively 4: one less. A single coefficient is impossible, as W is a net. So the syzygetic nets contribute at most $8 + 6 = 14 < 15$, respectively $10 + 6 = 16 < 18$. For

each of the five types of Z some net in $(I_Z)_3$ has $(\star)$, with 33 or 31 (part (A1)).

$F = X^5$. The tails jump ($18 + s'$ with R raw, $0 \le s' \le 2$), so we count all data, with R raw: $W \in \mathrm{Gr}(3, 9)$, $D_4/X^4 \in \mathrm{Gr}(2, 14)$, τ, R, E give $18 + 24 + 9 + 15 + 3 = 69$, with $18 + 9$ equations: the bound 42 is attained by $\sigma = 0$ ($18 + 6 + 18$), while $\sigma = 1, 2, 3$ give at most $12 + 8 + 20 = 40$, $10 + 10 + 20 = 40$, $4 + 12 + 20 = 36$. Some $W \subseteq I(p)_3$ has $(\star)$, with 29 (part (A1)). □

Theorem A.14. *Every local algebra with Hilbert function* $(1, 3, 6, 7, 2, 2)$ *is smoothable.*

Proof. We repeat [16, Theorem A.15] with one more cubic. By [15, Lemma 2.16], $D_k = \nu_k(\Gamma)$, $k = 4, 5$, $\ell(\Gamma) = 2$. The data Γ, $D_3/\nu_3(\Gamma)$, R_1, R_2 modulo D_4, E_1, E_2 have dimension $4+15+26+6 = 51$, and the four columns $\sum_k v_k \circ R_k$ of [15, Theorem C.10] lie in D_3: components have dimension at least $51 - 12 = 39$. With the bounds of [15, Theorem C.10] for (R_1, R_2) and $D_3/\nu_3(\Gamma) \in \mathrm{Gr}(5 - \varrho, 8 - \varrho)$, the strata of column rank $\varrho = 3, 2, 1, 0$ have dimension at most $4+21+6+6 = 37$, $4+15+9+6 = 34$, $4+10+12+6 = 32$, $4 + 4 + 15 + 6 = 29$, and $\varrho = 4$ gives an irreducible open set of dimension $4+26+3+6 = 39$ over reduced Γ, $3+26+3+6 = 38$ over double points. The anchor M_0 of [15, Theorem C.10], of length 19, lies in a complete intersection of local length 32. The cubics $G = XZ^2 - XY^2$ and $G' = XYZ - \frac{1}{2}XY^2$ are linearly independent modulo $D_3(M_0)$ and orthogonal to the cubic parts of its three equations, so adding them (Section 2.7, twice) keeps the complete intersection and raises the length to 21: a member (script `121_136722`). □

Theorem A.15. *Every local algebra with Hilbert function* $(1, 3, 6, 8, 2, 1)$ *is smoothable.*

Proof. As $S_1 \circ F \subseteq D_4$ for $D_5 = \mathbb{C}F$, F is a power or binary. If $F = X^5$, we follow [16, Theorem A.13]: with $D_4 = \langle X^4, Q \rangle$ and $\Phi = X^5 + R + E$, the space $C_0 = S_1 \circ D_4 + \langle y \circ R, z \circ R \rangle \subseteq D_3$ has dimension at most $1 + 3 + 2$, so $M = M' \oplus \langle G, G' \rangle$ with M' of Hilbert function $(1, 3, 6, 6, 2, 1)$, and the proof of [15, Theorem E.12], case $r = 1$, with [15, Lemma A.14] gives rays on a dense subset of a family through A.

Let $F \in \mathbb{C}[X, Y]_5$. Then $D_4 = S_1 \circ F$, $W \subseteq \mathrm{Ann}(F)_3$, closedness determines τ by R and says $R \perp zW$, and E is free; so the algebras with this F form a vector bundle over $\mathrm{Gr}(2, \mathrm{Ann}(F)_3)$, with coprime pencils dense ($g + z^3$ and zl^2 for a binary cubic $g \in \mathrm{Ann}(F)_3$ and $l \nmid g$). Integrating $v \circ F = P^4$ shows that a derivative of F is a fourth power only for $F = X^5 + Y^5$ or X^4Y in suitable coordinates. Otherwise, for coprime W a general quartic of I_4^* vanishes at no point of $\mathfrak{B}$, by (1), and [15, Lemma 3.3] gives $3 \cdot 3 \cdot 4 = 36$. For $F = X^5 + Y^5$ or X^4Y, $\mathrm{Ann}(F)_3 = J_3$ with $J = (z, xy)$ or (z, y^2), and the member with coprime $W \subseteq J_3$ and zero tails contains $w_1, w_2, q_4 + l^6$ for $q_4 \in J_4$, $l \in S_1$, of local length 40 (script `121_136821`, parts (A1), (A2)). □

Appendix B. Socle degree at least six: the remaining cases

B.1. This appendix proves the eighteen Hilbert functions of Section 7 other than $(1,3,6,3,3,2,1,1,1)$. Here $H_A(2)=6$, the socle degree s is at least six, and, as everywhere in the appendices, the length is twenty-one and all shorter schemes are smoothable. The first twelve Hilbert functions are settled by excess linkage, free quartic tails, rays and families with anchors, with linkage at infinity on a few strata. In the last six, from Section B.19 on, the work is concentrated in a few graded data over a point $[X]$ of the highest weight $s-3$, or over a line through it.

Two facts are used throughout. A coordinate change $v\mapsto v+q$, where v is a variable and q a form of degree at least two in the other variables, acts on inverse systems by $\exp(V\,q(\partial))$, where V is the variable dual to v; it keeps the graded data [16, Lemma B.4]. And when the cubic base scheme is a line $z=0$ and $D_3=\mathbb{C}[X,Y]_3$, every D_k with $k\geq 3$ consists of binary forms [16, proof of Proposition 3.6].

B.2. Three lemmas are used only here. The first lets a point leave along a line when H_A ends with a long tail of ones, one degree earlier than [15, Proposition 2.12(2)].

Lemma B.3. *Let A be a local algebra with $H_A(2)=6$ and socle degree s, and let $k_0\geq 3$ with $H_A(k)=1$ for $k_0<k\leq s$ and $s\geq 2k_0-1$. Then (y,z) is a ray of I in suitable coordinates.*

Proof. As $s\geq 2k_0-1\geq k_0+2$ (since $k_0\geq 3$), take $D_k=\mathbb{C}X^k$ for $k_0<k\leq s$, and let $\Phi=X^s+\Phi_{s-1}+\cdots$ be the lift of X^s, so that $M=(M\cap T_{\leq k_0})\oplus\langle x^j\circ\Phi : j<s-k_0\rangle$. Going down from $j=s-1$ to k_0+2, $y\circ\Phi_j, z\circ\Phi_j\in\mathbb{C}X^{j-1}$, and a gauge by $x^{s-j}\circ\Phi$ and the shears $y\mapsto y+ax^{s-j+1}$, $z\mapsto z+bx^{s-j+1}$ [15, Section 5.17] make $\Phi_j=0$; shears with exponent $s-k_0$ remove $X^{k_0}Y$, $X^{k_0}Z$ from Φ_{k_0+1}. Then $x^{s-k_0}\circ\Phi\in\frac{s!}{k_0!}X^{k_0}+T_{\leq 2k_0+1-s}$ with $2k_0+1-s\leq 2$, so $X^{k_0}\in M$; and $x^{k_0}y$, $x^{k_0}z$ kill $M\cap T_{\leq k_0}$ by degree and each $x^j\circ\Phi$, as the coefficient of $X^{k_0+j}Y$ in Φ_{k_0+j+1} vanishes. [15, Lemma 5.4] concludes. □

The next two adapt [15, Corollary D.4] to the top X^6Y and [15, Proposition C.6(a)] to two points.

Lemma B.4. *Let A be a local algebra of length $d\geq 20$ with $H_A(2)=6$, socle degree seven, $D_7=\mathbb{C}X^6Y$, $D_3=X^2T_1$ and $\mathfrak{B}=V(\mathfrak{m}_X^2)$. Then A is smoothable.*

Proof. Run the proof of [15, Lemma D.3] with $y\circ\Phi$, x^6, x^6y in place of $u\circ\Phi$, u^{s-1}, u^s and with $R=Y^4$, as [15, Corollary D.4] does with $z\circ\Phi$, x^5, x^5z; its space M'' contains $x\circ\Phi$, $z\circ\Phi$, and $x\circ R=z\circ R=0$. Then θ_i is a multiple of the coefficient of y^3 in $w_i\in(y,z)^2S_1$, nonzero as $y^3\in I_3^*$, and the cubics without y^3 have all binary quadrics as quadratic parts at $[X]$; so Steps 2 and 3 of [15, Theorem 3.12] with $q_6=x^6$ [15, Remark D.2] give $27+(6-3)\cdot 4=39<2d$. □

Lemma B.5. *Let A be a local algebra with $H_A(2) = 6$, socle degree seven, $D_7 = \mathbb{C}F$ with $F \in \langle X^7, Y^7\rangle$ and $\langle x^7, F\rangle \neq 0$, $D_6 \subseteq \langle X^6, Y^6\rangle$, $D_5 \subseteq \langle X^5, Y^5\rangle$ and $X^4 \in D_4 \subseteq \langle X^4, Y^4\rangle$. Then I has a ray along a curve tangent to the line of X.*

Proof. The proof of [15, Proposition C.6(a)] applies to $F = X^7$ or $X^7 + Y^7$, with $N = M_{\leq 5} \oplus \mathbb{C}\Psi$ if $Y^6 \in D_6$, where $\Psi \in M$ has part of degree six Y^6 ($\Psi = y \circ \Phi/7$ if $F = X^7 + Y^7$). As $x \circ \Psi$, $z \circ \Psi$ have degree at most four, the quintic part V_5 of Ψ lies in $\langle X^5, XY^4, Y^5, Y^4Z\rangle$, so no element of N contains X^4Y or X^4Z; and as $y \circ R_6 \in D_5$ or $y \circ R_6 = 7V_5$ for the sextic tail R_6 of F, the monomials of R_6 divisible by X^3 are still among X^6, X^5Y, X^5Z. □

B.6. Over the double point $\Gamma_0 = V(z, y^2)$ at $[X]$ let $E_0 = \nu_3(\Gamma_0) = \langle X^3, X^2Y\rangle$, and for $D \supseteq E_0$ let $\mathcal{A}_D = \{A \in T_4 : y \circ A, z \circ A \in D\}$. As $A \mapsto (y \circ A, z \circ A)$ has kernel $\mathbb{C}X^4$ and image $\{(u, v) \in D^2 : z \circ u = y \circ v\}$, $\dim \mathcal{A}_D = 1 + 2\dim D - \operatorname{rank} \beta_D$ for $\beta_D(u, v) = z \circ u - y \circ v$; and $\operatorname{rank} \beta_D \geq 3$ if $D \supseteq \nu_3(\Lambda)$ for a line Λ, as $y\circ$ or $z\circ$ maps $\nu_3(\Lambda)$ onto $\nu_2(\Lambda)$, or if $D \supseteq \nu_3(C_4)$ for a curvilinear $C_4 \supseteq \Gamma_0$ of length four (script `l21_1366221`, part (B)). If $D = X^2T_1 \oplus E$ with E spanned by forms $Xg_2 + g_3$, g_i binary, then $X^2Q + Xa_3 + a_4 \in \mathcal{A}_D$ if and only if $\partial_Y(a_3, a_4), \partial_Z(a_3, a_4) \in E$, and a binary form of positive degree is determined by its gradient; so $\dim \mathcal{A}_D = 6 + 2\dim E - r_E$ with $r_E = \dim(\partial_Z E + \partial_Y E)$.

Theorem B.7. *Every local algebra with Hilbert function $(1, 3, 6, 3, 3, 3, 1, 1)$ is smoothable.*

Proof. By [15, Lemma 2.16], $D_k = \nu_k(\Gamma)$ for $k = 3, 4, 5$, where $\Gamma = \mathfrak{B}$ has length three; $D_7 = \mathbb{C}X^7$, $D_6 = \mathbb{C}X^6$ and $[X] \in \Gamma$. If Γ is curvilinear, [15, Corollary 3.15] gives $27 + 3 \cdot 4 = 39$. Otherwise $\Gamma = V(\mathfrak{m}_X^2)$, and [15, Lemma D.3] applies by the remark after it: $27 + (6 - 3) \cdot 4 = 39$. □

Theorem B.8. *Every local algebra with Hilbert function $(1, 3, 6, 3, 3, 2, 2, 1)$ is smoothable.*

Proof. By [15, Lemma 2.16], $D_k = \nu_k(\Gamma)$ for $k = 3, 4$ with $\Gamma = \mathfrak{B}$ of length three, and $D_k = \nu_k(\Gamma')$ for $k = 5, 6$ with $\Gamma' \subseteq \Gamma$ of length two. The weights are 2 off Γ' and at most 4 on it, so a curvilinear Γ gives $27 + 4 \cdot 3 = 39$. If $\Gamma = V(\mathfrak{m}_X^2)$, then $D_7 = \mathbb{C}F$ with $F \in \nu_7(\Gamma')$, so $F = X^7$, where [15, Lemma D.3] gives 39, or $F = X^6Y$, where Lemma B.4 applies. □

Theorem B.9. *Every local algebra with Hilbert function $(1, 3, 6, 3, 2, 2, 2, 2)$ is smoothable.*

Proof. By [15, Lemma 2.16], $D_k = \nu_k(\Gamma)$ for $4 \leq k \leq 7$ with $\ell(\Gamma) = 2$. As $\dim I_3^* = 7 > 6$, $\mathfrak{B} \supseteq \Gamma$ is finite, of length at most $3^{\langle 3\rangle} = 3$. [15, Corollary 3.14] with $c = 3$, $s = 7$ gives $2s + c + 19 = 36$ if $\mathfrak{B} = \Gamma$ near Γ, and $3s + c + 15 = 39$ if $\mathfrak{B}$ is curvilinear of length three near Γ. There remains one graded datum: $\mathfrak{B} = V(\mathfrak{m}_X^2)$, $D_3 = X^2T_1$ and $D_k = \langle X^k, X^{k-1}Y\rangle$ for $4 \leq k \leq 7$. We apply Proposition 3.7 at $v = [X]$, the only point of $\mathfrak{B}$. Here

$\mathfrak{b}_3 = (\eta, \zeta)^2$ and $\mathfrak{b}_k = (\zeta, \eta^2)$ for $k = 4, 5, 6$, so $K_v(L_{\mathrm{def}}) = ((\eta, \zeta)^2, \rho\zeta, \rho^4)$. This ideal contains η^2, $\zeta(\rho + \eta)$, $\zeta^2 + \rho^4$, of colength 12: on the curve $\eta^2 = \zeta^2 + \rho^4 = 0$ the product $\zeta(\rho + \eta)$ of two parameters cuts out 8 on $\zeta = 0$ and 4 on $\rho = -\eta$. So $e(K_v(L_{\mathrm{def}})) \leq 12$, part (a) gives $e(L_{\mathrm{def}}) \leq 27 + 12 = 39 < 42$, and part (c) applies (script `l21_13632222_fat_param`, parts (K), (J), (W), (T)). In fact the three elements generate a reduction of $K_v(L_{\mathrm{def}})$, so $e(L_{\mathrm{def}}) = 39$ (part (E)). □

Theorem B.10. *Every local algebra A with $H_A = (1, 3, 6, 3, 2, 2, 2, 1, 1)$ is smoothable.*

Proof. By [15, Lemma 2.16], $D_k = \nu_k(\Gamma)$ for $k = 4, 5, 6$ with $\ell(\Gamma) = 2$, $D_8 = \mathbb{C}X^8$, $D_7 = \mathbb{C}X^7$, $[X] \in \Gamma$, and $\mathfrak{B} \supseteq \Gamma$ is finite of length at most three, as in Theorem B.9. The weights are 5 at $[X]$, 4 at the other point P of Γ, as $P^7 \notin D_7$, and 1 off Γ. For $\Gamma = \{X, P\}$ this gives at most $27 + 5 + 4 + 1 = 37$, $27 + 10 + 4 = 41$ or $27 + 5 + 8 = 40$, and for $\Gamma = \Gamma_0$ with $\mathfrak{B}$ of length two at $[X]$, at most $27 + 10 + 1 = 38$. There remain three graded data, $D_3 = \nu_3(\mathfrak{B})$ for $\mathfrak{B} = V(z, y^3)$, $V(xz - y^2, yz, z^2)$, $V(\mathfrak{m}_X^2)$. By Corollary 3.4 and (5) they are settled by

$$(x^2z,\ y^3 + x^8,\ z^3 + x^2y^2), \qquad (x^2z - xy^2,\ y^3 + x^8,\ z^3 + x^2y^2),$$
$$(xy^2 + x^6y,\ xz^2 + y^3 + x^8,\ xyz + z^3 + x^3z),$$

of local length 40, 40, 41 (script `l21_136322211`, part (A)). □

Theorem B.11. *Every local algebra A with $H_A = (1, 3, 6, 3, 2, 2, 1, 1, 1, 1)$ is smoothable.*

Proof. Lemma B.3 applies with $k_0 = 5$ and $s = 9 = 2k_0 - 1$. □

Theorem B.12. *Every local algebra A with $H_A = (1, 3, 6, 4, 2, 2, 1, 1, 1)$ is smoothable.*

Proof. We repeat [16, Theorem 7.1] with $D_3 = E_0 \oplus K$, $\dim K = 2$. As there, by a closedness computation independent of D_3, the line of X is a ray unless $\Gamma = \Gamma_0$ and the lift of X^8 has sextic part δX^4Y^2, $\delta \neq 0$; then M is closed if and only if $C_5 = c_1X^3YZ + c_2X^3Y^2 + c_3X^2Y^3 + c_4XY^4 + c_5Y^5$ and two cubics, $\rho_1 = (6c_4XY^2 + 10c_5Y^3)/\delta$ and one with linear part $z \circ H_4$ in the quartic tail H_4, lie in D_3 modulo E_0. As $z\circ\colon T_4 \to T_3$ is onto, the algebras form two irreducible families: $\rho_1 = 0$, $K \in \mathrm{Gr}(2, 8)$; and $0 \neq \rho_1 \in K$, $K/\mathbb{C}\rho_1 \in \mathbb{P}^6$. With weights 5 at $[X]$ and 1 elsewhere, $\mathfrak{B}$ of length two at $[X]$ gives $27 + 5 \cdot 2 + 2 = 39$ [15, Lemma A.11]. Length three at $[X]$ needs K to meet $\langle XY^2, X^2Z\rangle$ modulo E_0, which contains ν_3 of every length-three scheme at $[X]$ containing Γ_0: a proper closed condition in both families (in the second for $c_5 \neq 0$, as ρ_1 then contains Y^3). General members are linked. □

Theorem B.13. *Every local algebra with Hilbert function $(1, 3, 6, 4, 5, 1, 1)$ is smoothable.*

Proof. As $H_A(4) = 5 = 4^{\langle 3\rangle}$, Gotzmann's theorem gives $D_3 = \mathbb{C}[X,Y]_3$, $D_4 = \mathbb{C}[X,Y]_4$ in suitable coordinates [15, proof of Lemma A.11], and $D_k = \mathbb{C}X^k$ for $k = 5, 6$. For the lift $\Phi = X^6 + P + Q + R$, $y \circ P, z \circ P \in D_4$ give $P = P_{\mathrm{b}} + cX^4Z$ with P_{b} binary, and a gauge and shears with exponent two remove X^5, X^4Y, X^4Z from P. Then $x^2 \circ \Phi/30 \in X^4 + D_3 + T_{\leq 2}$, so $X^4 \in M$, while the quartic parts of $\langle y, z\rangle \circ M$ lie in $\mathbb{C}\, y \circ P \not\ni X^4$: a ray [15, Theorem 2.11(2)]. □

Theorem B.14. *Every local algebra with Hilbert function* $(1, 3, 6, 6, 3, 1, 1)$ *is smoothable.*

Proof. Rays settle dense subsets of a few incidences; strata of exactly the expected dimension are linked by excess, and one family by an anchor. Here $D_k = \mathbb{C}X^k$ for $k = 5, 6$ and $X^4 \in U = D_4$; as in [16, Theorem C.13], a gauge and shears make the quintic tail A of $\Phi = X^6 + A + B + C$ free of X^5, X^4Y, X^4Z (a space $\mathcal{A}$ of dimension 18), and then $A_Y = y \circ A$, $A_Z = z \circ A$ lie in U without X^4; let $r = \dim\langle A_Y, A_Z\rangle$. Two rays are used.

(R1) If $X^4 \in M$, the line of X is a ray [15, Theorem 2.11(2)], as the quartic parts of $\langle y, z\rangle \circ M$ lie in $\langle A_Y, A_Z\rangle \not\ni X^4$.

(R2) Let $r \geq 1$, with $A_Z = 0$ if $r = 1$. Suppose that a cubic g with $\langle g, X^3\rangle \neq 0$ kills $y \circ U + z \circ U + \langle A_{XY}, A_{XZ}\rangle$, and also B_Z if $r = 1$. Then some quartic p has $\langle p, A_Y\rangle = -\langle g, B_Y\rangle$ and $\langle p, A_Z\rangle = -\langle g, B_Z\rangle$, as A_Y, A_Z are independent if $r = 2$, and $A_Y \neq 0 = A_Z$ if $r = 1$; and $u = g + p$ has $yu, zu \in I$. Indeed, $g \circ A_Y = 0$, as g kills $S_1 \circ A_Y = \langle A_{XY}, A_{YY}, A_{YZ}\rangle$, so $(yu) \circ \Phi = g \circ A_Y + \langle g, B_Y\rangle + \langle p, A_Y\rangle = 0$; yu kills the lift of R if $r = 1$, as g kills $y \circ U$, and it kills $D_3 + T_{\leq 2}$ by degree; the same holds for zu. As $X^3 \in M$ is killed by y and z and $\langle u, X^3\rangle \neq 0$, we get $u \notin I + (y, z)$, and (y, z) is a ray (4).

If $r = 0$, then $A = 0$, $x^2 \circ \Phi/30 \in X^4 + T_{\leq 2}$, and (R1) applies.

$r = 2$. Closedness says $V = S_1 \circ U = \mathbb{C}X^3 + K \subseteq D_3$, $K = \langle A_{XY}, A_{XZ}, A_{YY}, A_{YZ}, A_{ZZ}\rangle$; B, C are free. The pairs (A, D_3) satisfy 20 conditions on $\mathcal{A} \times \mathrm{Gr}(5, 9)$, so the bound is $18 + 20 - 20 = 18$. Let $\rho = \dim V - 1$; $\rho \geq 2$, as $\dim V \leq 2$ would give $\dim U \leq \dim V$ by Macaulay's bound. If $\rho = 5$, then $\dim K = 5$, so $X^3 \notin K = y \circ U + z \circ U + \langle A_{XY}, A_{XZ}\rangle$, and (R2) holds. If $\rho = 2$, $V = \nu_3(\Gamma)$ with $\ell(\Gamma) = 3$ [15, Lemma 2.16], and a general $D_3 \supseteq V$ has base scheme Γ [15, Lemma A.16]; as $e(\Gamma) \leq 4$ and every weight is at most $s - 3 = 3$, Theorem 2.6 gives $27 + 3 \cdot 4 = 39$. The stratum $\rho = 2$ is closed, a $\mathrm{Gr}(3, 7)$-bundle over its image, so these D_3 are dense in it. If $\rho \leq 4$, respectively $\rho \leq 3$, then $\Pi \circ A \subseteq \mathbb{C}X^3$ for a subspace Π of dimension one, respectively two, of the space $L = \langle xy, xz, y^2, yz, z^2\rangle$ of quadrics without x^2. As $h \circ A \in \mathbb{C}X^3$ if and only if $A \perp hM_3$, where $M_3 = (y, z)S_2$, and as the quintics of $(y, z)^2$, which contain ΠM_3, pair perfectly with $\mathcal{A}$, these A form a space of dimension $18 - \dim \Pi M_3$: 9 for one quadric, and 3, 6 or 5 for a pencil Π without common factor, of the form $l\langle y, z\rangle$, or $l\langle p_1, p_2\rangle$ otherwise (then $l \in \langle y, z\rangle$). With the dimensions 4 of $\mathbb{P}(L)$ and 6, ≤ 2, ≤ 3 of these pencils, and the fibres $\mathrm{Gr}(1, 5)$ and $\mathrm{Gr}(2, 6)$ of D_3,

the strata $\rho = 4, 3$ have dimension at most $4 + 9 + 4 = 17$ and $9 + 8 = 17$, less than 18 (script `l21_1366311`, part (3b)). So every component has a dense subset with $\rho = 5$ or $\rho = 2$.

$r = 1$. After a change fixing X, A is binary and $A_Z = 0$; let $U_0 = \langle X^4, A_Y\rangle$, $A_0 = S_1 \circ U_0$ of dimension $d_0 \in \{2, 3\}$, and $U = U_0 \oplus \mathbb{C}R$. Closedness says $A_0 + \partial R + \mathbb{C}B_Z \subseteq D_3$, where $\partial R = S_1 \circ R$; C and the cubic tail of the lift of R are free. For fixed A, (R, D_3) satisfy 12 conditions in $(T_4 \setminus U_0) \times \mathrm{Gr}(6 - d_0, 10 - d_0)$: the bound is $15 + 4(6 - d_0) - 12 = 27 - 4d_0$. Let $q = \dim(A_0 + \partial R)/A_0$, so that $D_3/(A_0 + \partial R) \in \mathrm{Gr}(6 - d_0 - q, 10 - d_0 - q)$. If $q \leq 2$, $l \circ R \in A_0$ for some $l \neq 0$ ($d_0 + 7$ parameters). If $q \leq 1$, $P \circ R \subseteq A_0$ for a plane $P = L_0^\perp$, so $R \in \{R' : S_1 \circ R' \subseteq A_0\} + \mathbb{C}L_0^4$ if $L_0 \notin \langle X, Y\rangle$ ($d_0 + 3$ in all, with L_0); if $L_0 \in \langle X, Y\rangle$, R moves in dimension at most $d_0 + 2$. So the strata $q = 2$ and $q = 1$ have dimension at most $(d_0 + 7) + 4(4 - d_0)$ and $(d_0 + 3) + 4(5 - d_0)$, both equal to $23 - 3d_0 < 27 - 4d_0$. The stratum $q = 3$ is a $\mathrm{Gr}(3 - d_0, 7 - d_0)$-bundle over an open subset of T_4, irreducible of dimension $27 - 4d_0$.

For $d_0 = 3$, every component has its general point in $q = 3$ or $q = 0$. If $q = 3$, then $X^3 \notin S_0 = \langle A_{XY}, A_{YY}, R_Y, R_Z\rangle = y \circ U + z \circ U + \langle A_{XY}, A_{XZ}\rangle$, as a relation would put a combination of R_Y, R_Z in A_0, while $X^3 \notin \langle A_{XY}, A_{YY}\rangle$; so (R2) holds where $B_Z \notin S_0 + \mathbb{C}X^3$, a dense condition, as B_Z runs over D_3. If $q = 0$, then $U = \nu_4(\Gamma)$ for a collinear $\Gamma \ni [X]$ of length three [15, Lemma 2.16], a stratum of the expected dimension 15; a general $D_3 \supseteq \nu_3(\Gamma)$ has base scheme Γ [15, Lemma A.16], and Theorem 2.6 gives $27 + 3 \cdot 3 = 36$.

For $d_0 = 2$, $A = Y^5$ or X^3Y^2 (part (2)); then $q \geq 1$, as $R \notin U_0 = \{R' : S_1 \circ R' \subseteq A_0\}$ [15, Lemma 2.16], so the incidence of (R, D_3) is irreducible [15, Lemma 2.18]: its stratum $q = 3$ is a $\mathrm{Gr}(1, 5)$-bundle over an open subset of T_4, of the bound 19, and the others lie below it. For Y^5, $S_0 = \langle Y^3, R_Y, R_Z\rangle$ does not contain X^3 when $q = 3$, by the argument for $d_0 = 3$, so (R2) holds on the dense subset where $q = 3$ and $B_Z \notin \langle Y^3, R_Y, R_Z, X^3\rangle$. For X^3Y^2, $X^3 = A_{YY}/2$ lies in $y \circ U$, so (R2) is not available; the algebras form one irreducible family, a vector bundle over the incidence, with the member $R = XZ^3 - \frac{3}{2}X^2Y^2$, $B = C = 0$, zero tail on R and $D_3 = \langle X^3, X^2Y, XYZ, XZ^2, Y^2Z, 3XY^2 - Z^3\rangle$, whose ideal contains x^2z, $xy^2 + z^3 - x^4/60$, $y^3 + yz^2$, of local length $2 \cdot 9 + 12 = 30$: on $x = 0$, z^3 and $y(y^2 + z^2)$ meet in 9; on $z = 0$, $x(y^2 - x^3/60)$ and y^3 meet in $3 + 3 \cdot 3$ (part (5)). This member is a linkage anchor of the family.

In every case the smoothable algebras are dense in each component of the families above, and [15, Proposition 2.2(1)] concludes. □

Theorem B.15. *Every local algebra with Hilbert function* $(1, 3, 6, 6, 2, 2, 1)$ *is smoothable.*

Proof. We repeat [16, Theorem C.10] with one more cubic: $D_k = \nu_k(\Gamma)$, $k = 4, 5$, $\ell(\Gamma) = 2$, $D_6 = \mathbb{C}F$. If $\Gamma = \{X, Y\}$ and $F = X^6$, [15, Proposition C.6(b)] gives a ray. If F is not a power, the general member of the family of [16, Theorem C.10], now over $\mathrm{Gr}(6 - n, T_3/C)$, has a curvilinear base scheme of

length $n \le 3$ [15, Lemma A.16]: $27 + 3n \le 36$. Let $F = X^6$, $\Gamma = \Gamma_0$. The data of [16, Theorem C.10] have dimension $16 + 4 + 30 + 20 = 70$, with $4 \cdot 4$ conditions: the bound is 54, and the part over a set Σ of flags has dimension at most $\dim \Sigma + 4 + 2 \dim \mathcal{A}_D + 20$. By [16, proof of Theorem A.14, case $F = X^5$], $\mathfrak{B}$ is finite of length at most six or contains a line. With weights $3e_X$ and 1, good flags have $\mathfrak{B}$ finite with $\mathfrak{B}_X$ curvilinear of length 2, 3, 4: $27 + 6 + 5 = 38$, $27 + 9 + 4 = 40$, $27 + 12 + 2 = 41$ [16, Lemma 3.2]. The bad flags have $\dim \Sigma + 2 \dim \mathcal{A}_D \le 29$: $\mathfrak{B}$ containing a line $(8 + 2 \cdot 10)$; $\mathfrak{B}_X$ curvilinear of length at least five, $D_3 = \nu_3(C_5) \oplus \mathbb{C}G$ $(7+20)$; $D_3 = X^2 T_1 \oplus E$, $E \in \mathrm{Gr}(3, 7)$, $\dim \mathcal{A}_D = 12 - r_E$: $12 + 16$ if $r_E \ge 4$; $11 + 18$ if $r_E = 3$, a proper closed condition (script `121_1366221`, part (B)); $6 + 20$ if $r_E \le 2$, as E then meets the pencils of pairs killed by ∂_Z and by ∂_Y. The bad part has dimension at most $29 + 24 = 53 < 54$, and [16, Lemma C.6] concludes. □

Theorem B.16. *Every local algebra with Hilbert function* $(1, 3, 6, 5, 2, 2, 2)$ *is smoothable.*

Proof. We repeat [16, Theorem C.8] with one more cubic. If $\Gamma = \{X, Y\}$, [15, Proposition C.6(b)] gives a ray. Over Γ_0 the data of [16, Theorem C.8] have dimension $15 + 8 + 30 + 20 = 73$, with $4 \cdot 5$ conditions: the bound is 53, and the part over Σ has dimension at most $\dim \Sigma + 8 + 2 \dim \mathcal{A}_D + 20$. With weights $3e_X$ and 1 and [15, Lemma A.12], good flags have $\mathfrak{B}_X$ equal to Γ_0 $(27 + 6 + 4 = 37)$, curvilinear of length three $(27 + 9 + 2 = 38)$, $V(\mathfrak{m}_X^2)$ $(27 + 12 + 2 = 41)$, or curvilinear or a complete intersection of colength four $(27 + 12 + 1 = 40)$. Bad ones have $\dim \Sigma + 2 \dim \mathcal{A}_D \le 24$: $\mathfrak{B}_X$ curvilinear of length five $(3 + 16)$; $\mathfrak{B}_X \supseteq V(\mathfrak{m}_X^3 + \text{a pencil of quadrics})$, $D_3 = X^2 T_1 \oplus \mathbb{C}X g_2 \oplus \mathbb{C}G$ $(7+16)$; $\mathfrak{B}$ containing a line $(5+16)$. The bad part has dimension at most $24 + 28 = 52 < 53$; [16, Lemma C.6] concludes. □

Theorem B.17. *Every local algebra with Hilbert function* $(1, 3, 6, 5, 2, 2, 1, 1)$ *is smoothable.*

Proof. We repeat [16, Theorem C.9] with one more cubic. If $\Gamma = \{X, Y\}$, [15, Proposition C.6(a)] gives a ray. Over Γ_0, if the coefficient δ of X^4Y^2 in the lift of X^7 is not zero, the two families of [16, Theorem C.9] do not involve D_3, and its anchors, of local length 32 and 33 with cubic parts orthogonal to XYZ, give members of ours when XYZ is added (Section 2.7; script `121_13652211`, part (E)). If $\delta = 0$, the data have dimension $15 + 3 + 4 + 30 + 20 = 72$, with $4 \cdot 5$ conditions: the bound is 52, and the part over Σ has dimension at most $\dim \Sigma + 7 + 2 \dim \mathcal{A}_D + 20$. With weights $4e_X$ and 1, good flags have $\mathfrak{B}_X = \Gamma_0$ $(27 + 8 + 4 = 39)$ or curvilinear of length three $(27 + 12 + 2 = 41)$. Bad ones have $\dim \Sigma + 2 \dim \mathcal{A}_D \le 24$: $D_3 = X^2 T_1 \oplus E$, with $\dim \mathcal{A}_D = 10 - r_E$, which is 6 unless E contains the gradient of a pair $(a_3, a_4) \ne 0$, which leaves 8 parameters and $\dim \mathcal{A}_D \le 8$ $(10 + 12,\ 8 + 16)$; $\mathfrak{B}_X$ curvilinear of length four $(7 + 16)$; $\mathfrak{B}$ infinite $(5 + 16)$. The bad part has dimension at most $24 + 27 = 51 < 52$; [16, Lemma C.6] concludes. □

Theorem B.18. *Every local algebra with Hilbert function* $(1,3,6,4,2,2,2,1)$ *is smoothable.*

Proof. The supplement lists the cases of this proof with their certificates (file `COVERAGE_B18_B21_B24.md`). Here $D_k = \nu_k(\Gamma)$, $k = 4,5,6$, $\ell(\Gamma) = 2$, $D_7 = \mathbb{C}F$, and $\mathfrak{B}$ is finite of length at most four or a line [15, Lemma A.11]. If $\Gamma = \{X, Y\}$, Lemma B.5 gives a ray. Over Γ_0 the weight of $[X]$ is $4e_X$, and [15, Corollary 3.14] with $c = 4$ gives 37 if $\mathfrak{B}_X = \Gamma_0$ and 40 if $\mathfrak{B}_X$ is curvilinear of length three. If $F = X^6Y$, after two shears M is closed if and only if the sextic tail lies in $\langle X^6, X^5Y, X^4Y^2, X^3Y^3\rangle$, a binary cubic c_Y lies in D_3, and an affine condition on the quartic tail holds; $E_0 + \mathbb{C}c_Y = \nu_3(\mathfrak{B}_C)$ for a curvilinear $\mathfrak{B}_C$ on the line $z = 0$, and a general D_3 has base scheme $\mathfrak{B}_C$ [15, Lemma A.16]: at most 40. Let $F = X^7$. If $D_3 = X^2T_1 \oplus \mathbb{C}(Xg_2 + g_3)$ with $g_3 \neq 0$, then $\mathfrak{B}_X = V(\mathfrak{m}_X^2)$, as $g_3 \neq 0$ lets the cubics of I_3^* have every binary quadric as quadratic part at $[X]$, and [15, Lemma D.3] gives $27+3\cdot4+1 = 40$. The rest are seven orbits under the stabilizer of Γ_0 and X^7: $\mathfrak{B}_X$ curvilinear of length four (two); $D_3 = X^2T_1 \oplus \mathbb{C}Xg_2$ with g_2 a square (two) or not (two); $D_3 = \mathbb{C}[X,Y]_3$. For four, three elements of I^* have local length 41, and Proposition 2.16 with $L = I^*$ applies (script `121_13642221`, part (B)); for the other three, elimination with all 120 tail coefficients symbolic gives five irreducible families [15, Lemma 5.16], with members of local length 41, 39, 38, 36, 41 (part (A)). □

B.19. The last six Hilbert functions share one more device. When the lift of an extra quartic top form is a free generator, its top form can be moved, and with it the graded data.

Lemma B.20 (A free quartic)**.** *Let $M' \subseteq M$ be a closed subspace of codimension one that contains $T_{\leq 2} \oplus D_3$, with $M = M' \oplus \mathbb{C}\Psi$ for an element Ψ of degree four. Let G_4 be the top form of Ψ, let $D_4(M')$ be the space of quartic top forms of M', and let $\Omega \in T_4$ satisfy $S_1 \circ \Omega \subseteq D_3$. For every t with $G_4 + t\Omega \notin D_4(M')$, the space $M_t = M' \oplus \mathbb{C}(\Psi + t\Omega)$ is closed, with the graded data of A except D_4, which becomes $D_4(M') \oplus \mathbb{C}(G_4 + t\Omega)$. If the algebras with the graded data of M_t are smoothable for all but finitely many t, then A is smoothable.*

Proof. The derivatives of Ψ lie in M and have degree at most three, so they lie in $M \cap T_{\leq 3} = T_{\leq 2} \oplus D_3 \subseteq M'$; so do those of Ω. As $M \cap T_{\leq 3} \subseteq M'$, $G_4 \notin D_4(M')$, so at most one t is excluded, and $\dim M_t = \dim M$. By [15, Lemma 2.3(2)] the spaces M_t form a family over an open subset of $\mathbb{A}^1$ containing 0, and smoothability is closed [15, Proposition 2.2(1)]. □

Theorem B.21. *Every local algebra with Hilbert function* $(1,3,6,4,3,2,2)$ *is smoothable.*

Proof. Excess linkage settles the finite base schemes except two configurations; there, and on the common line, cubic tails at infinity settle all graded data but

three, which Lemma B.20 and families with anchors settle. The supplement lists the cases with their certificates (file `COVERAGE_B18_B21_B24.md`).

By [15, Lemma 2.16], $D_k = \nu_k(\Gamma)$ for $k = 5, 6$ and a scheme Γ of length two, so $\Gamma \subseteq \mathfrak{B}$; by [15, Lemma A.11], $\mathfrak{B}$ is finite of length at most four, or $D_3 = \mathbb{C}[X, Y]_3$. The points of Γ have weight $s - 3 = 3$, the others at most two.

Finite base scheme. As $\sum_p e_p \le 5$ [16, Lemma 3.2], Theorem 2.6 gives at most $27 + 3e_\Gamma + 2e' \le 41$, with $e_\Gamma = \sum_{p\in\Gamma} e_p$ and $e' = \sum_{p\notin\Gamma} e_p$, unless $e_\Gamma = 5$ and $e' = 0$. By [15, Lemma A.4] this leaves, up to coordinates: (O1) $\Gamma = \{[X], [Y]\}$, $\mathfrak{B} = V(\mathfrak{m}_X^2) \cup \{[Y]\}$, $D_4 = \langle X^4, Y^4, X^3L\rangle$ with $L \in \{Y, Z\}$; and (O2) a double point Γ_0 at $[X]$ in $\mathfrak{B} = V(y^2, yz, z^3)$, where substitutions preserving $\mathfrak{B}$ leave $\Gamma_0 = V(z, y^2)$ with $D_4 = \langle X^4, X^3Y, X^2Z^2\rangle$ or X^3T_1, and $\Gamma_0 = V(y, z^2)$ with $D_4 = \langle X^4, X^3Z, X^2Z^2\rangle$, $\langle X^4, X^3Z, X^3Y + X^2Z^2\rangle$ or X^3T_1. Excess gives 42 on all of them. Except for the two data with $D_4 = X^3T_1$, three elements of L_{def} have local length 40, 40 in (O1) and 38, 41, 41 in (O2) (script `121_1364322`, parts (A1), (A2)), for instance $xy^2 + z^3$, $x^6 + xyz$, $x^3z + y^3 + yz^2$ for $\Gamma_0 = V(z, y^2)$, and Corollary 3.4 applies.

The common line. Now all D_k are binary and Γ lies on the line $z = 0$. If $\Gamma = \{[X], [Y]\}$, then $D_4 = \langle X^4, Y^4, G_4\rangle$, and the torus and $X \leftrightarrow Y$ leave the family $G_\lambda = \lambda X^3Y + X^2Y^2 + XY^3$ and the data X^2Y^2, $X^3Y + XY^3$, X^3Y, XY^3. For G_λ, three elements of L_{def} with cubic and sextic parts independent of λ lie in $\mathrm{Ann}(M_Q)$ for one Q and all λ; their ideal $J \subseteq \mathbb{Q}[x, y, z, \lambda]$ has dimension one and $(J : \mathfrak{m}^\infty) + \mathfrak{m} = (x, y, z, p)$ with $p = (9\lambda - 4)(432\lambda^2 - 1080\lambda - 677)$ (script `121_1364322_r2`, part (U7)). So for $p(\lambda_0) \neq 0$ the zero set of J near $(0, \lambda_0)$ is the λ-line, its local ring is a one-dimensional complete intersection with the single minimal prime $\mathfrak{m}$, and the local length at λ_0 is the length 40 over $\mathbb{C}(\lambda)$; at the roots of p it is 41 (part (A7)). The data X^2Y^2 and $X^3Y + XY^3$ have anchors at infinity of length 40, and for XY^3, and symmetrically X^3Y, the proof of [15, Proposition C.6(b)] gives a ray, as it uses D_4 only through two monomial conditions (parts (A5), (U8)). If $\Gamma = V(z, y^2)$, then $Y \mapsto Y + uX$ and the torus leave $G_4 \in \{Y^4 + X^2Y^2, Y^4, XY^3, X^2Y^2\}$, the first three with anchors at infinity of length 41.

Three data with $e(L_{\mathrm{def}}) = 42$: N1, N2 = (O2) with $D_4 = X^3T_1$ and $\Gamma_0 = V(z, y^2)$, $V(y, z^2)$, and N3 = $\Gamma = V(z, y^2)$ on the common line with $G_4 = X^2Y^2$. Let M_6 be spanned by $T_{\le 2}$, D_3 and $S \circ \Phi$ for the lifts Φ of D_6 (derivatives of all orders, the lifts included). If $M_6 \neq M$, then $\dim M_6 = 20$, and Lemma B.20 with $\Omega = X^2Z^2$, X^2Z^2, Y^4 moves D_4, for $t \neq 0$, into the settled orbits of $\langle X^4, X^3Y, X^2Z^2\rangle$ (by $Z \mapsto Z - X/(2t)$), $\langle X^4, X^3Z, X^3Y + X^2Z^2\rangle$ and $Y^4 + X^2Y^2$. If $M_6 = M$, write $D_6 = \langle X^6, X^5L\rangle$, $D_4 = \langle X^4, X^3L, H\rangle$ with $(L, H) = (Y, X^3Z)$, (Z, X^3Y), (Y, X^2Y^2), and lifts $\Phi_1 = X^6 + A + B_1 + C_1$, $\Phi_2 = X^5L + P + B_2 + C_2$, $\Psi = H + \tau$. Let l be the variable dual to L and o the other variable among y, z. Closedness is equivalent to four linear conditions on the quintic tails (A, P), namely that the quartic parts of $l\circ\Phi_1$, $o\circ\Phi_1$, $o\circ\Phi_2$ and $l\circ\Phi_2 - x\circ\Phi_1/6$, which have degree

at most four, lie in D_4, and four cubic conditions, linear in (B_1, B_2, τ) with coefficients depending on (A, P); the cubic tails C_i are free. The solutions (A, P) form a ten-dimensional space, the sum of the seven-dimensional space reached by gauges $\Phi_j \mapsto \Phi_j - c\, x \circ \Phi_i$ and the shears $v \mapsto v + cx^2$ ($v = y, z$), $o \mapsto o + c'xl$, and of an explicit monomial complement $\langle r_1, r_2, r_3\rangle$ (script `l21_1364322_r1`, part (U10); the shears act by [16, Lemma B.4]); so we may take $(A, P) = \sum p_i r_i$. The coefficients of H in the quartic parts of the four derivatives above are linear in p, and the algebra is non-free, $M_6 = M$, exactly when one of them is nonzero. With p symbolic, the cubic system has constant rank on each irreducible component of the consistent non-free locus (part (U12)): two components for N1 and for N2, sharing the points $(0, 0, 1)$, resp. $(0, 1, 0)$, and one for N3, containing $(1, 0, 0)$. Over each the algebras form an affine bundle, and members at these points have triples of local length 39, 39, 38 (part (A12)); Section 2.4 concludes. □

Theorem B.22. *Every local algebra with Hilbert function* $(1, 3, 6, 4, 3, 3, 1)$ *is smoothable.*

Proof. Over a finite base scheme of multiplicity five, free quartic tails, the tangent cone, a rerun of [15, Corollary D.4] and one family with an anchor settle the strata where excess gives 42; on the common line a point leaves along a parabola.

By [15, Lemma 2.16], $D_k = \nu_k(\Gamma)$ for $k = 4, 5$ and $F \in \nu_6(\Gamma)$ for a scheme $\Gamma \subseteq \mathfrak{B}$ of length three, and by [15, Lemma A.11] $D_3 = \mathbb{C}[X, Y]_3$, or $\mathfrak{B}$ is finite of length at most four. In the finite case $\sum_p e_p \leq 5$ [15, Lemma A.4]; if $\sum_p e_p \leq 4$, excess gives $27 + 3 \cdot 4 = 39$, and otherwise, up to coordinates, $\mathfrak{B} = \mathfrak{B}_A$ with local ideal $(\eta^2, \eta\zeta, \zeta^3)$ at $[X]$, or $\mathfrak{B} = \mathfrak{B}_B = V(\mathfrak{m}_X^2) \cup \{[Y]\}$ (script `l21_1364331`). Then: (i) on $\mathfrak{B}_B$ with Γ fat, $k([Y]) = 4$ and excess gives $27 + 3 \cdot 4 + 1 = 40$; (ii) if $F = X^6$, free quartic tails apply, with rank three (part (D3)), and give 37; (iii) on $\mathfrak{B}_B$ with Γ a double point and $[Y]$, and $F \neq X^6$, the tangent cone (Proposition 2.16(b) with $L = I^*$) gives $27 + 10 + 3 = 40$, or 41 if $F = Y^6$, through Proposition 3.7(a), as the flag of base schemes does not depend on F (part (W)); (iv) on $\mathfrak{B}_A$ with $\Gamma = V(\eta + a\zeta^2, \zeta^3)$ curvilinear and $F \neq X^6$, it gives $27 + 14 = 41$; (v) on $\mathfrak{B}_A$ with Γ fat, $F = X^5 L$ modulo X^6. If $L \notin \langle X, Z\rangle$, we bring F to $X^5 Y$ and rerun the proof of [15, Corollary D.4] with $y \circ \Phi$, x^5, $x^5 y$, $R = Y^4$ in place of $z \circ \Phi$, x^5, $x^5 z$; the cubics of I_3^* without y^3 generate the local ideal of $\mathfrak{B}_A$, and the bound is $27 + 2 \cdot 5 = 37$ (part (D4)). If $L \in \langle X, Z\rangle$, then $F = X^5 Z$, the algebras form one family over $\mathbb{A}^{26}$ (part (F), elimination with constant pivots), and its member $\Phi = X^5 Z + Z^4$, $\Psi = X^4 Y$ contains $2y^3 + xyz$, $5z^3 - x^5$, $xy^2 - x^2 z^2$, of local length 38, a linkage anchor of the family.

The common line. Now $I_3^* = zS_2$, all D_k are binary, $\Gamma = V(z, q)$ for a binary cubic q, and we choose coordinates with $q(0, 1) \neq 0$. The lifts of the quartics and quintics are $v + Z(p_2 \circ v)$ and $v + g(v) + Z(p_2 \circ v) + \chi(v)$, with $g(v)$ binary, $\chi(v)$ divisible by Z, and one binary quadric p_2 (script `l21_1364331_line`, part (L)); the change $z \mapsto z - p_2(x, y)$, acting by

$\exp(-Zp_2(\partial))$, removes the terms $Z(p_2 \circ v)$. Then closedness is linear in all the tails, g, χ and those of the sextic lift $\Phi = F + A_5 + B_4 + C_3$, so for fixed (q, F) the normalized algebras form a linear space $L_{q,F}$ (part (S)). Choose a binary quartic p_4 and a binary quintic p_5 with $\langle xp_4, v\rangle = -\langle xq, g(v)\rangle$ for $v \in D_5$ and $\langle xp_5, F\rangle = -\langle xq, B_4\rangle - \langle xp_4, A_5\rangle$ (possible: $x\circ$ is injective on D_5, as $Y^5 \notin D_5$, and $F_X \neq 0$), and put $u = q + p_4 + p_5$. Comparing degrees, $xu \in I$, and zu and y^2u pair to zero with every basis element of M except Φ, with which they pair to k and h. If $k = \alpha h$ for some α, then $u \in (I : (x, z - \alpha y^2))$, and $\Psi = Y^3 + 6\alpha YZ \in M$ satisfies $x \circ \Psi = (z - \alpha y^2) \circ \Psi = 0 \neq \langle u, \Psi\rangle = 6q(0, 1)$, so $u \notin I + (x, z - \alpha y^2)$, and a point leaves along the parabola $x = 0$, $z = \alpha y^2$ (4). If $F = P^6$, then $k = 0$ for every tail. Otherwise h is a polynomial function on $L_{q,F}$, nonzero at an explicit point (part (P)), so the rays are dense in $L_{q,F}$, and [15, Proposition 2.2(1)] concludes. (Script `l21_1364331_line_u` checks the construction of u with all tails symbolic, for four sample cubics q covering the three root types, each with a sample non-power top and the top X^6.) □

Theorem B.23. *Every local algebra with Hilbert function* $(1, 3, 6, 4, 4, 2, 1)$ *is smoothable.*

Proof. Finite base schemes are treated as in Theorem B.22; on the common line a point leaves along a line or a parabola, or an irreducible family has an anchor.

Finite base scheme (script `l21_1364421`). By [15, Lemma A.11], $\mathfrak{B}$ is finite of length at most four or $D_3 = \mathbb{C}[X, Y]_3$. If $\sum_p e_p \leq 4$, excess gives 39. Otherwise $\mathfrak{B} = \mathfrak{B}_A$ or $\mathfrak{B}_B$ as in Theorem B.22, $D_3 = \nu_3(\mathfrak{B})$, $D_4 = \nu_4(\mathfrak{B})$, and $k(p) = 6$ or 5 according as $P^5 \in D_5$ or not. On $\mathfrak{B}_A$: if $X^5 \notin D_5$, excess gives $27 + 2 \cdot 5 = 37$; otherwise $F = aX^6 + X^5L$, and $F = X^6$ gives 37 by free quartic tails, a transverse L gives 37 by the rerun of [15, Corollary D.4], and $L = Z$ gives one family over $\mathbb{A}^{30}$ with an anchor of local length 33. On $\mathfrak{B}_B$: excess gives at most $27 + 2 \cdot 4 + 3 = 38$ if $X^5 \notin D_5$, and $27 + 3 \cdot 4 + 2 = 41$ if $X^5 \in D_5 \not\ni Y^5$; if $D_5 = \langle X^5, Y^5\rangle$, then $F = aX^6 + bY^6$, and $F = X^6$ gives 37, $ab \neq 0$ one family with an anchor of length 34, and $F = Y^6$ two irreducible components with anchors of length 35 and 33.

The common line, F not a power. Here $D_5 = S_1 \circ F$, $G = \langle F_{XX}, F_{XY}, F_{YY}\rangle \subseteq D_4$, and every algebra is $T_{\leq 2} + \mathbb{C}[X, Y]_3 + \{v + \varphi(v) : v \in D_4\} + \langle P_X, P_Y, P\rangle$ with $P = F + A_0 + Za_4 + B_4 + C_3$, where $A_0 \in \mathbb{C}[X, Y]_5$, $a_4 \in \mathbb{C}[X, Y]_4$, $B_4 \in T_4$, and C_3 and the values of φ are divisible by Z; closedness says that $a_4 \in D_4$, $\varphi(F_{ij}) = Z(a_4)_{ij}$ and $(B_4)_Z - \varphi(a_4)$ is binary, and the other tails are free. If $\dim G = 3$, the algebras form an affine bundle of constant rank over the irreducible incidence of pairs (F, D_4) with $G \subseteq D_4$ (its free data are $a_4 \in D_4$, φ on a complement of G, A_0, C_3, and the binary parts of B_4 and of $(B_4)_Z - \varphi(a_4)$); its member $F = X^3Y^3$, $D_4 = \langle X^4, X^3Y, X^2Y^2, XY^3\rangle$, $\varphi(X^4) = 12Y^2Z$ contains xyz, $x^2z + z^3 - y^4$, $y^2z + z^3 - x^4$, of local length 36. If $\dim G = 2$, then $F = X^6 + Y^6$ or X^5Y, a shear removes Za_4, and the algebras form one irreducible family, as

X^5Y is a limit of secant sextics, with a member of local length 40 (script `l21_1364421_line`).

The common line, $F = X^6$. Then $D_5 = \langle X^5, H\rangle$ with H binary, and a shear $z \mapsto z + \gamma' x^2$ makes the quintic tail of the lift of X^6 binary. Unless D_5 is the tangent pencil $\langle X^5, X^4Y\rangle$, a change $Y \mapsto Y + \beta X$ removes the term X^4Y from H, and a shear $y \mapsto y + \gamma x^2$ removes it from that quintic tail; then $X^4 \in M$ and x^4y, $x^4z \in I$, and [15, Lemma 5.4] with $k = 4$ gives a ray along the line of $[X]$. In the tangent pencil $D_5 = \langle X^5, X^4Y\rangle$, two shears bring M to the form $T_{\leq 2} + \mathbb{C}[X,Y]_3 + \{v + \varphi(v)\} + \langle P_X, Q, P\rangle$ with $P = X^6 + A + B + Zb + C$, $Q = X^4Y + g + \chi$, with A a binary quintic, B and g binary quartics, b a binary cubic, C and χ divisible by Z, $\varphi(X^4) = \varphi(X^3Y) = 0$ and $D_4 = (y^2\rho)^\perp \cap \mathbb{C}[X,Y]_4$ for a binary quadric ρ; closedness is $A_Y \in D_4$ and $\varphi(A_Y) = Zb_Y$. For $w = A_Y$ modulo $\langle X^4, X^3Y\rangle$, the loci $w = 0$ and $w \neq 0$ are images of irreducible parameter spaces, as the pairs (ρ, w) with $A_Y \in D_4$ form an irreducible quadric and, for fixed w, the condition $\varphi(A_Y) = Zb_Y$ is linear in (φ, b) of constant rank on each locus (script `l21_1364421_tangent`, part (I)); on a dense open subset of each, $\rho = l_1 l_2$ with distinct factors not proportional to y. For a factor l, with other factor l', a corrected cubic $u = y^2 l' + p_4 + c_5 x^5$ has $lu \in I$, and $(z - \alpha y^2)u \in I$ if and only if $k = \alpha h$ for two explicit numbers k, h. Both h vanish only if $w = 0$, and then both k vanish, as $\varphi(A_Y) = 0$ forces $b \in \mathbb{C}X^3$. So one factor gives a ray $(l, z - \alpha y^2)$, detected by $\Psi = L^3 + 6\alpha LZ$ for the binary linear form L with $l \circ L = 0$ and $y \circ L = 1$ (it exists, as l is not proportional to y; then $(z - \alpha y^2) \circ \Psi = 6\alpha L(1 - (y \circ L)^2) = 0$), and closedness on each stratum concludes (script `l21_1364421_tangent`, all tails symbolic). □

Theorem B.24. *Every local algebra with Hilbert function* $(1, 3, 6, 4, 3, 2, 1, 1)$ *is smoothable.*

Proof. Here $D_7 = \mathbb{C}X^7$, so $[X] \in \mathfrak{B}$ has weight four, and at most one other point has weight three. The supplement lists the cases with their certificates (file `COVERAGE_B18_B21_B24.md`).

A free cubic. If $D_3 \not\subseteq N = \mathfrak{m} \circ M + T_{\leq 2}$, then $M = M' \oplus \mathbb{C}G$ with M' of type $(1, 3, 6, 3, 3, 2, 1, 1)$ [15, Lemma A.15], whose cubic data are $\nu_3(\Gamma')$ for a scheme Γ' of length three [15, Lemma 2.16]. Replacing G by a general cubic gives an irreducible family [15, Lemma A.14] whose general member has $\mathfrak{B} = \Gamma'$ [15, Lemma A.16], and excess, with free quartic tails if Γ' is fat, gives at most 39.

Finite base scheme. Now $\ell(\mathfrak{B}) \leq 4$ and $\sum_p e_p \leq 5$. Excess, with free quartic tails where their rank, $1 + \sigma$ for σ the socle dimension of $\mathfrak{B}$ at $[X]$ (for $\ell(\mathfrak{B}) = 4$ by [15, Lemma A.3], for the fat point of length three directly), is three, gives at most 41 except in six configurations of length four, with bounds 42, 43 (script `l21_13643211`). On three of them the tangent cone gives at most 41 through Proposition 3.7(a), except on two special flags. Five residual strata remain: $\mathfrak{B}$ curvilinear of length four at $[X]$ with a special flag; $[X]$ with a fat point of weight three; a fat point at $[X]$ with a point of weight

three; (u^2, uv, v^3) at $[X]$; and (u^2, v^2) at $[X]$ with $D_4 = X^3T_1$. They have two normal forms each, seven for (u^2, uv, v^3). For each we write the lifts with tails avoiding the pivot monomials and eliminate closedness with constant pivots (scripts `l21_13643211_fam`, `l21_13643211_fam2`, `l21_13643211_fam3`); the residual equations, of the shape $a \cdot (\cdots) = 0$, cover the algebras by 24, resp. 27 for (u^2, uv, v^3), irreducible families, each with a member of local length between 34 and 40.

The common line. Now all D_k are binary and $D_5 = \langle X^5, T\rangle$ with no X^5 in T. Exactly one of the following holds: (R) $\langle X^4, T_X, T_Y\rangle$ has dimension three; (S) $T \equiv \lambda M^5$ modulo X^5, $M \notin \mathbb{C}X$; (T) $T \equiv X^4L$ modulo X^5, $L \notin \mathbb{C}X$ (script `l21_13643211_cl_ray`, part (0)). In (R), and in (S) with $T = Y^5$, let p_4 be a binary quartic with $\langle p_4, T_X\rangle = \langle p_4, T_Y\rangle = 0$ and $\langle p_4, X^4\rangle = 1$. After four shears $v \mapsto v + kx^j$ ($v = y, z$, $j = 2, 3$) normalizing the sextic and quintic tails of the lift of X^7, $p_4 \circ M \subseteq \mathbb{C}[X]$ (part (1), all coefficients symbolic); so $p_4 \in (I : (y, z))$, while $X^4 \in M$ (it is $x^3 \circ \Phi/210$ up to a binary cubic and lower terms, which lie in M) pairs to 1 with p_4 and to 0 with $I + (y, z)$: a ray, for every algebra. In (T), $D_5 = \langle X^5, X^4Y\rangle$ and $D_4 = \langle X^4, X^3Y, R\rangle$ with $R \in \{Y^4, X^2Y^2 + Y^4, XY^3, X^2Y^2\}$; for each R the normalized algebras lie in three irreducible families, the third in the closure of the first, with members of local length 39, 41, 40 (script `l21_13643211_cl_tangent`). □

Theorem B.25. *Every local algebra with Hilbert function* $(1, 3, 6, 5, 4, 1, 1)$ *is smoothable.*

Proof. Here $D_6 = \mathbb{C}X^6$, $D_5 = \mathbb{C}X^5$, so $[X] \in \mathfrak{B}$ has weight three and every other point weight at most two. Let $\Phi = X^6 + P + R + E$ be the lift of X^6; the shears $y \mapsto y + ax^2$, $z \mapsto z + bx^2$ add $30X^4(aY + bZ)$ to P, and a gauge removes X^5. If afterwards $x^2 \circ P \in D_3$ and $X^4 \notin \langle y \circ P, z \circ P\rangle$, then $X^4 \in M \setminus N_X \circ M$, a ray [15, Theorem 2.11(2)].

A common linear factor. By [15, Lemma A.12], $\mathfrak{B}$ is finite, or a line $\Lambda = V(z)$ with a point, isolated or embedded, or a line with $D_3 = \nu_3(\Lambda) \oplus \mathbb{C}G$. A quartic with all derivatives in D_3 has z-derivative a multiple of a cube L^3; so D_4 is binary, or the point of $\mathfrak{B}$ is $[L]$. In each case closedness restricts P to an explicit space, and after the shears the criterion gives a ray, except when the point is embedded at $[X]$; there $P = P_b + cX^3YZ + \frac{a}{2}X^3Z^2$ with P_b binary, and the ray needs $a = c = 0$. If $a \neq 0$, then the substitution $Z \mapsto Z - (c/a)Y$ makes $c = 0$, and the family with $L_t = X + tZ$, $D_3(t) = \mathbb{C}[X, Y]_3 \oplus \mathbb{C}(L_t^3 - X^3)/(3t)$, $P_t = P_b + a(L_t^5 - X^5 - 5tX^4Z)/(20t^2)$ and correspondingly moved quartic lifts (when $y \circ P_b \in \mathbb{C}X^4$, after correcting R by a regular right inverse of $R \mapsto y \circ R$ modulo $D_3(t)$) is closed for every t, equals M at $t = 0$, and has the point $[L_t]$ off the line for $t \neq 0$; if $a = 0 \neq c$, an explicit family moves A into the case $a \neq 0$ (scripts `l21_1365411`, `l21_1365411_line`; the rays by the colon criterion in `l21_1365411_rays`).

Finite base scheme. As $\sum_p e_p \leq \ell(\mathfrak{B}) + 1 \leq 6$ [16, Lemma 3.2], excess gives at most 41 unless the germ of $\mathfrak{B}$ at $[X]$ is the fat point with a residual scheme of length two, curvilinear of length five, (u^2, uw, w^3) with at most one

more point, non-curvilinear of length five and not a complete intersection, or a non-curvilinear complete intersection of length five. In the first, third and fourth cases with $\ell(\mathfrak{B}) = 5$, the socle of $\mathfrak{B}$ at $[X]$ has dimension two, so the residual of $\mathfrak{B}$ with respect to $\mathfrak{m}_X$ has length three, the space of quartics of [15, Lemma D.3] has dimension $15 - (10 - 3) = 8$, its kernel $\nu_4(\mathfrak{B})$ has dimension five, and free quartic tails give $27 + 2 \cdot 6 = 39$ (script `l21_1365411_d3rank`); for (u^2, uw, w^3) alone the rank is three as well, as otherwise the cubics of I_3^* would not generate the local ideal, and the bound is 37 (script `l21_1365411`, part (B)). If $\mathfrak{B}$ is curvilinear of length five, the ideal of the proper transforms of I^* at $[X]$ contains $(w', u^5, \rho^2 u, \rho^2 w', \rho^4)$, of multiplicity 14 by its Newton polygon, and the tangent cone gives $27+14 = 41$. The last case is Proposition 3.11. □

B.26. The last Hilbert function needs one more variant of free quartic tails. A quartic tail can add a prescribed quintic direction to the cubic equations; an operator α chooses that direction, and at a point where α does not vanish the surfaces separate in degree five. The price is that the cubic pencil must lie in the kernel of a linear form φ_R, so that its local intersection length ε_p replaces the multiplicity e_p. In this variant the operator u^5 of [15, Lemma D.3] is replaced by an arbitrary quintic α, no rank condition is imposed, and cubic tails ride on top as in Lemma 3.9.

Lemma B.27 (Quartic tails along an operator)**.** *Let A be a local algebra of length d with $H_A(2) = 6$, socle degree six, $D_6 = \mathbb{C}F$, $\dim I_3^* \geq 3$ and $\mathfrak{B}$ finite, and suppose that every finite subscheme of $\mathbb{A}^3$ of length less than d is smoothable. Let $\alpha \in S_5$ and $\ell \in S_1$ satisfy $c_0 = \langle \alpha\ell, F\rangle \neq 0$. Put $N_\alpha = \{v \in S_1 : \langle \alpha v, F\rangle = 0\}$ and $K_\alpha = \{R \in T_4 : N_\alpha \circ R \subseteq D_3\}$, and let $K' \subseteq K_\alpha$ be a subspace on which the linear form $\varphi_R \colon w \mapsto \langle \ell w, R\rangle$ on I_3^* is not identically zero. For $p \in \mathfrak{B}$ let ε_p be the colength at p of two general elements of $\ker \varphi_R$, for R general in K', and suppose that every ε_p is finite. Let $k'(p) = \min(k(p), 5)$ if α does not vanish at p, and $k'(p) = k(p)$ otherwise. If*

$$27 + \sum_{p \in \mathfrak{B}} (k'(p) - 3)\, \varepsilon_p < 2d,$$

then A is smoothable. If the forms φ_R, $R \in K'$, span a space of dimension at least three, then $\varepsilon_p = e_p$ for every p.

For $F = X^6$, $\alpha = x^5$, $\ell = x$ and $K' = K_\alpha$, the last sentence gives back [15, Lemma D.3] in socle degree six.

Proof. We follow the proofs of [15, Lemma D.3] and of Lemma 3.9, with the functionals α and $\alpha\ell$. Let $\Phi \in M$ have top form F, and $M'' = \{m \in M : \langle \alpha, m\rangle = \langle \alpha\ell, m\rangle = 0\}$. The two functionals take the values $(c_0, 0)$ on $\ell \circ \Phi$ and $(*, c_0)$ on Φ, so $M = M'' \oplus \mathbb{C}\ell \circ \Phi \oplus \mathbb{C}\Phi$. Elements of M'' have degree at most five, so M'' is closed, and it contains $T_{\leq 2} \oplus D_3$, the second derivatives of Φ and $N_\alpha \circ \Phi$. As $N_\alpha + \mathbb{C}\ell = S_1$, for $R \in K'$ and $C \in T_3$ the spaces

$$M_{\lambda,\mu} = M'' \oplus \mathbb{C}\big(\ell \circ \Phi + \lambda\, \ell \circ R + \mu\, \ell \circ C\big) \oplus \mathbb{C}\big(\Phi + \lambda R + \mu C\big)$$

are closed with the graded data of A, and they form a family over $\mathbb{A}^2$ through A [15, Lemma 2.3(2)]. Pairing with this basis shows that their ideal $I_{\lambda,\mu}$ contains $I \cap \mathfrak{m}^5$; the element $g - \lambda c_0^{-1}\langle q_4, R\rangle\,\alpha\ell$ for every $g \in I$ of order four with leading form q_4; and $f + \lambda(\theta\alpha + \theta'\alpha\ell) + \mu q$ for every $f \in I$ with leading form $w \in I_3^*$ and quartic part f_4, where $\theta = -\varphi_R(w)/c_0$, $\theta' = -(\langle f_4, R\rangle + \theta\langle \alpha, \Phi\rangle)/c_0$ and $q \in S_6$ has $\langle q, F\rangle = -\langle w, C\rangle$.

Let w_1, w_2, w_3 be general in I_3^*, let $f_i \in I$ have leading forms w_i, let R be general in K', and let θ_i, θ_i' be as above; then $\theta_3 \neq 0$. Choose $L \in S_1$ vanishing at no point of $\mathfrak{B}$, and C with $\langle w_i, C\rangle = -\theta_i\langle L^6, F\rangle$, so that $q_i = \theta_i L^6$; choose $g_4, g_5 \in I$ with general leading forms $q_4 \in I_4^*$, $q_5 \in I_5^*$, and put $a = -\langle q_4, R\rangle/c_0$. For $\lambda = \varepsilon^{-2}$ and $\mu = \varepsilon^{-3}$ the elements

$$f_i^\varepsilon = f_i + \theta_i(\varepsilon^{-1}g_4 + \varepsilon^{-2}(g_5 + \alpha) + \varepsilon^{-3}(a\,\alpha\ell + L^6)) + \varepsilon^{-2}\theta_i'\,\alpha\ell$$

lie in $I_{\lambda,\mu}$, and they have the form of Step 2 of the proof of [15, Theorem 3.12], with the forms q_4, $q_5 + \alpha$ and $a\,\alpha\ell + L^6$ in degrees 4, 5, 6. At $p \in \mathfrak{B}$ the first of these forms that does not vanish has degree $k'(p)$: by generality if $k(p) \leq 5$, and because α or L does not vanish at p if $k(p) = 6$. Step 2 uses of the cubics only that $w_i - (\theta_i/\theta_3)w_3$, $i = 1, 2$, cut out a finite scheme at p, and it bounds the contribution of p by $k'(p) - 3$ times its length. These two cubics span $\ker\varphi_R \cap \langle w_1, w_2, w_3\rangle$. Over the dense open set $K'^\circ = \{R \in K' : \varphi_R \neq 0\}$ the kernels $\ker\varphi_R$ form a vector bundle of rank $\dim I_3^* - 1$, whose relative Grassmannian of pencils is irreducible, and for general R and w_i the pair $(R, \ker\varphi_R \cap \langle w_1, w_2, w_3\rangle)$ is a general point of it (a pencil of $\ker\varphi_R$ and a cubic outside it span a net). So it lies in the open set where the finitely many colengths at the points of $\mathfrak{B}$ take their general values, and this length is ε_p. For general w_i the three surfaces have no common zero outside $\mathfrak{B}$, and Step 3 with [15, Remark D.2] concludes, as in [15, Lemma D.3]. If the φ_R span at least three dimensions, every pencil of I_3^* lies in some $\ker\varphi_R$, so $\varepsilon_p = e_p$. □

Theorem B.28. *Every local algebra with Hilbert function* $(1, 3, 6, 5, 3, 2, 1)$ *is smoothable.*

Proof. This profile uses every device. We first separate the algebras with a free cubic from those whose cubic top forms come from derivatives, $D_3 \subseteq N$, and treat the sector B3. For the others the cubic base scheme is finite or contains a line. For a finite base scheme, the test of Proposition 3.7(c) on the flags $\mathfrak{B}_3 \supseteq \mathfrak{B}_4 \supseteq \mathfrak{B}_5$ at the points of weight three leaves 26 of the 157 configurations, 10 in P2, 6 secant and 10 tangent; free quartic tails, Lemma B.27, or an irreducible family with an anchor settle them. When the base scheme contains a line, rays settle most pure tops; the rest form finitely many orbits and families, in which the free quartic lifts are separated from those that are derivatives of the sextic lift, and each irreducible family is paired with a linkage certificate (Lemma 3.13). Table 3 at the end indexes the cases, and the file `B28_COVERAGE.md` of the supplement names the certificate of each.

Sectors. As $\dim S_1 \circ D_6 \leq 2$, the top form F has at most two essential variables. If two, then $D_5 = S_1 \circ F$ and $\dim S_2 \circ F \in \{2,3\}$: the value 3 is the sector B3, and 2 means $F = X^6 + Y^6$ or X^5Y, with $D_5 = \nu_5(\Gamma')$ for $\Gamma' = \{[X],[Y]\}$ or the double point $V(z,y^2)$. If $F = X^6$, then $D_5 = \langle X^5, G\rangle$, and either $\dim(\mathbb{C}X^4 + S_1 \circ G) = 3$ (the sector P2), or D_5 is $\langle X^5, Y^5\rangle$ or $\langle X^5, X^4Y\rangle$, again $\nu_5(\Gamma')$. When $D_5 = \nu_5(\Gamma')$, $D_4 = \nu_4(\Gamma') \oplus \mathbb{C}R$; we put $P(R) = \nu_3(\Gamma') + S_1 \circ R$ and $q = \dim P(R)/\nu_3(\Gamma') \in \{1,2,3\}$. A point $[L]$ has weight three exactly when $L^5 \in D_5$: the points of Γ', $[X]$ in P2, and none in B3, where $S_1 \circ F$ contains no fifth power (if $v \circ F$ were one, integration along v would give $\dim S_2 \circ F \leq 2$).

B3 and a free cubic. In B3 a finite $\mathfrak{B}$ has length at most five [15, Lemma A.12], and excess gives $27 + 2 \cdot 6 = 39$; if $\mathfrak{B}$ contains a line, it is $z = 0$ and $D_3 = \mathbb{C}[X,Y]_3 \oplus \mathbb{C}G_3$. For the lift $\Phi = F + R_5 + R_4 + R_3$ of F, closedness then says

$$z \circ R_5 = q \circ F \ \text{ for some } q \in \mathbb{C}[x,y]_2, \qquad z \circ R_4 - q \circ R_5 \in D_3.$$

As $q \mapsto q \circ F$ is injective and $z\circ$ maps T_5 onto T_4 and T_4 onto T_3, the R_5 form a vector bundle of rank 9, the R_4 then an affine space of dimension 10, and R_3 is free: the algebras form a tower of affine bundles over an irreducible base, with an anchor (script `121_1365321_B3`). If $D_3 \not\subseteq N = \mathfrak{m} \circ M + T_{\leq 2}$, then $M = M_0 \oplus \mathbb{C}G_3$ with M_0 of type $(1,3,6,4,3,2,1)$ [15, Lemma A.15]. For a family of [16, Theorem C.11] containing M_0, the pairs (M', G') with M' in it and $G' \notin D_3(M')$ form an irreducible family containing M, an open subset of its product with T_3. Each case of [16, Theorem C.11] gives it a member in a complete intersection of local length at most 39, after adding a general cubic, or, at an anchor M_a of length 20 in $V(f_1,f_2,f_3)$, a cubic orthogonal to the three equations [16, Remark 3.7]. Such a cubic exists: as $f_i \in \mathfrak{m}^3$, the three cubic heads impose at most three conditions on T_3, which leaves a space of dimension at least seven, while $\dim D_3(M_a) = 4$; for G_a in this space outside $D_3(M_a)$, $M_a \oplus \mathbb{C}G_a$ has length 21 and lies in $V(f_1,f_2,f_3)$.

Finite base schemes (with $D_3 \subseteq N$). Here $\ell(\mathfrak{B}) \leq 5$ [15, Lemma A.12] and $\sum_p e_p \leq \ell(\mathfrak{B}) + 1$ [16, Lemma 3.2]. If the points of weight three have total multiplicity $E_3 \leq 2$, excess gives at most $27 + 2 \cdot 6 + E_3 \leq 41$. Otherwise Proposition 3.7(c), with the local values bounded over all analytic types of the flag $\mathfrak{B}_3 \supseteq \mathfrak{B}_4 \supseteq \mathfrak{B}_5$ at the points of weight three, where $\mathfrak{B}_k$ is the base scheme of I_k^* (157 configurations), gives at most 41 except in 26 configurations with a non-curvilinear germ of multiplicity at least four there: ten in P2, six with a secant and ten with a tangent pencil or top (script `121_1365321_wm`). Three devices settle them (script `121_1365321_cfg`).

First, when $\ell(\mathfrak{B}) = 5$ the rank in [15, Lemma D.3] is at least $1 + \sigma$, for σ the socle dimension of $\mathfrak{B}$ at $[X]$: the space of quartics there has dimension $5 + \dim(I_{\mathfrak{B}} : N_X)_2 \geq 5 + (6 - (5 - \sigma))$, and the kernel of $R \mapsto x \circ R$ modulo D_3 on it is $\nu_4(\mathfrak{B})$, of dimension five. So free quartic tails give 39 for pure tops with $\sigma \geq 2$ at $[X]$. The germ (u^2, uw, w^3) alone also gives rank three, as in Theorem B.25, and the bound 37. In P2, $D_4 = \mathbb{C}X^4 + S_1 \circ G$ is never

X^3T_1, as that would put G in X^4T_1 (part (U1)); so $\mathfrak{B}_4$ is not the fat point at $[X]$, and the remaining germ $(uw, u^2 - w^3)$ of P2 has the local value 14 (l21_1365321_wm, parts (U1), (U4)), 41 in all.

Second, Lemma B.27, with ε_p bounded through the conic that contains $\mathfrak{B}$ or by an exact computation on each orbit, gives at most 41 in the following cases, with $K' = K_\alpha$ unless said otherwise. For a pure top with the heavy germ at the point $[Y] \neq [X]$ of a secant pencil, take $\alpha = b^5$ and $\ell = b$ for a linear form b with $b(X)b(Y) \neq 0$ (part (A2)). For $F = X^6 + Y^6$, take the heavy germ at $[X]$, $\alpha = x^5$ and $\ell = x$; the φ_R span three dimensions by the rank formula above, so $\varepsilon_p = e_p$, and the bound is $27 + 2e_{[X]} + 3e_{[Y]} \leq 41$ (l21_1365321_tq_secant: each configuration is one orbit, part (U1); the φ_R span three dimensions, part (U2); bounds 40 and 41, part (A3)). For the pure tangent pencil and the germ $(uw, u^2 - w^3)$, take $\alpha = b^5$, $\ell = b$ with $b(X) \neq 0$, and $K' = \nu_4(Z)$ for the scheme $Z \supset \mathfrak{B}$ with the local ideal $\mathfrak{m}_X\mathfrak{b}$ at $[X]$; then $\varepsilon_{[X]} \leq 7$ (part (U8)). For $F = X^5Y$, take $\alpha = x^5 + \beta x^4 y$ and $\ell = y$, where $[\alpha \circ F]$ is a general point of the line of Γ' or a point of $\mathfrak{B}$ on it (part (A7)); this settles the configurations with $\ell(\mathfrak{B}) = 5$ in which this line meets the germ of $\mathfrak{B}$ at $[X]$ only in Γ'. Finally, for the tangent pencil and the tangent top, the germs (u^2, uw, w^3) alone and $(uw, u^2 - w^3)$ have local value $e(K_{[X]}(L_{\mathrm{def}})) \leq 14$ in Proposition 3.7 when $\mathfrak{B}_4$ is not the fat point at $[X]$, hence $e(L_{\mathrm{def}}) \leq 41$: up to automorphisms of the germ, the ideals of colength two and three between $\mathfrak{b}_3$ and $\mathfrak{m}$ other than $\mathfrak{m}^2$ are those of part (U1) of l21_1365321_wm (part (U4)), and a triple chosen with $\mathfrak{b}_4$ in place of $\mathfrak{b}_5 \supseteq \mathfrak{b}_4$ lies in $K_{[X]}(L_{\mathrm{def}})$.

Third, the remaining binary tangent configurations, namely (u^2, uw, w^3) with a further point, (uw, u^3, w^3), (u^2, uw^2, w^3) and (u^2, uw, w^4) with the line of Γ' meeting the germ of $\mathfrak{B}$ at $[X]$ in length three, and (u^2, uw, w^3) alone and $(uw, u^2 - w^3)$ with $\mathfrak{B}_4$ the fat point at $[X]$, all have $q \leq 2$, and we settle the whole sector $F = X^5Y$, $q \leq 2$, $D_3 \subseteq N$ by families, for every $\mathfrak{B}$. For instance, for $R \equiv X^2Y^2$ we have $P(R) = \langle X^3, X^2Y, XY^2 \rangle$ and $q = 1$, and by (3) below $D_3 \supseteq P(R) + \mathbb{C}A_{YY} = \mathbb{C}[X, Y]_3$, with A_{YY} binary and outside $P(R)$; so $\mathfrak{B}$ contains the line $z = 0$, and these algebras, which have no free cubic, belong to this sector and not to a finite base scheme. Here $D_4 = E_0 \oplus \mathbb{C}R$ with $E_0 = \langle X^4, X^3Y \rangle$, and the lifts are $\Phi = X^5Y + A + B + C$ and $\Psi = R + \tau$, with A, B, C of degrees 5, 4, 3 and $A_Z = c_1X^4 + c_2X^3Y + c_3R$. The closedness conditions are not jointly affine in all tails, but they are triangular, and we solve them in the order R, A, B.

(1) R is taken modulo E_0, in a locus with given q and given rank r_Z of $S_1 \circ R$ modulo binary cubics.
(2) For fixed R, $A_Z \in E_0 + \mathbb{C}R$ is a linear condition on A of constant rank 12, as $z\circ$ maps T_5 onto T_4; its solutions are $A = A_{\mathrm{bin}} + Z(e_1X^4 + e_2X^3Y) + c_3 \int R\, dZ$, with A_{bin} binary.
(3) D_3 is then the space N_3 of the cubic parts of the elements of degree three of N, namely $P(R) + \mathbb{C}A_{YY} + \mathbb{C}g$ if $c_3 = 0$, where

$g = B_Z - c_1A_{XY}/5 - c_2A_{XX}/20$, and $P(R) + \mathbb{C}A_{YY}$ if $c_3 \neq 0$; the other closedness conditions then hold, with τ determined modulo D_3 when $c_3 \neq 0$. The condition $\dim N_3 = 5$ splits the algebras into pieces: for $q = 1$, $c_3 = 0$ with A_{YY}, g independent modulo $P(R)$; for $q = 2$, (i) $c_3 = 0$, $A_{YY} \notin P(R)$ and $g \in P(R) + \mathbb{C}A_{YY}$, an affine condition on B of constant rank 5; (ii) $c_3 = 0$, $A_{YY} \in P(R)$, a linear condition on A_{bin} of constant rank for given r_Z, and $g \notin P(R)$; (iii) $c_3 \neq 0$ and $A_{YY} \notin P(R)$.

(4) C is free, and so is τ when $c_3 = 0$.

So each piece is an open subset of a tower of affine bundles of constant rank over a locus of R. These loci are irreducible over $\mathbb{C}$: the locus $q \leq 2$ is the image of a vector bundle of rank five over $\mathbb{P}^2$, as $v\circ$ maps T_4/E_0 onto $T_3/\langle X^3, X^2Y\rangle$ for every $v \neq 0$; the locus $r_Z = 0$ of binary R is linear; and the loci $q \leq 1$ and $\{q \leq 2,\ r_Z \leq 1\}$ are irreducible over $\mathbb{Q}$ (script `l21_1365321_b2t`, part (U6)) and contain a smooth rational point, which lies on only one component over $\mathbb{C}$, and on all its Galois conjugates (script `l21_1365321_irr`, parts (P1), (P2)). So Lemma 3.13 applies with one certified member per piece (`l21_1365321_b2t`, parts (A2), (A7)).

Base schemes containing a line. The same family method settles the binary tangent and secant tops with $q \leq 2$ (scripts `l21_1365321_b2t`, `l21_1365321_b2s`; for the secant top with $\langle X^4, Y^4\rangle$ and $\langle X^3, Y^3\rangle$ above) and with $q = 3$, where $D_3 = P(R)$ and the algebras form affine bundles over irreducible bases (`l21_1365321_b23`). For pure tops, [15, Lemma 5.13] with $k = 4$ gives a ray along a parabola $y = \beta_2x^2$, $z = \gamma_2x^2$ whenever G can be chosen without the terms X^4Y, X^4Z and $x^2 \circ \tilde{P} \in D_3$, where $\tilde{P}$ is the quintic tail of the lift of X^6 without X^5, X^4Y, X^4Z. Over a line the second condition always holds, and a change $x \mapsto x+ay$ achieves the first, except in three cases: the tangent pencil, the datum $D_5 = \langle X^5, X^4Y + Z^5\rangle$ of P2, and a point of $\mathfrak{B}$ embedded in the line (scripts `l21_1365321_rays`, `l21_1365321_r1s`). The datum $\langle X^5, X^4Y + Z^5\rangle$ forms one irreducible family for each of two normal forms, with members of local length 35 (`l21_1365321_p2g`): here $D_3 \subseteq N$ means that three cubics h_1, h_2, h_3, affine in the tails, span a plane modulo $S_1 \circ D_4$, and as the linear part of every nonzero combination $\sum \lambda_i h_i$ has rank seven (part (U1)), these algebras form an open subset of the image of a vector bundle over $\mathbb{P}^2$. With a point embedded in the line, P2 still has rays unless the point lies at $[X]$ and G contains X^4Z (`l21_1365321_r1s`), and the secant pencil has rays except for one datum, with an anchor at infinity of local length 41 (`l21_1365321_r1a`).

The rays leave the tangent pencil, and P2 with a point embedded at $[X]$ and X^4Z in G. Here the graded data form finitely many orbits and three families with moduli (rows 23–25 of Table 3). On an orbit the graded data stay fixed, except that some free algebras are moved into other orbits. The orbits certified in `l21_1365321_r1a`, `l21_1365321_r1b` and `l21_1365321_r1f2` have anchors at infinity of local length 40 or 41 (row 20). In seven further

orbits the graded tests give 42–44 (rows 21, 22): $R = Z^4$, Y^4, X^2Y^2 over the line $z = 0$ and the point $[Z]$; $R = X^3Z$, Y^4, X^2Y^2 over this line with a point embedded at $[X]$; and $R = X^3Z$ over the line $y = 0$ with a point embedded at $[X]$. In the two orbits $R = Y^4$, X^2Y^2 over the line $z = 0$ with a point embedded at $[Y]$, where $q = 1$, they give 44 and 42; these two belong to row 23. There Lemma B.20 moves the *free* algebras, whose quartic lift is not a derivative of the sextic one, into settled orbits (`l21_1365321_r1fq`). The non-free ones form irreducible families, for fixed R and every $D_3 \subseteq N$, with members of local length 35–39 (`l21_1365321_p1t`; the $\mathbb{P}^1$-bundles: `l21_1365321_bundle`, parts (U0), (U1)). Indeed, $D_3 \subseteq N$ means that two cubics h_3, h_4, linear in the tails, are independent modulo $P(R)$ and a third cubic h' lies in their span, or that h_3, h_4 are dependent and h' does not. Here h' involves the quartic tail of the sextic lift through a nonzero operator $\mu' y - \lambda' z$, and h_3, h_4 are dependent exactly when $\mu h_3 - \lambda h_4 \in P(R)$ for some $[\lambda : \mu] \in \mathbb{P}^1$, which involves the quartic tail of the lift of X^4Y through $\mu y - \lambda z$. As every nonzero $\mu y - \lambda z$ maps T_4 onto T_3, both pieces are open subsets of vector bundles over irreducible bases, the second one over $\mathbb{P}^1$.

The families with moduli are settled by Lemma 3.13, with D_3 moving. First, let R be binary and $D_3 \supseteq \mathbb{C}[X, Y]_3$ in the tangent pencil; this covers the line $z = 0$ in case (d) of [15, Lemma A.12] and the family $R = lX^2Y^2 + XY^3 + Y^4$ with a point embedded at $[Y]$. Here $q(R) = 2$ unless R is Y^4 or X^2Y^2 up to coordinates. For $q(R) = 1$ the non-free algebras lie in the families of `l21_1365321_p1t` above (anchors 35 and 36), and Lemma B.20 with $\Omega = X^2Y^2$ or Y^4 moves the free ones to $q(R) = 2$ (`l21_1365321_F1F8_f1fr`, part (U9)). For $q(R) = 2$ the non-free algebras form two families, irreducible over $\mathbb{C}$ by the same $\mathbb{P}^1$-argument (over $\mathbb{A}^1 \times \mathbb{P}^1$ for the family in l), with members of local length 40 (`l21_1365321_F1F8_f1nf`, `l21_1365321_f4`; the bundles: `l21_1365321_bundle`, parts (U1)–(U3)); the free algebras are the images of four components of an incidence variety in eleven variables, each of dimension six (`l21_1365321_F1F8_f1fr`, part (U7)), with members of local length 40, 40, 41, 40. Each component is geometrically integral of dimension six. One is linear. For each of the other three, inverting one of the eleven coordinates gives a graph over six free coordinates, and the component ideal is the contraction of this graph ideal (script `l21_1365321_F1F8_abs`, part (U4); its parts (U2), (U3), absolute primary decompositions, confirm this). Second, for `P2` with $G = lX^3Y^2 + X^2Y^3 + Y^5 + X^4Z$, the algebras form two irreducible families over the l-line, open subsets of vector bundles over irreducible bases, with members of local length 34 (`l21_1365321_F2_family`). Third, in the tangent pencil over the line $y = 0$ with a point embedded at $[X]$ and $R_m = mX^3Z + X^2Z^2 + Z^4$, the non-free algebras form two families over the m-line, irreducible by the $\mathbb{P}^1$-argument over $\mathbb{A}^1 \times \mathbb{P}^1$, with members of local length 35. The free ones move with m by Lemma B.20 with $\Omega = X^3Z$, as $R_m + tX^3Z = R_{m+t}$; and Lemma 3.9, with $\ell = x + y$, the quartic X^2YZ

in place of R, and $C = 0$, gives the elements

$$xyz + y^3 + v_a - v_b - \tfrac{1}{360}(x^5 + 5x^4y), \qquad xy^2 - yz^2 - v_b, \qquad y^2z + v_a$$

of $\mathrm{Ann}(M_\infty)$, with $v_a = 2x^3z - 3mx^2z^2$ and $v_b = 6x^2z^2 - z^4$, of local length 40 at $m = 17/5$ and so for all but finitely many m (`l21_1365321_F1F8_f8`).

Case index. Table 3 lists the cases above, in the order of the proof; each algebra belongs to the first row that applies, and the last column names the script parts with the certificates (a script name `_x` there stands for `l21_1365321_x`). The computational supplement lists every configuration and family with its data (file `B28_COVERAGE.md`). The 157 flags and the 26 configurations index the local possibilities over a finite $\mathfrak{B}$; the 25 rows of the table form an ordered list that also covers infinite base schemes and the families with moduli.

Table 3: The cases of Theorem B.28. Here E_3 is the multiplicity of $\mathfrak{B}$ at the points of weight three, σ the socle dimension of $\mathfrak{B}$ at $[X]$, and the bound is the local length of the complete intersection used.

	case	method	bound	certificate
all sectors				
1	$D_3 \not\subseteq N$ (a free cubic)	reduction to $(1,3,6,4,3,2,1)$ [15, Lemma A.15], [16, Theorem C.11], [16, Remark 3.7]	≤ 39	—
2	B3, $\mathfrak{B}$ finite	excess; no point of weight three	≤ 39	—
3	B3, $\mathfrak{B} \supseteq V(z)$	one family, tower of affine bundles; anchor	< 42	`_B3` (A)
$D_3 \subseteq N$, *sectors other than B3,* $\mathfrak{B}$ *finite*				
4	$\mathfrak{B}$ finite, $E_3 \leq 2$	excess	≤ 41	—
5	$\mathfrak{B}$ finite, $E_3 \geq 3$, flag not among the 26	Proposition 3.7(c), local values over the 157 flags	≤ 41	`_wm` (U1)–(U3)
the 26 *configurations left by row 5*				
6	pure top, $\ell(\mathfrak{B}) = 5$, $\sigma \geq 2$ at $[X]$	free quartic tails, rank $\geq 1 + \sigma$	39	`_cfg` (S3)
7	pure top, (u^2, uw, w^3) alone	free quartic tails, rank three	37	`_cfg` (S6)
8	P2, $(uw, u^2 - w^3)$	$\mathfrak{B}_4$ not fat; Proposition 3.7(c)	41	`_cfg` (U1), `_wm` (U1), (U4)
9	pure secant pencil, heavy germ at $[Y]$	Lemma B.27, $\alpha = b^5$, $\ell = b$	≤ 41	`_cfg` (A2)
10	$F = X^6 + Y^6$, the remaining binary secant configurations	Lemma B.27, $\alpha = x^5$, $\ell = x$; $\varepsilon_p = e_p$	≤ 41	`_tq_secant` (U1), (U2), (A3)
11	pure tangent pencil, $(uw, u^2 - w^3)$	Lemma B.27, $K' = \nu_4(Z)$, $\varepsilon_{[X]} \leq 7$	41	`_cfg` (U8)
12	tangent pencil or top, (u^2, uw, w^3) alone or $(uw, u^2 - w^3)$, $\mathfrak{B}_4$ not fat	Proposition 3.7(c), local value ≤ 14	≤ 41	`_wm` (U1), (U4)

Table 3, continued

	case	method	bound	certificate
13	$F = X^5Y$, $\ell(\mathfrak{B}) = 5$, the line of Γ' meets $\mathfrak{B}_{[X]}$ only in Γ'	Lemma B.27, $\alpha = x^5 + \beta x^4 y$, $\ell = y$	≤ 41	`_cfg` (A7)
binary tops, every base scheme not treated above				
14	binary top, $q \leq 2$, not above (all other binary tangent flags have $q \leq 2$)	families in the order R, A, B; Lemma 3.13	≤ 41	`_b2t` (U6), (A2), (A7); `_b2s` (U1), (A2); `_irr`
15	binary top, $q = 3$, $\mathfrak{B}$ infinite	affine bundles over $\mathcal{S}_1$, $\mathcal{S}_2$; Lemma 3.13	41	`_b23` (U1), (U2), (A3)
pure tops, $\mathfrak{B}$ containing a line				
16	pure top, $\mathfrak{B} \supseteq$ line, none of the next rows	ray along a parabola [15, Lemma 5.13]	ray	`_rays` (U1)–(U3), `_r1s` (U)
17	P2, $D_5 = \langle X^5, X^4Y + Z^5 \rangle$	two families over $\mathbb{P}^2$-bundles	35	`_p2g` (U1), (A2)
18	P2, embedded point, unless it is at $[X]$ and $X^4Z \in G$	rays	ray	`_r1s` (P2eX), (P2eY)
19	secant pencil, embedded point	rays; one datum by an anchor at infinity	ray, 41	`_r1s`, `_r1a` (Sx)
20	tangent pencil, or P2 with the point at $[X]$ and $X^4Z \in G$: the orbits with an anchor in the last column	anchors at infinity	40, 41	`_r1a`, `_r1b`, `_r1f2`
21	the seven orbits with graded value 42–44 listed in the text, free	Lemma B.20 into settled orbits	—	`_r1fq`
22	these seven orbits, non-free	families over $\mathbb{P}^1$-bundles	35–39	`_p1t` (U1), (A2); `_bundle` (U0), (U1)
23	tangent pencil, R binary, $D_3 \supseteq \mathbb{C}[X, Y]_3$	$q = 1$ ($R = Y^4$, X^2Y^2): non-free, the families of row 22; free, Lemma B.20 into $q = 2$; $q = 2$: two non-free families and four free components	35, 36; 40, 41	`_p1t` (U1), (A2); `_F1F8_f1fr` (U9); `_F1F8_f1nf`, `_F1F8_f1fr` (U7), (U7b), (A8); `_F1F8_abs` (U4); `_f4`; `_bundle` (U1)–(U3)
24	P2, $G = lX^3Y^2 + X^2Y^3 + Y^5 + X^4Z$	two families over the l-line	34	`_F2_family`
25	tangent pencil, $R_m = mX^3Z + X^2Z^2 + Z^4$	non-free: two families; free: Lemmas B.20 and 3.9	35, 40	`_F1F8_f8`

□

Appendix C. One quadric: the remaining cases

C.1. This appendix proves Theorem 1.1 for the twenty Hilbert functions with $H_A(2) = 5$ that Section 8 leaves open. Throughout, s is the socle degree,

$I_2^* = \mathbb{C}x^2$, $f \in I$ has leading form x^2, and K_m, θ_m, $J_k = \pi(I_k^*)$, $\kappa(v)$, μ_v are as in Sections 2.8, 2.9, 2.11 and 4.2; on the blow-up we use the chart and the notation $g^\natural$, g'' of the proof of Proposition 4.4, and the length formulas of [16, Lemma 3.14]. When $\mathfrak{m}^7 \subseteq I$ we make the reductions of Section 4.2 without comment: the binary case $I_3^* = x^2 S_1 \oplus xW_2$, $\gamma_f = 0$, and f with cubic part in $x^2 S_1$. Each bound below is the local length of a complete intersection containing the general member of a flat family through A; as it is less than 42, that member is smoothable (Section 2.4), and so is A [15, Proposition 2.2(1)].

C.2. A turning operator must satisfy $u \circ M \subseteq T_{\leq 1}$. One quadratic defect can be allowed, at the price of one more moving class: we keep the corresponding quadratic multiple of the moving equation as an extra class. The condition $XR_0 \in D_3$ below keeps the products of the fixed cubic classes out of that class, and this is what keeps the family an ideal of constant colength; without it, flatness can fail.

Lemma C.3 (Turning past a quadratic defect)**.** *Let $A = S/I$ be a local algebra with $I_2^* = \mathbb{C}x^2$ and $\mathfrak{m}^N \subseteq I$ for some $N \geq 5$, and let $f \in I$ have leading form x^2. Let $R \in T_2$ be nonzero, let $R_0 = R(0, Y, Z)$, and suppose that $XR_0 \in D_3$. Let $u \in \mathfrak{m}^3$ satisfy $u \circ M \subseteq T_{\leq 1} + \mathbb{C}R$. Then there is an ideal $\mathcal{I} \subseteq S[t]$ such that $S[t]/\mathcal{I}$ is a free $\mathbb{C}[t]$-module of rank $\ell(A)$, $\mathcal{I}|_{t=0} = I$, and $\mathcal{I}|_{t=t_0}$ contains $f + t_0 u$ and $\mathfrak{m}^N$ for every $t_0 \in \mathbb{C}$.*

Proof. For $h \in S$ we have $(hu) \circ G = h \circ (u \circ G)$. So $\mathfrak{m}^3 u \subseteq I$, and $\rho u \in I$ for every $\rho \in S_2$ with $\langle \rho, R \rangle = 0$. The map $x^2 \circ$ sends $X^2 T_2$ isomorphically onto T_2; let $\Xi \in X^2 T_2$ satisfy $x^2 \circ \Xi = R$, and let $\epsilon(h) = \langle h, \Xi \rangle$ for $h \in S_4$. Then $\epsilon(x^2 \rho) = \langle \rho, R \rangle$; fix $\rho_0 \in S_2$ with $\langle \rho_0, R \rangle = 1$.

Let $W \subseteq I_3^*$ be the space of elements of I_3^* with no monomial divisible by x^2; it is a complement of $x^2 S_1$. For $w \in W$ and $l \in S_1$ we have $\epsilon(lw) = \langle w, l \circ \Xi \rangle = 0$. Indeed, write $\Xi = X^2 \Xi'$. Then $y \circ \Xi$ and $z \circ \Xi$ lie in $X^2 T_1$, which pairs to zero with w. Comparing X-free parts in $R = x^2 \circ (X^2 \Xi')$ shows that the X-free part of Ξ' is $R_0/2$, so $x \circ \Xi \equiv XR_0$ modulo $X^2 T_1$, and $\langle w, XR_0 \rangle = 0$ as $XR_0 \in D_3$.

Choose $g_w \in I$ with cubic part exactly w, linearly in $w \in W$, and let

$$K = \operatorname{span}\{g_w : w \in W\} + \{h \in I \cap \mathfrak{m}^4 : \ \epsilon(h_4) = 0\}.$$

It is an ideal, by the previous paragraph and since ϵ vanishes on $\mathfrak{m}^5$. It contains $\mathfrak{m}^N$, and $I = K \oplus \operatorname{span}\{f, xf, yf, zf, \rho_0 f\}$, as $\epsilon(\rho_0 f) = 1$. Put $Q = f + tu$ and $\mathcal{I}|_t = K + \mathbb{C}Q + S_1 Q + \mathbb{C}\rho_0 Q$. For $\rho \in S_2$ with $\langle \rho, R \rangle = 0$ we have $\rho f \in K$, as $\epsilon(\rho f) = \langle \rho, R \rangle = 0$, and $\rho u \in I \cap \mathfrak{m}^5 \subseteq K$; and $\mathfrak{m}\rho_0 Q \subseteq I \cap \mathfrak{m}^5 \subseteq K$. So $\mathcal{I}|_t$ is an ideal. The sum is direct for every t, by comparing the parts of degree two and three and the value of ϵ on the quartic parts; so the colength is constant, and [15, Lemma 2.3(1)] gives freeness. $\square$

With $R = 0$ the same construction, without ρ_0, is the turning family [15, Lemma 4.2]. Without $XR_0 \in D_3$, ϵ does not vanish on S_1W, and K is not an ideal.

Corollary C.4. [15, Theorem 4.5], [15, Corollaries 4.6 and 4.8] *and* [15, Lemma F.4] *hold for every* $u \in \mathfrak{m}^3$ *as in Lemma* C.3 *in place of an element of* $(I : \mathfrak{m}^2) \cap \mathfrak{m}^3$.

Proof. Their proofs use u only through the members of the turning family: a flat family through I whose member at t_0 contains $f + t_0 u$ and $\mathfrak{m}^N$. Lemma C.3 provides such a family. (In [15, Corollary 4.8], u is also replaced by $u + bz^5$, which again satisfies the hypothesis of Lemma C.3, as $\mathfrak{m}^7 \subseteq I$.) □

Corollary C.5. *Let* $A = S/I$ *be a local algebra with* $I_2^* = \mathbb{C}x^2$, $\mathfrak{m}^8 \subseteq I$, $D_7 = \mathbb{C}Y^7$ *and* $XY^2 \in D_3$, *in coordinates with* $[Y] = V(x, z)$.

(a) *If some* $f \in I$ *with leading form* x^2 *has* $\gamma_f = kz^3$ *with* $k \neq 0$, *then* A *is smoothable.*

(b) *If* $\gamma_f = 0$ *and some* $u \in \mathfrak{m}^3$ *with* $u \circ M \subseteq T_{\leq 1} + \mathbb{C}Y^2$ *has* $\pi(u_3) \neq 0$, *then* A *is smoothable.*

No induction is used.

Proof. Let $\Phi = Y^7 + \cdots$ lie in M, so that $M = \mathbb{C}\Phi \oplus (M \cap T_{\leq 6})$. Every element of M of degree at most six is mapped by y^5 into $T_{\leq 1}$, and $y^5 \circ \Phi \in 2520\, Y^2 + T_{\leq 1}$; so $u = y^5$ satisfies Lemma C.3 with $R = Y^2$. Every binary quintic T_5 with $T_5(Y) = 0$ lies in $(I : \mathfrak{m}^2)$, as $T_5 \circ Y^7$ is a multiple of $T_5(Y)Y^2$.

(a) In the coordinates of [15, Theorem 4.5], which fix y and z, we have $u(0, y, z) = y^5$ and $\partial_x u = 0$, so $G_t = g + ty^5$ has cubic part kz^3. If the coefficient of y^4 or of $y^3 z$ in g is nonzero, case (2) or (3) of the theorem, with y and z exchanged, holds for all t; otherwise case (4) holds for $t \neq -g_{y^5}$. Corollary C.4 concludes.

(b) For $s \in \mathbb{C}$ and T equal to y^5 or to a quintic T_5 as above, $u_s = u + sT$ still satisfies Lemma C.3 with $R = Y^2$, and $\partial_x u_s = \partial_x u$. The cubic part of G_t is $t\,\pi(u_3)$. If $\pi(u_3)$ is not a cube, case (1) holds for $t \neq 0$. If $\pi(u_3) = l^3$ with root $[V] \neq [Y]$, take $T = T_5$ with $T_5(V) \neq 0$, and argue as in the proof of [15, Corollary 4.8] in coordinates (l, l') with $l'(V) \neq 0$: the coefficients of l'^4 and ll'^3 in G_t are polynomials in t; if one of them is not identically zero, case (2) or (3) holds for all but finitely many t, and otherwise the coefficient of l'^5 in case (4) is $c(t) + st\, T_5(V)c'$ with $c' \neq 0$, a nonzero polynomial in t for all but one s. If $\pi(u_3) = cz^3$ with $c \neq 0$, the same argument works with $T = y^5$. In each case Corollary C.4 concludes. □

Lemma C.6 (Bad roots)**.** *Let* $\mathfrak{m}^6 \subseteq I$, *in the binary case with* $\gamma_f = 0$, *and let* $0 \neq g \in J_4$, $0 \neq m \in W_2$. *For a root* v *of* g *let* $\omega_v = \min(\operatorname{ord}_v m, \operatorname{ord}_v \theta_m)$. *The general member of a turning family of* A *lies in a complete intersection of local length at most* $32 + 4\sum_v \omega_v$, *which is at most* 40.

Proof. As in [16, Corollary B.2(c)], the turning family of z^4, $z \in \bar{S}_1$ vanishing at no root of g, contains $f + tz^4$, K_m and $G + \nu z^6$, $G \in I \cap \mathfrak{m}^4$ with leading

form of binary part g, which give $2 \cdot 3 \cdot 4$ plus $2\mu_v$ at each root with $\omega_v = 0$. At a root v with $\omega = \omega_v > 0$, in the analogous chart $x = \rho u'$, $y = \rho\eta$, $z = \rho$, $(f + tz^4)^\natural$ is, for general t, a unit times $(u' - \rho\alpha_+)(u' - \rho\alpha_-)$. On the branch $u' = \rho\alpha_\pm$, $K_m^\natural = \rho H_\pm$, where $H_\pm(\eta, 0) = \alpha_\pm(\eta, 0)m + \theta_m$ vanishes to order exactly ω at v, so the branch contributes $\mu_v + \ell(B_\pm/(G^\natural + \nu\rho^2)) \leq \mu_v + 2\omega$, by [16, Lemma 3.14(b)] for $B_\pm = \mathbb{C}\{\eta - c, \rho\}/(H_\pm)$, free of rank ω. So $\lambda_v \leq 2\mu_v + 4\omega_v$, and $\sum_v \omega_v \leq \deg m = 2$. □

Lemma C.7 (Weights 4, 4, 5)**.** *Let $\mathfrak{m}^6 \subseteq I$, in the binary case with $\gamma_f = 0$. Let $G \in I$ have order five and a leading form with nonzero binary part g, and let $m \in W_2$ satisfy $(m(v), \theta_m(v)) \neq (0, 0)$ at every root v of g. Then the general member of a turning family of A lies in a complete intersection of local length at most $4 \cdot 4 \cdot 5/2 = 40$.*

Proof. As in the proof of Proposition 2.12: for $b \in \bar{S}_1$ vanishing at no root of g, the turning family of b^4 contains $f + tb^4$, K_m and G, with weighted initial forms $x^2 + \beta + tb^4$, $xm + \theta_m$ and g (x of weight two, β the binary part of f_4). At a common zero $(x_0, v) \neq 0$ we have $v \neq 0$ and $g(v) = 0$, the second form fixes x_0, and the first fails for general t. [16, Lemma 3.11(a)] concludes. □

Lemma C.8 (Two quartics)**.** *Let $P, Q \in I$ have order four and leading forms p, q with $\pi(p) \neq 0$, and let $N \in I$ have order $k \geq 5$ and a leading form vanishing at no common root v of $\pi(p)$ and $\pi(q)$. Then for general ν, $V(f, P, Q + \nu N)$ has local length at most $32 + 2(k-4)\sum_v \mu_v$, where μ_v is the multiplicity of v as a root of $\pi(p)$.*

Proof. As in [16, Corollary B.2(b)]: the tangent cones meet at the common roots, which gives $2 \cdot 4 \cdot 4$; at such a root the local ring of $V(f^\natural, P^\natural)$ is free of rank $2\mu_v$ over $\mathbb{C}\{\rho\}$ (proof of Proposition 4.7); and $(Q+\nu N)^\natural = Q^\natural + \nu\rho^{k-4}N^\natural$ with $N^\natural$ a unit, so [16, Lemma 3.14(b)] gives $\lambda_v \leq 2(k-4)\mu_v$. □

Lemma C.9 (Quintic radial costs)**.** *Let $\mathfrak{m}^7 \subseteq I$, $I_3^* = xS_2$, $\gamma_f = 0$, let $P \in I$ have order five and a binary leading form p, let $b \in \bar{S}_1$ vanish at no root of p, and let $K \in I$ have cubic part xb^2.*

(a) *If a binary quartic $h \in (I : \mathfrak{m}^2)$ vanishes at no root of p, then for general t the member of the turning family of h lies in $V(f + th, K, P)$, of local length at most $30 + 2 \cdot 5 = 40$.*

(b) *Let $z \in \bar{S}_1$ vanish at a root v_0 of p, $b^3 z \in (I : \mathfrak{m}^2)$, and $\mu_{v_0} = 1$ or $p_6(v_0) \neq 0$ for the sextic part p_6 of P. For general (σ, τ) the general member of the turning family of $\sigma b^3 z + \tau b^5$ lies in a complete intersection of local length at most $30 + 2 \cdot 5 + 1 = 41$.*

Proof. The tangent cones meet at the roots of p, which gives $2 \cdot 3 \cdot 5$. In the chart $x = \rho u'$, $b = \rho$, $z = \rho w$, K eliminates u', $\mathbb{C}\{w - c, \rho\}/(P'')$ is free of rank μ_v, and $f'' = \rho^2\psi$. (a) $(f + th)'' = \rho^2(\psi + t\, h(1, w))$ with $h(1, w)$ a unit, so $\lambda_v \leq 2\mu_v$. (b) $(f + \sigma b^3 z + \tau b^5)'' = \rho^2 H$, $H = \psi + \sigma w + \tau\rho$, a unit at the roots $w \neq 0$ for general σ. At v_0 ($w = 0$), if $\psi(0, 0) = 0$, then for general σ we get $H = (\text{unit})(w - \varphi(\rho))$ with $\varphi'(0) = -(\partial_\rho\psi(0, 0) + \tau)/(\partial_w\psi(0, 0) + \sigma)$,

and $\lambda_{v_0} = 2\mu_{v_0} + \operatorname{ord}_\rho P''(\varphi(\rho), \rho)$. As $P'' = p(1,w) + \rho\,\pi(p_6)(1,w) + O(\rho^2)$, the coefficient of ρ here is $\partial_w p(1,0)\varphi'(0) + \pi(p_6)(1,0)$, nonzero for general τ under either hypothesis. So $\lambda_{v_0} \le 2\mu_{v_0} + 1$ off a proper closed set of (σ, τ), which the line $t(\sigma_0, \tau_0)$ meets in finitely many points for general (σ_0, τ_0). □

C.10. *Socle degree five.* Here $\mathfrak{m}^6 \subseteq I$; a quintic orthogonal to D_5 lies in I, and a cubic h with $h \circ D_5 = 0$ lies in $(I : \mathfrak{m}^2)$ [15, remark before Corollary 4.9].

Theorem C.11. *Every local algebra with Hilbert function* $(1,3,5,7,3,2)$ *is smoothable.*

Proof. Two quartics link A, unless their common factor is a cubic turning operator. Here $I_3^* = x^2 S_1$, and $\dim J_4 \ge 12 - 10 = 2$, as $\dim I_4^* = 12$ and $\dim xS_3 = 10$. Let $h = \gcd J_4$, of degree $\delta \le 3$. If $\delta \le 2$, two general elements of J_4 have only the roots of h in common, of total multiplicity δ, and Lemma C.8 with $N = l^6$, l vanishing at no root of h, gives $32 + 4\delta \le 40$. If $\delta = 3$, then $J_4 = h\bar{S}_1$ has dimension two, so $\dim(I_4^* \cap xS_3) = 10$ and D_4 is binary; so is D_5, as an X-term in $F \in D_5$ would give one in $y \circ F$ or $z \circ F$. Hence $hS_2 \subseteq xS_4 + S_1 \cdot h\bar{S}_1 \subseteq I_5^*$, that is, $h \circ D_5 = 0$: h is a turning operator of type (b). □

Theorem C.12. *Every local algebra with Hilbert function* $(1,3,5,6,4,2)$ *is smoothable.*

Proof. As $\dim I_3^* = 4$, $W_2 = \mathbb{C}r_2 \ne 0$; as $\dim I_4^* = 11 > 10 = \dim xS_3$, $J_4 \ne 0$. Lemma C.6 applies. □

Theorem C.13. *Every local algebra with Hilbert function* $(1,3,5,5,5,2)$ *is smoothable.*

Proof. A slide moves A off the one stratum where weighted linkage fails. Here W_2 is a pencil. If $J_4 \ne 0$, Lemma C.6 applies. If $J_4 = 0$, then $I_4^* = xS_3$ (both have dimension 10), so $D_4 = \bar{T}_4$, D_5 is binary, θ_m is well defined, and $J_5 = D_5^\perp \cap \bar{S}_5 \subseteq I_5^*$ has dimension four. Lemma C.7 applies with $0 \ne g \in J_5$ and m vanishing at no root of g, if W_2 has no common zero, or if $W_2 = z\bar{S}_1$, $z(Y) = 0$ and $g(Y) \ne 0$ for some g; and with a general $m = zl'$ if some $\theta_m(Y) \ne 0$. There remains $W_2 = z\bar{S}_1$, $D_5 = \langle Y^5, G\rangle$, $\theta_{yz}(Y) = \theta_{z^2}(Y) = 0$. If $z \circ G \in \mathbb{C}Y^4$, $yz^2 \circ D_5 = 0$, and yz^2 is a turning operator of type (b). Otherwise take $Z_4 \in \bar{S}_4$ with $\langle Z_4, Y^4\rangle \ne 0 = \langle Z_4, z \circ G\rangle$. As xZ_4, $zZ_4 \perp D_5$ lie in I, $\varphi(yz) = Z_4$, $\varphi(z^2) = 0$ satisfies Lemma 4.6 with $E = 0$. By Lemma 4.6 the members have the tangent cone of A, hence the Hilbert function H_A and $J_4 = 0$; for $\tau \ne 0$, $\theta_{yz} + \tau Z_4$ does not vanish at $[Y]$, so they are smoothable by the case above, and so is A. □

Theorem C.14. *Every local algebra with Hilbert function* $(1,3,5,5,4,3)$ *is smoothable.*

Proof. Here W_2 is a pencil and $J_4 \ne 0$, as $\dim I_4^* = 11 > 10$. Lemma C.6 applies. □

Theorem C.15. *Every local algebra with Hilbert function* $(1,3,5,4,5,3)$ *is smoothable.*

Proof. Here $I_3^* = xS_2$, $D_4 = \bar{T}_4$, and $J_5 = D_5^\perp \cap \bar{S}_5 \subseteq I_5^*$ has dimension $6-3=3$. Lemma C.7 applies with $0 \neq g \in J_5$ and $m \in \bar{S}_2$ vanishing at no root of g. □

Theorem C.16. *Every local algebra with Hilbert function* $(1,3,5,4,4,4)$ *is smoothable.*

Proof. Here $I_3^* = xS_2$ and $H_A(4) \leq 4$, so Proposition 2.12 applies (local length 32). □

C.17. *Socle degree six.* Here $D_6 = \mathbb{C}F$ and $M = \mathbb{C}\Phi \oplus (M \cap T_{\leq 5})$ for a lift Φ of F; so a sextic orthogonal to F lies in I, and $u \in \mathfrak{m}^3$ lies in $(I : \mathfrak{m}^2)$ if and only if its cubic part kills D_5 and $u \circ \Phi \in T_{\leq 1}$. If $H_A(4) \leq 4$, some $P \in I$ has a quartic leading form p_4 with $p = \pi(p_4) \neq 0$, as $\dim I_4^* > 10 = \dim xS_3$.

Theorem C.18. *Every local algebra with Hilbert function* $(1,3,5,6,3,2,1)$ *is smoothable.*

Proof. Here $W_2 = \mathbb{C}r_2$. If F is binary, we slide: let $\varphi(r_2) = w$, a binary quintic vanishing at no root of p; as $xw \perp F$, $xw \in I$, and Proposition 4.7 with $k=5$ gives $24 + 8 \cdot 2 = 40$. Otherwise a corrected cubic is a turning operator. As $\dim S_1 \circ F \leq 2$, a linear change keeping $\mathbb{C}x^2$ and the binary case gives $F = XY^5$, $D_5 = \langle Y^5, XY^4 \rangle$, and $r_2 \in \{yz, z^2\}$, as $xr_2 \circ F = 0$. Then $T = T_3 - \frac{A_0}{60}xy^3 - \frac{B_0}{120}y^4$ is a turning operator of type (b) if T_3 is a cubic divisible by z with $\pi(T_3) \neq 0$ and $T_3 \circ P_5 = A_0Y^2 + B_0XY$ for the quintic tail P_5 of Φ, as $xy^3 \circ XY^5 = 60Y^2$ and $y^4 \circ XY^5 = 120XY$. As $z \circ P_5 \in D_4$, $x^2 \circ P_5 \in D_3$ and $x^2, xr_2 \in I^*$, the operators x^4, x^2z, xr_2z kill P_5, and so, modulo D_5,

$$P_5 = \lambda_0 X^3Y^2 + \mu_0 X^2Y^3 + X(\nu_0 Y^4 + \sigma Y^3 Z + \rho Z^4) + B_5,$$

with B_5 binary, if $r_2 = yz$, and the same with ρY^2Z^2 in place of ρZ^4 if $r_2 = z^2$ (part (N1)). Write $h \circ B_5 = \beta_1 Y^2 + \beta_2 YZ + \beta_3 Z^2$ for a binary cubic h, and take h and T_3 as follows:

	$h \in$	conditions	T_3
$\rho = 0$	$z\bar{S}_2$	$\beta_2 = \beta_3 = 0$	h
$\rho \neq 0,\ r_2 = yz$	$\langle y^2z, yz^2 \rangle$	$\beta_2 = 0$	$12\rho\, h - \beta_3\, xz^2$
$\rho \neq 0,\ r_2 = z^2$	$\langle yz^2, z^3 \rangle$	$\beta_3 = 0$	$4\rho\, h - \beta_2\, xyz$

In each row there are fewer homogeneous linear conditions than unknowns, so some $h \neq 0$ satisfies them; then $T_3 \circ P_5 \in \mathbb{C}Y^2 + \mathbb{C}XY$, and $\pi(T_3)$ is h, $12\rho\, h$, $4\rho\, h$, which is nonzero (script `121_1356321`, part (N)). □

Theorem C.19. *Every local algebra with Hilbert function* $(1,3,5,5,4,2,1)$ *is smoothable.*

Proof. We turn by a power, find a turning operator, or slide. Here W_2 is a pencil. If it has no common zero, Proposition 4.4 gives $24 + 4 \cdot 3 = 36$. Let $W_2 = z\bar{S}_1$ with $z(Y) = 0$, so $D_k \subseteq \bar{T}_k \oplus \mathbb{C}XY^{k-1}$. If F is not binary, then $F = XY^5$ as in the proof of Theorem C.18, P_5 has no term XZ^4 or XY^2Z^2, as $z \circ P_5 \in D_4 \subseteq \bar{T}_4 \oplus \mathbb{C}XY^3$, and the operator T there works, with $0 \neq T_3 \in z\bar{S}_2$ (script `l21_1355421`, part (b)). If F is binary, take $d \in \bar{S}_1 \setminus 0$ with $d \circ F$ zero or not a fifth power ($d(L) = 0$ if $F = L^6$; otherwise the first derivatives of F span a plane, which the cone of fifth powers does not contain). Then $(d \circ F)^\perp \subseteq \bar{S}_5$ has no base point and contains T_5 vanishing at no root of p, and xT_5, $dT_5 \perp F$ lie in I. For a linear form λ on $\bar{S}_1$ with kernel $\mathbb{C}d$, $\varphi(zl) = \lambda(l)T_5$ satisfies Lemma 4.6: if $\sum_i c_i z l_i = 0$, then $d \mid \sum_i \lambda(l_i)c_i$. Proposition 4.7 with $E = \mathbb{C}p_4$ and $k = 5$ gives 40 (script `l21_1355421_slide`). □

Theorem C.20. *Every local algebra with Hilbert function* $(1, 3, 5, 5, 3, 3, 1)$ *is smoothable.*

Proof. As for $(1, 3, 5, 5, 4, 2, 1)$, with three normal forms of F. Here W_2 is a pencil. If it has no common zero, Proposition 4.4 gives 36. Let $W_2 = l_0\bar{S}_1$ with zero $[V_0]$; then $x \circ F \in \mathbb{C}V_0^5$, as $(xl_0l) \circ F = 0$. If F is binary, $\varphi(l_0l) = l\ell_4$, with $\ell_4 \in \bar{S}_4$ vanishing at no root of p, satisfies Lemma 4.6, as $xl\ell_4 \perp F$, and Proposition 4.7 with $m = l_0l_1$, l_1 vanishing at no root of p, gives 40. Otherwise take $V_0 = Y$, so $F = XY^5 + G_6$ with G_6 binary. As D_4, of dimension three, contains Y^4 and $y^2 \circ F \notin \bar{T}_4$, the quintic $H = z \circ G_6$ has $\dim\langle Y^4, y \circ H, z \circ H\rangle \leq 2$, so it is zero, $aY^5 + bL^5$ or $aY^5 + bY^4L$. Integrating, after $X \mapsto X + \alpha Y + \beta Z$ (which keeps $\mathbb{C}x^2$, I_3^* modulo x^2S_1, the cubic part of f in x^2S_1 and $\gamma_f = 0$) and a change of y fixing $[Y]$, F is $XY^5 + Z^6$, $XY^5 + Y^4Z^2$ or XY^5. A turning operator of type (b) is then $T_3 + T_4$, with $T_3 \in \langle y^2z, yz^2\rangle$, resp. $\langle yz^2 - \frac{2}{5}xy^2, z^3\rangle$, subject to one linear condition, and $T_4 \in S_4$; for $F = XY^5$ it is yz^2 or z^3, as D_5 contains Z^5 or Y^4Z modulo $\langle Y^5, XY^4\rangle$ (script `l21_1355331`, part (N)). □

Theorem C.21. *Every local algebra with Hilbert function* $(1, 3, 5, 4, 5, 2, 1)$ *is smoothable.*

Proof. As $I_3^* = xS_2$ and $D_4 = \bar{T}_4$, there is no binary quartic equation; we use Lemma C.9, with $P \in I$ of leading form p in the four-dimensional $J_5 = D_5^\perp \cap \bar{S}_5$. A binary quartic lies in $(I : \mathfrak{m}^2)$ if and only if it kills F, which is binary. If F is not a sixth power, $D_5 = S_1 \circ F$. If moreover J_5 has no common zero, take $0 \neq h \in \bar{S}_4$ with $h \circ F = 0$ and p vanishing at no root of h: Lemma C.9(a) gives 40. If J_5 has a common zero $[V]$, then $V^5 \in S_1 \circ F$, so $F = Y^6 + Z^6$ or Y^5Z in suitable coordinates, and the turning operator of the proof of [16, Theorem B.10] applies verbatim: that proof uses only $D_3 = \bar{T}_3$, $D_4 \subseteq \bar{T}_4$ and $D_5 = S_1 \circ F$ (script `120_135_s6` of [16], part (421)). Let $F = Y^6$, $z(Y) = 0$; then J_5 vanishes at $[Y]$, and $b^3z \in (I : \mathfrak{m}^2)$. If some $p \in J_5$ has a simple root at $[Y]$, Lemma C.9(b) gives 41. Otherwise $J_5 = z^2\bar{S}_3$, both being of dimension four, $D_5 = \langle Y^5, Y^4Z\rangle$, and the quintic

tail of Φ is $\alpha XY^4 + B_5$ with B_5 binary, as $x \circ \Phi$ and $z \circ \Phi$ have binary tops. If $B_5 \in D_5$, yz^2 is a turning operator of type (a). If not, take p with $\langle p, B_5\rangle \neq 0$; pairing $P = p + p_6 + \cdots$, whose quintic part is p, with Φ gives $\langle p, B_5\rangle + 720\, p_6(0,1,0) = 0$, so $p_6(Y) \neq 0$, and Lemma C.9(b) gives 41. □

C.22. *Socle degree at least seven.* If the transverse cubics $\gamma_f + \pi(I_3^*)$ of the elements of I with leading form x^2 are all cubes, then $\pi(I_3^*)$ is zero, the *binary branch*, or a line $\mathbb{C}l^3$ containing γ_f, the *cube branch* [15, proof of Theorem 4.12]; otherwise [15, Theorem 2.9] applies.

Theorem C.23. *Every local algebra with Hilbert function* $(1,3,5,5,3,2,1,1)$ *is smoothable.*

Proof. We remove nonzero transverse cubes; in the binary branch the method depends on the common zero v_0 of W_2 and on J_4. Here $D_7 = \mathbb{C}Y^7$, $D_6 = \mathbb{C}Y^6$ [15, Lemma 2.16], $[Y] = V(x,z)$, J_4 vanishes at $[Y]$ as $Y^4 \in D_4$, and $\dim J_4 \geq 12 - 10 = 2$. Let $g = \gcd J_4$, of degree δ.

Nonzero cubes. Let $\gamma_f = \alpha l^3 \neq 0$ (in the cube branch, for some f). If the root $[V]$ of l is not $[Y]$, the proof of [16, Lemma 3.9] works with a binary quintic T_5, $T_5(Y) = 0 \neq T_5(V)$, in place of z^5: $T_5 \in (I : \mathfrak{m}^2)$, as $\langle qT_5, Y^7\rangle$ is a multiple of $q(Y)T_5(Y) = 0$ for $q \in S_2$, and in coordinates (l, l') on the line with $l'(V) = 1$ the jet $G_t = g + tT_5$ of [15, Theorem 4.5] has l'^5-coefficient $g_{l'^5} + tT_5(V)$. If $[V] = [Y]$ and some element of I_3^* has a term xy^2, [15, Lemma F.4] or [15, Theorem 4.5] applies, as in the reduction of [16, Theorem B.16]; otherwise $XY^2 \in D_3$ and Corollary C.5(a) applies.

Binary branch, $\gamma_f = 0$. Here W_2 is a pencil and, after the shear of Section 4.2, f has cubic part in $x^2 S_1$. (a) If W_2 has no common zero, or J_4 does not vanish at its common zero v_0, Proposition 4.4 gives $24 + 4 \cdot 4 = 40$.

(b) If J_4 vanishes at $v_0 \neq [Y]$, we slide at v_0 and turn elsewhere. Let $W_2 = c_0 \bar{S}_1$, $p \in J_4$ general, the leading form p_4 of $P \in I$ restricting to it, and $\ell_4 \in \bar{S}_4$ vanishing at $[Y]$ and at no other root of p. Then $\varphi(c_0 l) = l\ell_4$ satisfies Lemma 4.6 with $E = \mathbb{C}p_4$: the septics of $\mathfrak{m}^2 l\ell_4$ are orthogonal to Y^7, and $xl\ell_4$ annihilates Y^7 and the sextic tail $P_6 = \alpha XY^5 + B_6$, B_6 binary, of its lift Φ (the part of degree four of $f \circ \Phi = 0$ gives $x^2 \circ P_6 = 0$, and $x \circ P_6, z \circ P_6 \in D_5$ force $x \circ P_6 \in \mathbb{C}Y^5$). As the members contain $\mathfrak{m}^8$, the turning family of b^6 in a member, b vanishing at no root of p, contains $f + tb^6$, $K_m + \tau\varphi(m)$ and P, for $m = c_0 l_1$, l_1 vanishing at no root of p. The proofs of Propositions 4.4 and 4.7 give $4\mu_v$ at a root $v \neq v_0$, where $m(v) \neq 0$, and $2 \cdot 2\mu_{v_0}$ at v_0, each on a nonempty open set of (t, τ): in total $24 + 4 \cdot 4 = 40$.

(c) If $v_0 = [Y]$, then $W_2 = z\bar{S}_1$ and $z \mid g$. If $\delta = 1$, general $p, q \in J_4$ share only the simple root $[Y]$, and Lemma C.8 with $N = y^8$ gives $32 + 2 \cdot 4 = 40$.

(d) If $v_0 = [Y]$ and $\delta = 2$, or $\delta = 3$ and $z^2 \nmid g$, we slide by forms of order six, which need not vanish at $[Y]$. Let z^μ be the power of z in g, $p = gr \in J_4$ with r general, $q \in J_4$ vanishing at no root of r, $P, Q \in I$ with leading forms p_4, q_4 restricting to p, q, and $\ell_5 \in \bar{S}_5$ vanishing neither at $[Y]$ nor at the roots of p. Then $\varphi(zl) = l\ell_5$ satisfies Lemma 4.6 with $E = \langle p_4, q_4\rangle$, as $xl\ell_5 \perp Y^7$,

and the members contain f, P, Q, the $K_m + \tau\varphi(m)$, and, for each root $v \neq [Y]$ of g, some $Q_v \in I \cap \mathfrak{m}^5$ of order $\kappa(v)$ with leading form not vanishing at v, where $\kappa(v) \in \{5, 6\}$ as J_4 vanishes at v and $V^6 \notin D_6$. For $m = zl_1$, l_1 vanishing neither at $[Y]$ nor at the roots of p, and $\tilde{f} = f + \sigma Q + \sum_v \sigma_v Q_v$, the proof of Proposition 4.7, with $\rho^3\varphi(m)^\natural$, gives 6μ at $[Y]$, and that of Proposition 4.4 gives $2\mu_v$ at a root of r and $(\kappa(v)-2)\mu_v \leq 4\mu_v$ at a root of g; each bound holds on a nonempty open set of (σ, σ_v, τ). As $\mu \leq 2$, and $\mu = 1$ if $\delta = 3$, the total is at most $24+6\mu+4(\delta-\mu)+2(4-\delta) = 32+2\mu+2\delta \leq 40$.

(e) Otherwise $g \in \{z^3, z^2y\}$ after a change of y; $J_4 = g\bar{S}_1$ has dimension two, so $D_4 = \{G \in \bar{T}_4 : g \circ G = 0\}$, and D_5 is binary and killed by g. After shears and a gauge the lift of Y^7 is $\Phi = Y^7 + B_6 + XA_4 + B_5 + \cdots$ with B_5, B_6 binary, $B_6 = dY^4Z^2 + eY^3Z^3$, resp. $AY^4Z^2 + BZ^6$, and $A_4 = a_0Y^4 + a_1Y^3Z + a_2Y^2Z^2$, resp. $a_0Y^4 + a_1Y^3Z + a_4Z^4$, in D_4. If $g = z^2y$ and $A \neq 0 = B$, then $a_4 = 0$: the element $(z^2 - \frac{A}{105}y^3) \circ \Phi$ of M has degree at most three, and its cubic part is $12a_4XZ^2$ plus a binary cubic, identically in the tails; it lies in $D_3 = \bar{T}_3 \oplus \mathbb{C}XY^2$, which contains no element with a term XZ^2. Two kinds of operators $u \in \mathfrak{m}^3$ are used for turning. As $T_{\leq 1}$ is the orthogonal of $\mathfrak{m}^2$, $u \in (I : \mathfrak{m}^2)$ exactly when $u \circ M \subseteq T_{\leq 1}$: an ordinary colon operator, as in the cube branches below. A *quadratic-defect operator* is allowed one quadratic remainder, $u \circ M \subseteq T_{\leq 1} + \mathbb{C}Y^2$, and it need not lie in $(I : \mathfrak{m}^2)$; Lemma C.3 with $R = Y^2$ absorbs the remainder, as $XY^2 \in D_3$ for $W_2 = z\bar{S}_1$. In each of the nine charts below there is a quadratic-defect operator with $\pi(u_3) \neq 0$, and Corollary C.5(b) does the work. The constants c_i are explicit in the tails. Where u contains a free pair (β, γ), one homogeneous linear condition on it remains, the vanishing of the coefficient of Z^2, resp. YZ, in $u \circ \Phi$: with $B_5 = \sum_{i=0}^{5} b_i Y^{5-i}Z^i$, it reads $(12b_4 - \frac{24}{105}d^2)\beta + 60b_5\gamma = 0$, $(12b_4 - \frac{24}{105}A^2)\beta + 60b_5\gamma = 0$, resp. $12b_2\beta + 12b_3\gamma = 0$ in rows 3, 6 and 8. It has a nonzero solution, also when it vanishes identically, and $\pi(u_3)$ depends injectively on (β, γ), so $\pi(u_3) \neq 0$. For instance, in the third row $u = \beta yz^2 + \gamma z^3 - \frac{d\beta}{105}y^4 - \frac{b_3\beta + 2b_4\gamma}{4d}y^3z$, and $u \circ \Phi \equiv \big((12b_4 - \frac{24}{105}d^2)\beta + 60b_5\gamma\big)Z^2$ modulo $T_{\leq 1} + \mathbb{C}Y^2$ (script `121_13553211_0`, part (U3), with all tails symbolic). The lift of Y^6 is a derivative of Φ, so its image under u has degree at most one; every other lift has degree at most five, so the quadratic part of its image is $u_3 \circ G$ for its top form $G \in D_5$, and $u_3 \circ D_5 \subseteq \mathbb{C}Y^2$ in every row (part (U3)).

g	chart	u	$\pi(u_3)$
z^3	$e \neq 0$	$z^3 + c_1y^4 + c_2y^3z + c_3y^2z^2$	z^3
z^3	$e = 0 \neq d$, $a_2 \neq 0$	$z^3 + c_1xy^2 + c_2y^3z$	z^3
z^3	$e = 0 \neq d$, $a_2 = 0$	$\beta yz^2 + \gamma z^3 + c_1y^4 + c_2y^3z$	$\beta yz^2 + \gamma z^3$
z^3	$B_6 = 0$	z^3	z^3
z^2y	$A \neq 0 \neq B$	$yz^2 + c_1y^4 + c_2y^3z + c_3z^4$	yz^2
z^2y	$A \neq 0 = B$	$\beta yz^2 + \gamma z^3 + c_1y^4 + c_2y^3z$	$\beta yz^2 + \gamma z^3$
z^2y	$A = 0 \neq B$, $a_1 \neq 0$	$yz^2 + c_1xy^2 + c_2z^4$	yz^2
z^2y	$A = 0 \neq B$, $a_1 = 0$	$\beta y^2z + \gamma yz^2 + c_1z^4$	$yz(\beta y + \gamma z)$
z^2y	$B_6 = 0$	yz^2	yz^2

Corollary C.5(b) applies in each chart. □

Theorem C.24. *Every local algebra with Hilbert function* $(1,3,5,5,2,2,2,1)$ *or* $(1,3,5,5,2,2,1,1,1)$ *is smoothable, without induction.*

Proof. Both Hilbert functions satisfy the hypotheses $H_A(2) = 5$, $H_A(4) = H_A(5) = 2$ and $H_A(6) + H_A(7) + \cdots \leq 3$ of [16, Theorem B.14], whose statement places no restriction on $H_A(3)$, and whose proof uses no induction. □

C.25. From now on $H_A(3) = 4$. In the binary branch $I_3^* = xS_2$, $D_k \subseteq \bar{T}_k$ for $k \geq 3$, and we may assume $\gamma_f = 0$, as for $\gamma_f = \alpha l^3 \neq 0$ [15, Lemma F.4] applies to K_q, $q \in \bar{S}_2$ not vanishing at the root of l. In the cube branch, the reduction in the proof of [16, Theorem B.16] leaves $I_3^* = x^2S_1 + \langle xyz, xz^2, z^3\rangle$, so $D_k \subseteq \langle Y^k, Y^{k-1}Z, Y^{k-2}Z^2, XY^{k-1}\rangle$ for $k \geq 3$, and adding to f a multiple of $g_0 \in I$ with cubic part z^3 gives $\gamma_f = \lambda z^3 \neq 0$; let $g_1 \in I$ have cubic part xyz.

For $h \in I$ with leading form x^2 and cubic part $\lambda z^3 + x(ky^2 + k'yz + k''z^2) + x^2(\cdots)$, $\lambda \neq 0$, a formal change of x makes $h = x'^2 + g(y,z)$, where $g = h(\xi, y, z)$ and $\partial_x h(\xi, y, z) = 0$ (proof of [15, Theorem 4.5]); as $\xi = -\frac{1}{2}(ky^2 + k'yz + k''z^2) + O(\mathfrak{m}^3)$, with $[\,\cdot\,]$ a coefficient of h,

$$(7)\quad [y^4]g = [y^4]h - \tfrac{k^2}{4}, \qquad [y^3z]g = [y^3z]h - \tfrac{kk'}{2}, \qquad [y^5]g = [y^5]h \ \text{ if } k = 0.$$

By [15, Lemma 4.4], with y and z exchanged, h has type E_6 if $[y^4]g \neq 0$, E_7 if $[y^4]g = 0 \neq [y^3z]g$, and E_8 if $[y^4]g = [y^3z]g = 0 \neq [y^5]g$; then A is smoothable (Section 4.1).

C.26. *The binary branch when* $H_A(3) = 4$ *and* $H_A(4) \leq 4$. Proposition 4.4 applies with $0 \neq p \in J_4 = D_4^{\perp} \cap \bar{S}_4$ and r, b vanishing at no root of p. For $s = 7$ its bound is $24 + 4 \cdot 4 = 40$. For $s = 8$, $D_7 = \mathbb{C}Y^7$ and $D_8 = \mathbb{C}Y^8$, so $[Y]$ costs $\min(9,7) - 2 = 5$ per unit of multiplicity, a root v with $\kappa(v) \leq 7$ costs $\kappa(v) - 2$, and a simple root of a general factor costs 2. If $H_A = (1,3,5,4,2,2,2,1,1)$, then $D_k = \nu_k(\Gamma)$, $4 \leq k \leq 6$, for $\Gamma \ni [Y]$ of length two on the line; for $p = h_\Gamma q$, h_Γ the equation of Γ on the line and q general, the bound is $24 + 5 + 5 + 2 + 2 = 38$, as the second point $[Q]$ of a reduced Γ has $\kappa = 7$ ($Q^7 \notin D_7$), or $24 + 2 \cdot 5 + 2 + 2 = 38$ if $\Gamma = 2[Y]$. If $H_A = (1,3,5,4,3,2,1,1,1)$,

then J_4 vanishes at $[Y]$; for $h = \gcd J_4$, of degree $\delta \le 3$, z^μ the power of z in h, and $p = hh'$ with h' general, a root $v \ne [Y]$ of h has $\kappa(v) \le 6$, as $V^6 \notin D_6 = \mathbb{C}Y^6$, so the bound is $24{+}5\mu{+}4(\delta{-}\mu){+}2(4{-}\delta) = 32{+}\mu{+}2\delta \le 41$. This settles the binary branch below; the cube branches need no induction, and the scripts named check their operator identities with all lower tails symbolic.

Theorem C.27. *Every local algebra with Hilbert function* $(1,3,5,4,4,2,1,1)$ *is smoothable.*

Theorem C.28. *Every local algebra with Hilbert function* $(1,3,5,4,3,3,1,1)$ *is smoothable.*

Proof of Theorems C.27 *and* C.28. In the cube branch $D_7 = \mathbb{C}Y^7$ and $D_6 = \mathbb{C}Y^6$, as $D_7 \subseteq \langle Y^7, Y^6Z, Y^5Z^2, XY^6\rangle$. The cube-branch argument of the proof of [16, Theorem B.16] uses only the lift Φ of Y^7, $M = \mathbb{C}\Phi{+}\mathbb{C}\,y{\circ}\Phi{+}(M{\cap}\ T_{\le 5})$ and I_3^*, so it applies verbatim (scripts `121_13544211`, `121_13543311`, part (R)). □

Theorem C.29. *Every local algebra with Hilbert function* $(1,3,5,4,3,2,2,1)$ *is smoothable.*

Proof. In the cube branch $D_5 = \nu_5(\Gamma)$, $D_6 = \nu_6(\Gamma)$ for a double point Γ at $[Y]$, with $I_\Gamma = (x, z^2)$ or (z, x^2) after $z \mapsto z + cx$, and then F is Y^7 or Y^6Z, resp. Y^7 or XY^6 (script `121_13543221_cube`, parts (U0), (UF) and (UY)). Let first F be Y^7 or XY^6. Then $y^5 \circ \Phi \in 2520Y^2 + T_{\le 1}$, resp. $720XY + T_{\le 1}$, and $y^5 \circ (M \cap T_{\le 6}) \subseteq T_{\le 1}$; as $XY^2 \in D_3$ (needed only for $R = Y^2$), Lemma C.3 applies with $R = Y^2$, resp. XY, and Corollary C.4 gives an equation of type E, as in the proof of Corollary C.5(a). If $F = Y^6Z$, then $XYZ \notin D_3$ and Lemma C.3 does not apply; instead, except in the last case below, I itself contains an equation of type E. After shears and gauges, $\Phi = Y^6Z + \delta Y^3Z^3 + \epsilon Y^2Z^4 + P_5 + \cdots$; the parts of degrees three and two of $g_0 \circ \Phi = f \circ \Phi = g_1 \circ \Phi = 0$ give $[y^4]g_0 = -\epsilon/15$, $[y^3z]g_0 = -\delta/20$, and, if $\delta = \epsilon = 0$, $[y^4]f = [y^4]g_1 = [y^3z]g_1 = 0$, $[y^3z]f = -e_1/60$, $[y^5]g_0 = -e_2/30$ and $[y^5]g_1 = -e_3/180$, where e_1, e_2, e_3 are the coefficients of X^2Y^3, YZ^4, XY^2Z^2 in P_5. By (7), with $k = [xy^2]f$, in turn: if $(\delta,\epsilon) \ne 0$, $f + tg_0$ has type E_6 or E_7 for general t; if $k \ne 0$ or $e_1 \ne 0$, f has type E_6 or E_7; if $e_2 \ne 0$, resp. $e_3 \ne 0$, $f + tg_0$, resp. $f + tg_1$, has type E_8 for general t; and if $e_1 = e_2 = e_3 = 0$, then the X-part of P_5 is $X(AY^4 + BY^3Z)$, and

$$U = xy^2 - \tfrac{A}{30}\,y^4z - \tfrac{B}{120}\,y^5$$

maps Φ into $T_{\le 1}$, whatever the lower tails: its quadratic part is $12AY^2 + 6BYZ - 12AY^2 - 6BYZ = 0$ (script `121_13543221_cube`, part (UY)). This suffices: the derivatives of Φ span M modulo $M \cap T_{\le 5}$, U commutes with differentiation, and on an element of degree at most five the only possible part of degree two of its image is the action of the cubic part xy^2 on $D_5 = \langle Y^5, Y^4Z\rangle$, which is zero. So $U \circ M \subseteq T_{\le 1}$, $U \in (I : \mathfrak{m}^2)$, and by

[15, Lemma F.4], with y and z exchanged, a member of its turning family contains an equation of type E_6. □

Theorem C.30. *Every local algebra with Hilbert function* $(1,3,5,4,2,2,2,2)$ *is smoothable.*

Proof. Here $D_k = \nu_k(\Gamma)$ for $4 \le k \le 7$; in the cube branch Γ is a double point at $[Y]$, the only P with $P^4 \in \langle Y^4, Y^3Z, Y^2Z^2, XY^3\rangle$, and $I_\Gamma = (z, x^2)$ or (x, z^2) after $z \mapsto z + cx$. The first derivatives of the lifts Φ, Ψ of the two top forms span D_6, so $u \in \mathfrak{m}^3$ with $u_3 \in (I_\Gamma)_3$ lies in $(I : \mathfrak{m}^2)$ once $u \circ \Phi, u \circ \Psi \in T_{\le 1}$. After shears and gauges this holds for $u = yz^2$ if $I_\Gamma = (z, x^2)$, and for yz^2 corrected by multiples of y^4, y^5 and y^4z if $I_\Gamma = (x, z^2)$ (script `121_13542222`, parts (O) and (I)). As $\pi(u_3) = yz^2$ is not a cube, u is a turning operator of type (a). □

Theorem C.31. *Every local algebra with Hilbert function* $(1,3,5,4,2,2,2,1,1)$ *is smoothable.*

Proof. In the cube branch Γ is a double point at $[Y]$, as in Theorem C.30, and M is spanned by $M \cap T_{\le 5}$ and the derivatives of the lifts Φ of Y^8 and Ψ of the second top form of D_6. If $I_\Gamma = (x, z^2)$, after shears and a gauge an element U with cubic part $xy^2 \in (I_\Gamma)_3$, corrected by multiples of y^4z and y^6, maps Φ and Ψ into $T_{\le 1}$; so $U \in (I : \mathfrak{m}^2)$, and [15, Lemma F.4], with y and z exchanged, applies. If $I_\Gamma = (z, x^2)$, shears remove all terms with Z from the sextic tail of Φ and the quintic tail of Ψ, and yz^2 is a turning operator of type (a) (script `121_135422211`, parts (I) and (O)). □

Theorem C.32. *Every local algebra with Hilbert function* $(1,3,5,4,3,2,1,1,1)$ *is smoothable.*

Proof. In the cube branch, for the lift $\Phi = Y^8 + P_7 + P_6 + \cdots$ of Y^8, whose derivatives span M modulo $M \cap T_{\le 5}$, gauges and shears give $P_7 = 0$ and $P_6 = aX^2Y^4 + \sigma XY^4Z + \beta Y^4Z^2 + \epsilon Y^3Z^3$, with $\sigma^2 = 4a\beta$ and $a\epsilon = \sigma\epsilon = 0$, as $x \circ P_6$, $z \circ P_6 \in D_5$ (script `121_135432111`, part (C)). If $a \ne 0$, then $P_6 = aY^4(X + cZ)^2$, and $z \mapsto z + c'x$, which keeps I_3^*, makes $P_6 = aX^2Y^4$; then, and also if $a = \beta = \epsilon = 0$, $y^3z \in (I : \mathfrak{m}^2)$, and in [15, Theorem 4.5], $G_t = g + ty^3z$ falls in case (2) or (3) for general t. Otherwise $a = \sigma = 0$, and $f \circ \Phi = g_0 \circ \Phi = g_1 \circ \Phi = 0$ give $[y^4]f = [y^4]g_0 = [y^4]g_1 = 0$ and $[y^5]g_0 = -\epsilon/1120$; if $\epsilon = 0 \ne \beta$, also $D_5 = \langle Y^5, Y^4Z\rangle$, $P_5 = \eta X^2Y^3 + X(AY^4 + BY^3Z + CY^2Z^2) + B_5$, $[y^5]f = -\eta/3360$ and $4C + 48\beta[y^3z]g_1 = 0$. By (7), in turn: f has type E_6 if $[xy^2]f \ne 0$; $f + tg_0$ has type E_7 or E_8 if $\epsilon \ne 0$; f has type E_7 or E_8 if $\eta \ne 0$; $f + tg_1$ has type E_7 if $C \ne 0$. It remains the case $[xy^2]f = \epsilon = \eta = C = 0$, where $\beta \ne 0$ and the X-part of P_5 is $X(AY^4 + BY^3Z)$. Then

$$U = xy^2 - \tfrac{B}{8\beta}\, y^3z - \tfrac{A}{1680}\, y^6$$

maps Φ into $T_{\le 1}$, whatever the lower tails: its quadratic part is $12AY^2 + 6BYZ - 6BYZ - 12AY^2 = 0$ (script `121_135432111`, part (C)); its cubic

part xy^2 kills D_5, so, as the derivatives of Φ span M modulo $M \cap T_{\leq 5}$ and U commutes with differentiation, $U \circ M \subseteq T_{\leq 1}$. So $U \in (I : \mathfrak{m}^2)$, and [15, Lemma F.4], with y and z exchanged, applies. □

Theorem C.33. *Every local algebra with Hilbert function* $(1, 3, 5, 4, 2, 2, 1, 1, 1, 1)$ *is smoothable.*

Proof. We use the mechanism of [16, Theorem B.14]. By [15, Lemma 2.16], $D_k = \mathbb{C}Y^k$ for $6 \leq k \leq 9$ and $D_k = \nu_k(\Gamma)$ for $k = 4, 5$, with $[Y] \in \Gamma$ and $I_\Gamma = (x, yz)$, (x, z^2) or (z, x^2). An element $u \in \mathfrak{m}^3$ with $u_3 \in (I_\Gamma)_3$ lies in $(I : \mathfrak{m}^2)$ once $u \circ \Phi \in T_{\leq 1}$ for the lift $\Phi = Y^9 + Q_8 + \cdots$, normalized by gauges and shears to $Q_8 = Q_7 = 0$ and Q_6 a multiple of Z^6, Y^4Z^2, X^2Y^4, respectively. Script `121_1354221111` gives, with all tails symbolic, such u with $\pi(u_3)$ equal to yz^2, $yz(p_1z + p_2y)$ or $z^2(p_0z + p_1y)$, $p_1 \neq 0$, so that [15, Corollary 4.6] applies, except when $I_\Gamma = (x, z^2)$, $Q_6 \neq 0$, and Q_5 has a term YZ^4 but no term Z^5. There it gives u with cubic part z^3 and $V \in (I : \mathfrak{m}^2)$ with cubic part xy^2 and $\partial_x V = y^2$. The degree-six part of $f \circ \Phi = 0$ is $f_3 \circ Y^9$, so γ_f has no term y^3, and by [15, Theorem 2.9] we may assume $\gamma_f = cz^3$; in [15, Theorem 4.5] for $u + V$, with y and z exchanged, G_t has cubic part $(c + t)z^3$ and y^4-coefficient $c_0 + c_1t - \frac{t^2}{4}$: case (2) holds. □

Appendix D. Computations

D.1. This appendix lists the computer calculations, in SageMath 10.7. All are exact, over $\mathbb{Q}$, polynomial rings over $\mathbb{Q}$, explicitly specified localizations of them, function fields over $\mathbb{Q}$, or number fields; whenever a parameter is inverted, the excluded locus is treated separately. Every local length used as a linkage bound is checked twice in Singular, by a local standard basis and by stabilization (Section 2.4); fixed truncation colengths, Hilbert functions and Milnor numbers are computed separately, as the manifest specifies. Identities "with all tails symbolic" hold in a polynomial ring in all tail coefficients. We mark *universal* calculations, valid for all values of the parameters or exhaustive (U), *exact anchors and representatives*, which certify one member while exhaustion and irreducibility are proved in the text (A), and *sanity checks*, not used in the proofs (S). A universal calculation is valid on the stated coefficient rings and charts, and the text explains why these exhaust the case; agreement with a reference output certifies reproducibility, not a universal statement. The scripts, their outputs and a manifest accompany this paper as the archive `twenty_one_points_scripts.tar.gz` on the author's web page, with SHA-256 checksum `6a4ac632cc8927f5d638d7dee6b4739b3da6bb6edf389c834bb88b353914bb59`; a reader verifies them from the extracted archive with `./run_all.sh`, which first checks every delivered file against `SHA256SUMS`, and ends with `ALL RUNS OK` only if the internal checks of every script passed and every output agrees with its reference output character by character, except in the elapsed-time fields. It compares the outputs only with private copies of the checksummed reference outputs and refuses another reference directory; before anything is deleted, it refuses an output

directory or log location that coincides with, contains or aliases files of the release. `./run_all.sh --full` also runs `l21_1365411_qt` and `l21_hard_n4`, about twenty minutes each. Scripts are named without the suffix `.sage`, and parts by the labels that the scripts print. The files `B28_COVERAGE.md` and `COVERAGE_B18_B21_B24.md` list the cases of Theorems B.28, B.18, B.21 and B.24 with their certificates.

script	result	establishes
`l21_classify`	Tables 1, 2	U: the 254 Hilbert functions of length twenty-one and their rows
`l21_general_s6`	Cor. 3.4	S: the limit of the magnified line on actual algebras
`l21_general_WM`	Prop. 3.7	S: formula (a) against global computations (F); A: local bounds on fourteen graded data (B)
`l21_1365411_qt`	Prop. 3.11	A: $\ell(J) = 41$ for seven (α, β) (J); U+S: the normal form (N), and $J \subseteq \mathrm{Ann}(M_\infty)$ on ten actual algebras (E)
`l21_1365411_testc`	Prop. 3.11	A: Proposition 3.7(c) gives 38 for a general D_4 (A38) and 41 when $X^3Z \in D_4 \not\ni X^3Y$ (A41); U: at least 42 when $X^3Y, X^3Z \in D_4$ (U42)
`l21_explicit_triples`	Prop. 3.11, Thm. 7.1	U: the normal form of the cubic data, $\alpha = 1$, $\beta = 0$ (N); A: J annihilates M_∞ for four D_4, local length 41 (J41); the explicit triples of Proposition 3.7(c), 38 (A38) and 41 (A41); the two active-D_5 triples of Theorem 7.1, 39 each (C)
`l21_map_135_turn`	Prop. 4.4	A: model triples of local length $24 + 4(s-3)$ for $s = 5, 6, 7$ (T1)
`l21_general_N1`	Prop. 4.4	A: graded extreme members, 36 for $s = 6$ (G); S: actual algebras (T)
`l21_general_LD`	Prop. 6.2	A: the triples of local length $36 + \delta$, $\delta \leq 4$ (A), and twenty actual algebras (C); U: the reduction identities for the five monomial models (H); S: $S_2Q = S_4$ for two $Q = (x \circ G)^\perp$, proved in the text (H)
`l21_map_LD`	Prop. 6.2	S: twelve random triples of L_4^0, of local length 40 (B), and a special triple of local length 42 (A)
`l21_136452`, `l21_136443`	Thms. 6.4, 6.5	S: the proposition on actual algebras
`l21_136332111`	Thm. 7.1	U: excess bounds, the normal form of an active D_5, closedness with all tails symbolic (F); A: random triples 39 for the active data (A), the triple 41 at infinity (W), the anchor 39 of the fat stratum (A1)
`l21_1354422`, `l21_1354431`, `l21_1354332`	Thm. 8.1	S: turned members (≤ 36) against radial costs (up to 42)
`l21_1355322`	Thm. 8.3	A+S: slid ideals, triples of local length 40 (B); U: the four differentiation constants; A: the shapes and the corrected operator on five seeded algebras and one explicit algebra (N)
`l21_13692`, `l21_13683`, `l21_13674`, `l21_13665`, `l21_13656`	Thm. A.5	U+S: the counts and the ray conditions; A: an anchor of local length 36 (`l21_13683`, part (B))

script	result	establishes
`l21_136551`, `l21_136542`, `l21_136533`	Thms. A.7–A.9	U: tail dimensions at the orbit representatives; A: the curves moving the embedded point; S: $q^5 \notin D_5$ when $r = 4$, on random pencils (`l21_136542`, (R4)), proved in the text
`l21_136641_0`	Thm. A.11	S: ranks and rays on the strata of expected dimension (D), (R1), (R2); A: the fat-point anchor (W)
`l21_136632`, `l21_136632_fat`, `l21_136632_line`, `l21_136632_line2`, `l21_136632_line3`, `l21_136632_line4`, `l21_136632_line5`, `l21_136632_k2count`	Thm. A.12	U: the seven fat orbits and their tails, kernel dimensions, the ingredients of the count; A: anchors 36–41, triples 37–41; A: part (F) of `l21_136632_line` at one seeded hyperplane (generic values)
`l21_136632_kernel`	Thm. A.12	U: $k = 1$, N on the line, item (iii): $\dim \mathcal{K}_V = 1$ for every square point N, every Q with $\dim S_1 Q = 9$ and every pencil $V \ni N^5$ with $r = 3$, all parameters symbolic
`l21_136632_moves`	Thm. A.12	U: ρ constant along the four moves of N off the line, including $t = 0$ (M); S: the tails formula against the closedness equations (T), actual closed algebras (C)
`l21_136731`, `l21_136722`, `l21_136821`	Thms. A.13–A.15	U: the arithmetic of the counts of `l21_136731` (C); A: members with triples of local length 27–40
`l21_13633311`, `l21_13633221`, `l21_13632222`, `l21_1363221111`, `l21_136422111`, `l21_1364511`, `l21_1365222`	App. B	U+S: the hand arguments on representatives and actual algebras
`l21_13632222_fat_param`	Thm. B.9	U: the fat stratum, $K_v(L_{\mathrm{def}})$ (K), and a reduction of it, so $e(K_v) = 12$ (E); A: a parameter ideal of colength 12 in it (J), $e(L_{\mathrm{def}}) = 39$ (W), and a triple of local length 39 (T)
`l21_136322211`	Thm. B.10	A: triples at infinity of local length 40, 40, 41 (A); S: sampled undeformed triples of local length greater than 41 (S)
`l21_1366311`, `l21_1366221`	Thms. B.14, B.15	U: normal forms, ranks; A: anchor 30
`l21_13652211`, `l21_13642221`	Thms. B.17, B.18	U: elimination; A: anchors 32–41
`l21_1364322`, `l21_1364322_r1`, `l21_1364322_r2`	Thm. B.21	U: excess arithmetic, orbits, saturation in $\mathbb{Q}[x, y, z, \lambda]$, normalization and the symbolic systems; A: anchors 38–41
`l21_1364331`, `l21_1364331_line`	Thm. B.22	U: graded data, ranks, local bounds, elimination, the lower module; A: anchor 38
`l21_1364331_line_u`	Thm. B.22	U in the tails: the corrected cubic u with all tails symbolic, for sample (q, F); S: actual algebras of the non-normalized family
`l21_1364421`, `l21_1364421_line`, `l21_1364421_tangent`	Thm. B.23	U: graded data, eliminations, the tangent pencil with all tails symbolic; A: anchors 33–40; S: rays on actual algebras

script	result	establishes
`l21_13643211`, `l21_13643211_fam`, `l21_13643211_fam2`, `l21_13643211_fam3`	Thm. B.24	U: configurations, normal forms, eliminations; A: anchors 34–40
`l21_13643211_cl_ray`, `l21_13643211_cl_tangent`	Thm. B.24	U: the three sectors, $p_4 \circ M \subseteq \mathbb{C}[X]$, eliminations; A: anchors 39–41
`l21_1365411`	Thm. B.25	U: the local bound 14, rank three; A: representatives
`l21_1365411_d3rank`, `l21_1365411_line`, `l21_1365411_rays`	Thm. B.25	S: ranks on random embeddings; the deformations of the common line and the rays on actual algebras
`l21_1365321_B3`	Thm. B.28	U: sufficiency of the closedness conditions of sector B3 with all data symbolic, and the ranks 9 and 10 (F); S: sampled members and negative controls (F), no fifth power in $S_1 \circ F$ (L); A: the anchor (A)
`l21_1365321_wm`, `l21_1365321_cfg`	Thm. B.28	U: the 157 configurations, the 26 configurations; A: anchors
`l21_1365321_irr`	Thm. B.28	U: smooth rational points on the R-loci (P1), (P2); the locus $q \leq 2$ (V)
`l21_1365321_wm` (U3), (U4)	Thm. B.28	U: the list of the 157 configurations with their local values and bounds (U3); the ideals of colength 2 and 3 of the germs (u^2, uw, w^3) and $(uw, u^2 - w^3)$ up to automorphisms, local value at most 14 when $\mathfrak{B}_4$ is not the fat point (U4)
`l21_1365321_tq_secant`	Thm. B.28	U: the two binary secant configurations are single orbits (U1), the φ_R span three dimensions (U2); A: Lemma B.27 with $\alpha = x^5$, bounds 40 and 41 (A3)
`l21_1365321_F1F8_abs`	Thm. B.28	U: one absolute prime for each of the four components of dimension six
`l21_1365321_bundle`	Thm. B.28	U: constant ranks over $\mathbb{P}^1$ (resp. $\mathbb{A}^1 \times \mathbb{P}^1$) for the non-free families of `l21_1365321_p1t`, `l21_1365321_f4`, `l21_1365321_F1F8_f1nf`, so these loci are images of vector bundles
`l21_1365321_b2t`, `l21_1365321_b2s`, `l21_1365321_b23`, `l21_1365321_p1t`, `l21_1365321_p2g`, `l21_1365321_f4`, `l21_1365321_F2_family`, `l21_1365321_F1F8_f1nf`, `l21_1365321_F1F8_f1fr`, `l21_1365321_F1F8_f8`	Thm. B.28	U: closedness, the loci and their irreducibility; A: certified members (34–41)
`l21_1365321_rays`, `l21_1365321_r1s`, `l21_1365321_r1a`, `l21_1365321_r1b`, `l21_1365321_r1f2`	Thm. B.28	U: ray criteria, spaces of quintic tails, substitutions; A: anchors 40, 41; S: rays on actual algebras
`l21_1365321_r1fq`	Thm. B.28	U: $S_1 \circ W \subseteq D_3$; for every $t \neq 0$ a diagonal change (an identity modulo $\lambda^k = t^{\mp 1}$) maps the seven data to certified anchors; the substitution identity for the family with a point embedded at $[Y]$ over $\mathbb{Q}[a, b, c, t, 1/t]$; S: sampled substitutions and actual free algebras

script	result	establishes
`l21_general_N234`	Lemmas C.6–C.8	A: extreme models of local length 40 (A); S: actual algebras (T), and seeded algebras with the turning operator z^3 of Theorem C.11, whose proof is the universal argument (D3)
`l21_general_Q5MC`	Lemma C.9	A: models of local length 40 and 41; S: 42 without the hypothesis of (b) (Q); its part (M) checks an earlier variant of the sliding of Theorem C.19
`l21_135732`, `l21_135642`, `l21_135543`, `l21_135453`, `l21_135444`	App. C	A+S: certified members and actual algebras; `l21_135543` follows the route of [16, Theorem B.7], and Lemma C.6 is checked in `l21_general_N234`; for `l21_135453` the universal argument is Lemma C.7, and its check symbolic in t is S
`l21_135552`, `l21_1356321`, `l21_1355331`	App. C	U: the operators with all tails symbolic (`l21_1356321`, `l21_1355331`, part (N)) and the criterion for Z_4 (`l21_135552`, part (U)); A: slid members of local length 40 (`l21_135552`, `l21_1355331`); S: actual algebras
`l21_1355421`	Thm. C.19	U: the operator T with all tails symbolic (b); S: an earlier variant of the moving family (c), actual algebras
`l21_1355421_slide`, `l21_1355421_slide_bad`	Thm. C.19	S: the sliding family of Lemma 4.6 built as in its proof, flat over $\mathbb{Q}[\tau]$ with colength 21, and the triple of Proposition 4.7 (≤ 40), also on the stratum $\theta_m(Y) = 0$
`l21_1354521`, `l20_135_s6` (of [16])	Thm. C.21	U: the operator of [16, Theorem B.10]; S: actual algebras
`l21_13553211`, `l21_13553211_O`, `l21_13553211_F`, `l21_13553211_slide6`, `l21_hard_n4`	Thm. C.23	U: the nine operators of the case $\delta = 3$, and the emptiness of the tenth chart; A: the slides, of local length 40; S: families of the defect lemma
`l21_13544211`, `l21_13543311`, `l21_13543221`, `l21_13543221_cube`	App. C	U: cube branches; forced coefficients, symbolic in part (UY) of `l21_13543221_cube`, which prints $2e_1$, e_2, e_3 as C_v, f', C_u, and by their contraction constants in part (R) of `l21_13544211`, `l21_13543311`; A, S: binary branches
`l21_13542222`, `l21_135422211`, `l21_135432111`, `l21_1354221111`	App. C	U: normal forms of the tails and the operators, with all lower tails symbolic; the leading effects of the shears and gauges, and the contraction constants of the forced coefficients
`l21_13552221`, `l21_135522111`, `l20_13542221`, `l20_135422111` (the last two of [16])	Thm. C.24	U: the proof of [16, Theorem B.14], in `l20_13542221` and `l20_135422111`; S: `l21_13552221` and `l21_135522111` test its conclusion on actual algebras with $H_A(3) = 5$
`l21_135_s7plus_engine_sanity`, `l21_135_s7plus_cube_sanity`	App. C	S: both branches on random actual algebras

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