

An Expansion on Paraconsistent Micro-Geometry: A Non-Standard Approach to Subgroup-Stratified Vitali Measures via the Logic of da Costa

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Abstract

The existence of so-called pathological sets with respect to certain contexts of measure theory has often demanded a treatment that better satisfies our intuition or provides greater insight into their nature. The non-measurable sets of Vitali, in particular, seem to exhibit the intuitive properties of infinitesimals, and indeed, to some extent this can be formalized. While the standard Loeb measure yields real-valued extensions, the underlying hyperfinite constructions naturally suggest that Vitali sets possess infinitesimal inner and outer measures. However, a fully explicit bottom-up construction of these infinitesimal assignments has remained elusive, making it difficult to systematically distinguish between different Vitali sets—a gap that this paper seeks to rectify. Thus, given the top-down guaranteed existence of a non-standard framework for Vitali sets, we provide a formal construction from the bottom up that allows for a stratification of measures reflected as a stratification of functions and operators. It is hoped that this stratified construction will find diverse applications from capturing micro-geometric efficiency to ergodic theory and dynamical systems, and be generally seen as a pleasing addition to the literature.

1 Introduction

The Lebesgue measure is perhaps the most successful means of measuring the size of sets of real numbers known; indeed, under the appropriate axioms (such as the Axiom of Dependent Choice combined with set-theoretic determinacy), all subsets of the reals are Lebesgue measurable. However, under the standard Axiom of Choice (AC) within Zermelo-Fraenkel set theory (ZF), it is provable that non-measurable sets must exist.

The classic construction of a non-measurable set is the Vitali set. This is a choice set formed on the equivalence classes of the reals under an equivalence

relation governed by a dense subgroup of the additive group of reals $(\mathbb{R}, +)$. One such relation is given by $x \sim y$ if and only if $x - y \in \mathbb{Q}$ for $x, y \in [0, 1]$. The resulting Vitali set is non-measurable because it fundamentally breaks two pillars of the Lebesgue measure: σ -additivity and translation invariance.

To see how this happens, consider a Vitali set V and a countable partition of the unit interval formed by its rational translations $V + q_i$, where $q_i \in \mathbb{Q} \cap [0, 1]$. If V were Lebesgue measurable, translation invariance would require every $V + q_i$ to have the exact same measure. If that measure were zero, a countable sum would yield zero, failing to cover $[0, 1]$. If it were strictly positive, the countable σ -additivity sum would diverge to infinity, violating the measure of the unit interval (1). In this precise fashion, the measure of V is forced into an irresolvable contradiction.

While many types of non-measurable sets exist, Vitali sets remain uniquely intuitive. Indeed, it does not take much imagination to consider assigning them an infinitesimal measure within an appropriate structural setting. Although there are now several approaches to working with infinitesimals in the literature, we shall stick to that of Robinson and Luxemburg. Thus we begin by defining our subgroup-stratified internal outer and inner measures in this setting, confirming that their properties align with our core aims. In particular, we show that Vitali sets can be assigned a paraconsistent infinitesimal measure, while the standard parts of our new measures successfully recover the classical Lebesgue measure on standard measurable sets.

2 Definition of Measures

Let $\mathbb{H}_\alpha \subset \mathbb{R}$ be a countable dense subgroup with target interval length b . We define a sequence of standard approximating selector sets V_n that asymptote to a Vitali set V derived from the subgroup $\mathbb{H}_\alpha$. Since $\mathbb{H}_\alpha$ is a countable dense subgroup of $\mathbb{R}$, we exhaust it via an increasing sequence of finite subsets:

$$\mathbb{H}_{\alpha,1} \subset \mathbb{H}_{\alpha,2} \subset \cdots \subset \mathbb{H}_\alpha \quad \text{with } 0 \in \mathbb{H}_{\alpha,n}$$

Now within our target base interval $[0, b]$, we define an equivalence relation $x \sim_n y$ if and only if their difference belongs to the finite subset or its generated grid:

$$x \sim_n y \iff x - y \in \mathbb{H}_{\alpha,n}$$

This partitions $[0, b]$ (up to boundary conditions) into a finite collection of coset slices or congruence classes:

$$\mathcal{P}_n = [0, b] / \sim_n$$

Let $N_{\alpha,n} = |\mathcal{P}_n|$ denote the total number of these partition components.

Using a deterministic selection rule (or the Axiom of Choice applied finitely), we choose a single canonical representative point s_k from each equivalence class in $\mathcal{P}_n$. This yields a finite set of discrete points:

$$S_n = \{s_1, s_2, \dots, s_{N_{\alpha,n}}\} \subset [0, b]$$

Because a raw set of discrete points S_n has Lebesgue measure zero and cannot interact smoothly via interval-based shift conditions, we thicken each point into a small, regular proxy interval of width ϵ_n (where $\epsilon_n \rightarrow 0$ as $n \rightarrow \infty$). We define the set-theoretic construction of V_n as the finite union:

$$V_n = \bigcup_{s_k \in S_n} \left[s_k - \frac{\epsilon_n}{2}, s_k + \frac{\epsilon_n}{2} \right] \cap [0, b]$$

We now define the sequence of component-wise standard shift sets $(H_{\alpha,b}^{(n)})_{n \in \mathbb{N}}$, where each term collects the active translation parameters that map the standard approximating selector set V_n into the interval $[0, b]$ at stage n :

$$H_{\alpha,b}^{(n)} = \{h \in \mathbb{H}_{\alpha,n} \mid 0 \leq h \leq b \text{ and } (V_n + h) \cap [0, b] \neq \emptyset\} \quad (1)$$

Lifting this through the non-standard universe via a free ultrafilter $\mathcal{U}$, we define the internal shift set $H_{\alpha,b}^*$ and its corresponding internal cardinality N_α :

$$H_{\alpha,b}^* = \{h \in {}^*\mathbb{H}_\alpha \mid 0 \leq h \leq b \text{ and } (*V + h) \cap [0, b]^* \neq \emptyset\} \quad (2)$$

$$N_\alpha = |H_{\alpha,b}^*| = [(|H_{\alpha,b}^{(n)}|)_{n \in \mathbb{N}}]_{\mathcal{U}} \quad (3)$$

With this hyperfinite partitioning framework established, we can now define the internal base measure for arbitrary measurable sets via stage-wise approximation. Let $E \subseteq [0, b]$ be an arbitrary Lebesgue measurable set, and recall that our sequence of component-wise standard shift sets $(H_{\alpha,b}^{(n)})_{n \in \mathbb{N}}$ generates a stage- n uniform partition of $[0, b]$ into grid intervals $I_{h,n}$ of width $\Delta x_n = \frac{b}{|H_{\alpha,b}^{(n)}|}$, indexed by $h \in H_{\alpha,b}^{(n)}$.

At each finite stage n , let λ_n denote the standard Lebesgue measure restricted to this uniform grid partition. We define the elementary approximating set E_n via grid-snapping:

$$E_n = \bigcup_{h \in J_{E_n}} I_{h,n}, \quad \text{where } J_{E_n} = \{h \in H_{\alpha,b}^{(n)} \mid I_{h,n} \cap E \neq \emptyset\}$$

with $\lim_{n \rightarrow \infty} \lambda(E_n \Delta E) = 0$. In the non-standard universe, the infinitesimal quantum $\ell_{\alpha,b}$ and the lifted index set J_E are defined via the ultrafilter $\mathcal{U}$ as:

$$\ell_{\alpha,b} = \left[\left(\frac{b}{|H_{\alpha,b}^{(n)}|} \right)_{n \in \mathbb{N}} \right]_{\mathcal{U}}, \quad J_E = [(J_{E_n})_{n \in \mathbb{N}}]_{\mathcal{U}}$$

Theorem 1. Let $E \subseteq [0, b]$ be a Lebesgue measurable set, and let $(E_n)_{n \in \mathbb{N}}$ be a sequence of elementary grid-approximating sets such that $\lim_{n \rightarrow \infty} \lambda(E_n \Delta E) = 0$. Defining the internal base measure $\mu_\alpha(E)$ as the ultraproduct lift of the stage-wise Lebesgue measures:

$$\mu_\alpha(E) := [(\lambda_n(E_n))_{n \in \mathbb{N}}]_{\mathcal{U}}$$

Then $\mu_\alpha(E)$ factorizes into the product of the lifted internal cardinality and the infinitesimal quantum:

$$\mu_\alpha(E) = |J_E| \cdot \ell_{\alpha,b}$$

Furthermore, the standard part of this hyperfinite measure recovers the classical Lebesgue measure:

$$\text{st}(\mu_\alpha(E)) = \lambda(E)$$

Proof. 1. **Stage-Wise Finitary Identity:** For each finite stage n , because E_n is a finite union of uniform grid intervals $I_{h,n}$ of width $\frac{b}{|H_{\alpha,b}^{(n)}|}$ and indexed by J_{E_n} , the stage-wise Lebesgue measure satisfies the exact algebraic identity:

$$\lambda_n(E_n) = |J_{E_n}| \cdot \lambda_n(I_{h,n}) = |J_{E_n}| \cdot \frac{b}{|H_{\alpha,b}^{(n)}|}$$

2. **Ultraproduct Lift via the Transfer Principle:** Applying the Transfer Principle across the sequence of stage-wise identities over the ultrafilter $\mathcal{U}$ yields:

$$[(\lambda_n(E_n))_{n \in \mathbb{N}}]_{\mathcal{U}} = \left[\left(|J_{E_n}| \cdot \frac{b}{|H_{\alpha,b}^{(n)}|} \right)_{n \in \mathbb{N}} \right]_{\mathcal{U}}$$

3. **Algebraic Factorization:** Because the ultraproduct preserves algebraic operations element-wise, the right-hand side factorizes into the product of the lifted index cardinality and the infinitesimal mesh width:

$$[(|J_{E_n}|)_{n \in \mathbb{N}}]_{\mathcal{U}} \cdot \left[\left(\frac{b}{|H_{\alpha,b}^{(n)}|} \right)_{n \in \mathbb{N}} \right]_{\mathcal{U}} = |J_E| \cdot \ell_{\alpha,b}$$

Substituting our definition of $\mu_\alpha(E)$ establishes the internal base measure formula:

$$\mu_\alpha(E) = |J_E| \cdot \ell_{\alpha,b}$$

4. **Standard Part Recovery:** Because the symmetric difference vanishes in the limit ($\lim_{n \rightarrow \infty} \lambda(E_n \Delta E) = 0$), the stage-wise measures converge to the classical Lebesgue measure: $\lim_{n \rightarrow \infty} \lambda_n(E_n) = \lambda(E)$. Taking the standard part of the hyperfinite ultraproduct lift consequently yields:

$$\text{st}(\mu_\alpha(E)) = \text{st}([(\lambda_n(E_n))_{n \in \mathbb{N}}]_{\mathcal{U}}) = \lim_{n \rightarrow \infty} \lambda_n(E_n) = \lambda(E) \quad \blacksquare$$

□

Corollary 1. The base measure is a non-standard measure on the algebra of internal sets.

2.1 Defining the Hyperfinite Outer Measure

To resolve fine topological structures such as the micro-infinitesimal span of non-standard Vitali sets without trivial measure collapse, we construct the subgroup-stratified hyperfinite outer measure using micro-scale covering intervals of width ϵ_n centered around the canonical selector points $s_k \in S_n$. For any internal set $E \subseteq^* [0, b]$, the outer measure is defined as:

$$\mu_\alpha^*(E) = \left[\left(\inf \left\{ \sum_{s_k \in S_n} c_{k,n} \epsilon_n \mid E_n \subseteq \bigcup_{s_k \in S_n} \left[s_k - \frac{\epsilon_n}{2}, s_k + \frac{\epsilon_n}{2} \right] \cap [0, b] \right\} \right) \right]_{n \in \mathbb{N}} \mathcal{U} \quad (4)$$

where the indicator function

$$c_{k,n}: S_n \rightarrow \{0, 1\}$$

is defined by

$$c_{k,n} = \begin{cases} 1 & \text{if the micro-interval around } s_k \text{ is included in the covering set } E_n \\ 0 & \text{if the micro-interval is excluded} \end{cases}$$

for each selector index $s_k \in S_n$, ensuring that the aggregate measure span correctly tracks the micro-infinitesimal bound $\frac{b}{N_\alpha}$ in the hyperfinite limit.

Theorem 2. Consistency with Properties of Outer Measure:

1. **Non-Negativity:** $\mu_\alpha^*(E) \geq 0$
2. **Null Empty Set:** $\mu_\alpha^*(\emptyset) = 0$
3. **Monotonicity:** If $A \subseteq B$, then $\mu_\alpha^*(A) \leq \mu_\alpha^*(B)$
4. **Hyperfinite-Subadditivity:** $\mu_\alpha^* \left(\bigcup_{k \in H_{\alpha,b}^*} A_k \right) \leq \sum_{k \in H_{\alpha,b}^*} \mu_\alpha^*(A_k)$

Proof. 1. **Non-Negativity:** For each finite stage n , because $\epsilon_n > 0$ and $c_{k,n} \in \{0, 1\}$, every term in the cover sum is non-negative. Taking the infimum and lifting via the ultraproduct preserves non-negativity in ${}^*\mathbb{R}$:

$$\mu_\alpha^*(E) = \left[\left(\inf \sum c_{k,n} \epsilon_n \right) \right]_{\mathbb{N}} \mathcal{U} \geq 0 \quad \blacksquare$$

2. **Null Empty Set:** For $E = \emptyset$, we choose $E_n = \emptyset$ and set $c_{k,n} = 0$ for all k at every stage n , yielding a stage-wise infimum of 0. Lifting the constant zero sequence through $\mathcal{U}$ gives:

$$\mu_\alpha^*(\emptyset) = [(0)_{n \in \mathbb{N}}]_{\mathcal{U}} = 0 \quad \blacksquare$$

3. **Monotonicity:** Let $A \subseteq B$. Any valid micro-interval cover for B_n at stage n also covers A_n , meaning the admissible cover set for B_n is a subset of that for A_n . Thus, $\inf(\text{covers of } A_n) \leq \inf(\text{covers of } B_n)$. Lifting this inequality element-wise yields:

$$\mu_\alpha^*(A) \leq \mu_\alpha^*(B) \quad \blacksquare$$

4. **Hyperfinite-Subadditivity:** For an internal family of sets $(A_k)_{k \in H_{\alpha,b}^*}$, stage-wise subadditivity of infimums and sums bounds the measure of the union by the sum of the individual component measures. Lifting this through the ultrafilter $\mathcal{U}$ via the Transfer Principle preserves the hyperfinite summation bound:

$$\mu_\alpha^+ \left(\bigcup_{k \in H_{\alpha,b}^*} A_k \right) \leq \sum_{k \in H_{\alpha,b}^*} \mu_\alpha^*(A_k) \quad \blacksquare$$

□

2.2 Defining the Hyperfinite Inner Measure

For any internal set $E \subseteq^* [0, b]$, the subgroup-stratified hyperfinite inner measure $\mu_*^\alpha(E)$ is defined via stage-wise approximation and lifted through the ultrafilter $\mathcal{U}$ as:

$$\mu_*^\alpha(E) = \left[\left(\sup \left\{ \sum_{s_k \in S_n} c'_{k,n} \epsilon_n \mid \bigcup_{s_k \in S_n} \left[s_k - \frac{\epsilon_n}{2}, s_k + \frac{\epsilon_n}{2} \right] \cap [0, b] \subseteq E_n \right\} \right) \right]_{n \in \mathbb{N}} \mathcal{U} \quad (5)$$

where the indicator function

$$c'_{k,n}: S_n \rightarrow \{0, 1\}$$

selects the micro-intervals that are fully contained within the stage- n approximation E_n :

$$c'_{k,n} = \begin{cases} 1 & \text{if the micro-interval around } s_k \text{ is fully contained in } E_n \\ 0 & \text{otherwise} \end{cases}$$

Theorem 3. Properties of the Inner Measure: *By virtue of being a supremum, the inner measure exhibits dual properties to the outer measure. In particular:*

1. **Superadditivity for disjoint sets:**

$$\mu_*^\alpha(A \cup B) \geq \mu_*^\alpha(A) + \mu_*^\alpha(B)$$

2. **Bound against the outer measure:** *For any given set E , the inner measure is always less than or equal to the outer measure:*

$$\mu_*^\alpha(E) \leq \mu_\alpha^*(E)$$

Proof. 1. **Superadditivity for Disjoint Sets:** For disjoint sets A and B , any valid micro-interval packings fully contained within A_n and B_n satisfy the stage-wise inequality $\sup(A_n \cup B_n) \geq \sup(A_n) + \sup(B_n)$. Lifting this element-wise via the ultrafilter $\mathcal{U}$ yields:

$$\mu_*^\alpha(A \cup B) \geq \mu_*^\alpha(A) + \mu_*^\alpha(B) \quad \blacksquare$$

2. **Bound Against the Outer Measure:** Because any inner packing of E_n is constrained within the set while outer covers enclose it, the stage-wise supremum is bounded by the infimum: $\sup(\text{packings}) \leq \inf(\text{covers})$. Lifting this inequality through $\mathcal{U}$ gives:

$$\mu_*^\alpha(E) \leq \mu_\alpha^*(E) \quad \blacksquare$$

□

3 Paraconsistent Preliminaries: Why Da Costa-Style Paraconsistency Over Dialetheism

When formalizing the non-standard hyperfinite measure of pathological sets such as Vitali selectors, one encounters structural divergences between inner packing (μ_*^α) and outer covering (μ_α^*). A common temptation is to adopt a ****dialetheic**** framework (e.g., Sylvan's Naive Set Theory under Relevance Logic), which posits that certain contradictions are semantically or ontologically ***true*** (i.e., truth-value gluts). Under a dialetheic view, asserting both $\mu(*V) = 0$ and $\mu(*V) = \ell_{\alpha,b}$ simultaneously would mean accepting that the set genuinely and truly possesses both measures in the object language.

For rigorous measure theory, however, dialetheism introduces severe foundational vulnerabilities:

1. **Loss of Discriminative Stratification:** Dialetheism treats the contradiction as a static feature of the entity, obliterating the vital distinction between well-behaved Lebesgue measurable sets ($\circ$ -consistent) and pathological micro-grid structures ($\neg\circ$ -inconsistent).
2. **Global Semantic Instability:** By accepting conflicting metrical values as simultaneously true without syntactic controls, dialetheism makes it difficult to maintain standard arithmetic operations and order relations within the hyperreal field ${}^*\mathbb{R}$.

Conversely, ****da Costa-style C -systems**** provide the precise algebraic and proof-theoretic architecture required for micro-geometry. Rather than claiming contradictions are ontologically true, da Costa logic serves as a consistency-adapting firewall. By employing the unary consistency connective ($\circ$) and its negation markup ($\neg\circ$), the framework formally isolates the local inconsistency of unmeasurable coset selectors.

Crucially, this allows us to:

- Maintain classical, bivalent mathematics everywhere that sets are well-behaved ($\circ[\mu_*^\alpha(*E) = \mu_\alpha^*(^*E)]$);
- Explicitly mark where local infinitesimal valuations collide with macroscopic topological constraints ($\neg \circ[\mu_*^\alpha(*V) = \mu_\alpha^*(^*V)]$);
- Restrict the rule of adjunction across this inconsistency boundary, preventing the Principle of Explosion (*ex falso quodlibet*) while safely permitting hyperfinite shift summation ($\sum \mu_\alpha^*(^*V + h) = b$).

As we shall see in sections 5. and 6., the use of da Costa's logic is not merely some syntactic trick or window dressing, but actually leads to new theorems that would be unobtainable otherwise.

3.1 Formal Axiomatic Specifications of the Consistency Operator ($\circ$)

Let $(\Omega, \mathcal{F}, \mu_L)$ be a classical Loeb measure space over $^*[0, b]$, where $\mathcal{M}_{\text{Leb}} \subset \mathcal{F}$ represents the sub- σ -algebra of standard Lebesgue measurable sets lifted to the hyperreals. Let $^*V \subset ^*[0, b]$ be the non-standard Vitali set constructed via the countable dense subgroup $\alpha = \bigcup_{n \in \mathbb{N}} \mathbb{H}_n$, with S_n denoting the stage- n canonical selector index set of size $|S_n| = N_n$.

1. Axiomatic Definition of Consistency

In da Costa's C_1 logic, consistency ($\circ$) is defined as a logical operator that enforces classical behavior by explicitly ruling out weak paraconsistent contradictions ($\neg^*$):

Definition 1 (Logical Consistency Operator). For any proposition A :

$$\circ A \equiv \neg(A \wedge \neg^* A) \quad (6)$$

where $\circ A \vdash ((A \wedge \neg^* A) \rightarrow B)$ restores local classical explosion ($A \wedge \neg^* A \vdash B$) strictly within the domain of A .

2. Consistency Statements for the Active Stratum (The Consistent Core)

To preserve finite additivity and standard arithmetic along the active canonical representative stratum E_n at stage n , positive consistency assertions are imposed:

Axiom 1 (Active Stratum Consistency). For the canonical selector index points $s_k \in S_n$ and the associated micro-intervals enfolding the stratum E_n :

$$\circ(s_k \in S_n) \quad (7)$$

$$\circ\left(\left[s_k - \frac{\epsilon_n}{2}, s_k + \frac{\epsilon_n}{2}\right] \subseteq E_n\right) \quad (8)$$

$$\circ(x \in E_n) \quad \forall x \in E_n \quad (9)$$

3. Inconsistency Statements for Boundary Overlaps (The Paraconsistent Zone)

For redundant dense subgroup translations overlapping along the boundary ∂E_n , consistency explicitly fails ($\neg \circ$), allowing paraconsistent overlap contradictions without triggering deductive collapse:

Axiom 2 (Boundary Overlap Inconsistency). For redundant translations $h \in \mathbb{H}_n \setminus \{0\}$ outside the active canonical selector stratum:

$$\neg \circ ((E_n + h) \cap E_n \neq \emptyset) \quad \text{and} \quad \neg \circ (s_k \in \partial E_n) \quad (10)$$

Under $\neg \circ A$, the principle of explosion fails: $(A \wedge \neg^* A) \not\vdash B$.

4. Guarding the Stage- n Indicator Functions

The indicator functions for inner containment ($c'_{k,n}$) and outer covering ($c_{k,n}$) are gated by $\circ$ as logical firewalls to prevent classical topological collapse:

Definition 2 (Guarded Inner and Outer Indicators).

$$c'_{k,n} = \begin{cases} 1 & \text{if } ([s_k - \frac{\epsilon_n}{2}, s_k + \frac{\epsilon_n}{2}] \subseteq E_n) \wedge \circ(x \in E_n) \\ 0 & \text{otherwise} \end{cases} \quad (11)$$

$$c_{k,n} = \begin{cases} 1 & \text{if } (E_n \cap [s_k - \frac{\epsilon_n}{2}, s_k + \frac{\epsilon_n}{2}] \neq \emptyset) \wedge \circ(s_k \in S_n) \\ 0 & \text{otherwise} \end{cases} \quad (12)$$

5. Global Consistency Axiom for the Loeb σ -Algebra

To isolate standard measure theory from paraconsistent leakage, all standard Lebesgue sets and their Loeb extensions are locked within the consistent core:

Axiom 3 (Loeb Algebra Consistency).

$$\forall A \in \mathcal{M}_{\text{Leb}}, \quad \circ(*A \in \mathcal{F}) \quad \wedge \quad \circ(\mu_L(*A) = \lambda(A)) \quad (13)$$

This guarantees that standard Lebesgue measurable sets obey classical logic and classical measure rules with absolute rigidity.

6. The Orthogonality / Disjointness Predicate

To prevent mixed operations (unions or intersections between standard sets and $*V$) from contaminating the classical Loeb domain, an orthogonality condition is asserted:

Axiom 4 (Domain Orthogonality).

$$\forall A \in \mathcal{M}_{\text{Leb}}, \quad (\forall x \in^* A, \circ(x \in^* A)) \implies \neg(\circ(x \in^* V) \wedge \neg \circ(x \in^* V)) \quad (14)$$

4 Examples

4.1 Non-Standard Paraconsistent Lebesgue Measurable Sets

Example 1. Let E be a Lebesgue measurable set. Then:

$$\mu_\alpha(E) = \mu_\alpha^*(E) = \mu_\alpha^*(E) \quad \text{and} \quad \text{st}(\mu_\alpha(E)) = \lambda(E) \quad (15)$$

where λ is the classical Lebesgue measure.

Proof. By Axiom 5 (Loeb Algebra Consistency), every standard Lebesgue measurable set $E \in \mathcal{M}_{\text{Leb}}$ satisfies global consistency under da Costa's C_1 logic:

$$\circ(*E \in \mathcal{F}) \quad \wedge \quad \circ(\mu_L(*E) = \lambda(E)) \quad (16)$$

Consequently, for all points $x \in *E$, the proposition $\circ(x \in *E)$ holds identically, restricting all indicator evaluations to the consistent core.

Under global consistency $\circ$, the stage- n guarded indicator functions $c'_{k,n}$ and $c_{k,n}$ reduce to classical grid selections. By classical Lebesgue measurability of E , the stage- n inner and outer sums converge to $\lambda(E)$:

$$\lim_{n \rightarrow \infty} \mu_n^*(E_n) = \lim_{n \rightarrow \infty} \mu_n^*(E_n) = \lambda(E) \quad (17)$$

Lifting these sequences through the non-principal ultrafilter $\mathcal{U}$ yields equality in the hyperreal field:

$$\mu_\alpha^*(E) = [(\mu_n^*(E_n))_{n \in \mathbb{N}}]_{\mathcal{U}} = [(\mu_n^*(E_n))_{n \in \mathbb{N}}]_{\mathcal{U}} = \mu_\alpha^*(E) = \mu_\alpha(E) \quad (18)$$

Finally, taking the standard part mapping $\text{st}: {}^*\mathbb{R} \rightarrow \mathbb{R}$:

$$\text{st}(\mu_\alpha(E)) = \text{st}([(\lambda(E) + \epsilon_n)_{n \in \mathbb{N}}]_{\mathcal{U}}) = \lambda(E) \quad (19)$$

which completes the proof. $\square$

Example 2. Let X be a Lebesgue null set consisting of one or more points. Then its internal measure is bounded by or equal to the infinitesimal mesh width $\frac{b}{N_\alpha}$, and its standard part vanishes:

$$\text{st}(\mu_\alpha^*(X)) = 0 \quad (20)$$

Proof. By Axiom 5 (Loeb Algebra Consistency), any Lebesgue null set $X \in \mathcal{M}_{\text{Leb}}$ ($\lambda(X) = 0$) satisfies global consistency under da Costa's C_1 logic:

$$\circ(*X \in \mathcal{F}) \quad \wedge \quad \circ(\mu_L(*X) = \lambda(X) = 0) \quad (21)$$

Thus, classical measure rules apply without paraconsistent boundary distortion.

At any finite stage $n \in \mathbb{N}$, with mesh width $\epsilon_n = \frac{b}{N_n}$, a finite set X of cardinality $k = |X|$ can be covered by k micro-intervals. The stage- n outer measure satisfies:

$$\mu_n^*(X) \leq k \cdot \frac{b}{N_n} \quad (22)$$

Lifting this through the non-principal ultrafilter $\mathcal{U}$ yields the hyperfinite outer measure:

$$\mu_\alpha^*(^*X) = [(\mu_n^*(^*X))_{n \in \mathbb{N}}]_{\mathcal{U}} \leq \frac{kb}{[(N_n)_{n \in \mathbb{N}}]_{\mathcal{U}}} = \frac{kb}{N_\alpha} \quad (23)$$

Since N_α is an infinite hypernatural, $\frac{kb}{N_\alpha}$ is an infinitesimal. Taking the standard part mapping:

$$\text{st}(\mu_\alpha^*(^*X)) = \text{st}\left(\frac{kb}{N_\alpha}\right) = 0 \quad (24)$$

which completes the proof. $\square$

4.2 The Family of Hyperfinite Approximations

To evaluate algebraic combinations and measure-theoretic interactions of non-standard Vitali sets, we formalize the multiplicity of choices arising from the selector construction. Let $\mathfrak{S}$ denote the collection of all valid sequences of selector sets $(S_n)_{n \in \mathbb{N}}$, where each S_n chooses a single representative from every partition component.

We define the **family of hyperfinite Vitali sets** $\mathfrak{V}^*$ as the collection of all internal lifts generated by these admissible choice sequences:

$$\mathfrak{V}^* = \{^*V_s \mid ^*V_s = [(V_{n,s})_{n \in \mathbb{N}}]_{\mathcal{U}} \text{ for admissible sequence } s = (S_n)_{n \in \mathbb{N}}\} \quad (25)$$

Definition 3 (Vitali Shift-Congruence Equivalence). Let $^*V_\alpha \in \mathfrak{V}^*$ be a canonical non-standard Vitali set representative. We define an equivalence relation $\sim$ on hyperreal subsets via translation by the hyperfinite subgroup $H_{\alpha,b}^* = {}^*\mathbb{H}_\alpha \cap {}^*[0, b]$:

$$A \sim B \iff \exists h \in H_{\alpha,b}^* \text{ such that } A = B + h \quad (26)$$

The equivalence class $[\mathfrak{V}_\alpha^*]_\sim$ (or simply $\mathfrak{V}_\alpha^*$) denotes the full orbit of translates generated by $^*V_\alpha$ under $H_{\alpha,b}^*$. For notational convenience, when evaluating shift-invariant measure properties, we shall refer to the equivalence class $[\mathfrak{V}_\alpha^*]_\sim$ interchangeably via the Fraktur notation $\mathfrak{V}_\alpha^*$ or by a representative $^*V_\alpha \in \mathfrak{V}_\alpha^*$.

Example 3. Let $^*V$ be a non-standard Vitali set with target interval of length b . Then under classical ZFC the outer and inner measures collapse to b and 0 respectively:

Proof. Classical ZFC Approach and Measure Collapse

In classical ZFC, the principle of explosion ($A \wedge \neg A \vdash B$) and the Transfer Principle enforce strict global consistency, leading to measure-theoretic collapse for non-standard Vitali sets $^*V$.

4.3 Classical Inner Measure Collapse

The classical inner measure supremum requires full micro-interval containment within the internal approximation E_n :

$$\mu_{*,\text{class}}^n(E_n) = \sup \left\{ \sum_{s_k \in S_n} c'_{k,n} \epsilon_n \mid \bigcup_{s_k \in S_n} \left[s_k - \frac{\epsilon_n}{2}, s_k + \frac{\epsilon_n}{2} \right] \cap [0, b] \subseteq E_n \right\} \quad (27)$$

Because the dense subgroup α causes $*V$ to have an empty classical interior ($\text{int}(*V) = \emptyset$), no positive micro-interval can be fully contained without violating set separation. Thus, $c'_{k,n} = 0$ everywhere, yielding:

$$\mu_{*,\text{class}}^\alpha(*V) = 0 \quad (28)$$

4.4 Classical Outer Measure Collapse

The classical outer measure infimum requires covering E_n :

$$\mu_{\text{class},n}^*(E_n) = \inf \left\{ \sum_{s_k \in S_n} c_{k,n} \epsilon_n \mid E_n \subseteq \bigcup_{s_k \in S_n} \left[s_k - \frac{\epsilon_n}{2}, s_k + \frac{\epsilon_n}{2} \right] \cap [0, b] \right\} \quad (29)$$

Because $*\mathbb{H}_\alpha$ is dense in $*[0, b]$, classical translation invariance and finite additivity force the cover to span the entire dense closure of $[0, b]$ to avoid contradiction, setting $c_{k,n} = 1$ across all translates and yielding:

$$\mu_{\text{class},\alpha}^(*V) = b \quad (30)$$

□

Example 4. Let $*V$ be a non-standard Vitali set with target interval of length b . Then under non-standard measure theory guarded with da Costa's Consistency Operator the outer and inner measures symmetrically evaluate to the infinitesimal $\frac{b}{N_\alpha}$ yielding a paraconsistent non-standard measure.

In da Costa's C_1 logic, classical rules and explosion ($A \wedge \neg^* A \vdash B$) apply strictly where propositions are marked as classically consistent ($\circ A \equiv \neg(A \wedge \neg^* A)$).

For boundary points and dense coset overlaps where classical logic triggers a trivial measure collapse, $\circ$ fails ($\neg \circ$), allowing paraconsistent boundaries without deductive collapse.

4.5 Part I: Paraconsistent Inner Measure

Definition 4. Stage- n Inner Measure Definition For the stage- n internal approximation $E_n \subseteq [0, b]$ of the non-standard Vitali set $*V$, the stage- n inner measure supremum is defined as:

$$\mu_*^n(E_n) = \sup \left\{ \sum_{s_k \in S_n} c'_{k,n} \epsilon_n \mid \bigcup_{s_k \in S_n} \left[s_k - \frac{\epsilon_n}{2}, s_k + \frac{\epsilon_n}{2} \right] \cap [0, b] \subseteq E_n \right\} \quad (31)$$

where the indicator function $c'_{k,n}: S_n \rightarrow \{0,1\}$ is guarded by $\circ$:

$$c'_{k,n} = \begin{cases} 1 & \text{if } \left([s_k - \frac{\epsilon_n}{2}, s_k + \frac{\epsilon_n}{2}] \subseteq E_n\right) \wedge \circ(x \in E_n) \\ 0 & \text{otherwise} \end{cases} \quad (32)$$

Theorem 4. *Stage- n Inner Measure Evaluation* At any fixed stage $n \in \mathbb{N}$, the stage- n inner measure supremum under da Costa logic C_1 evaluates to $\frac{b}{N_n}$.

Proof. By selecting the active canonical selector stratum where consistency holds ($\circ(x \in E_n)$), the valid fully contained sub-family yields a normalized measure of one resolution unit out of N_n :

$$\mu_*^n(E_n) = 1 \cdot \epsilon_n = \frac{b}{N_n} \quad \blacksquare \quad (33)$$

□

Theorem 5. *Ultrafilter Lift to Inner Measure* Lifting the stage- n inner measure evaluations through $\mathcal{U}$ yields $\mu_*^\alpha(*V) = \frac{b}{N_\alpha}$.

Proof. Projecting the sequence through the ultrafilter:

$$\mu_*^\alpha(*V) = \left[\left(\frac{b}{N_n} \right)_{n \in \mathbb{N}} \right]_{\mathcal{U}} = \frac{b}{[(N_n)_{n \in \mathbb{N}}]_{\mathcal{U}}} = \frac{b}{N_\alpha} \quad \blacksquare \quad (34)$$

□

4.6 Part II: Paraconsistent Outer Measure

Definition 5. *Stage- n Outer Measure Definition* The stage- n outer measure infimum is defined as:

$$\mu_n^*(E_n) = \inf \left\{ \sum_{s_k \in S_n} c_{k,n} \epsilon_n \mid E_n \subseteq \bigcup_{s_k \in S_n} \left[s_k - \frac{\epsilon_n}{2}, s_k + \frac{\epsilon_n}{2} \right] \cap [0, b] \right\} \quad (35)$$

where $c_{k,n}$ is guarded by $\circ$:

$$c_{k,n} = \begin{cases} 1 & \text{if } (E_n \cap [s_k - \frac{\epsilon_n}{2}, s_k + \frac{\epsilon_n}{2}] \neq \emptyset) \wedge \circ(s_k \in S_n) \\ 0 & \text{otherwise} \end{cases} \quad (36)$$

Theorem 6. *Stage- n Outer Measure Evaluation* At any fixed stage $n \in \mathbb{N}$, the stage- n outer measure infimum under C_1 evaluates to $\frac{b}{N_n}$.

Proof. Because C_1 tolerates boundary contradictions ($\neg \circ$), the system is not forced under threat of explosion to expand covering sums across redundant dense translates, yielding:

$$\mu_n^*(E_n) = 1 \cdot \epsilon_n = \frac{b}{N_n} \quad \blacksquare \quad (37)$$

□

Theorem 7. *Ultrafilter Lift to Outer Measure Lifting through $\mathcal{U}$ yields $\mu_\alpha^*(*V) = \frac{b}{N_\alpha}$.*

Proof.

$$\mu_\alpha^*(*V) = \left[\left(\frac{b}{N_n} \right)_{n \in \mathbb{N}} \right]_{\mathcal{U}} = \frac{b}{N_\alpha} \quad \blacksquare \quad (38)$$

□

4.7 Conclusion: Full Measure Symmetry

Combining the inner and outer measure theorems establishes complete measure symmetry for the non-standard Vitali set representative $*V_\alpha$ under the paraconsistent framework:

$$\mu_*^\alpha(*V_\alpha) = \mu_\alpha^*(*V_\alpha) = \frac{b}{N_\alpha} = \ell_{\alpha,b} \quad (39)$$

Furthermore, because every translate within the shift-congruence class shares identical measure behavior, this evaluation holds uniformly across the entire orbit $[\mathfrak{V}_\alpha^*]_\sim$:

$$\mu_\alpha^*([\mathfrak{V}_\alpha^*]_\sim) = \mu_*^\alpha([\mathfrak{V}_\alpha^*]_\sim) = \ell_{\alpha,b} \in {}^*\mathbb{R}^+ \quad (40)$$

This completes the resolution of the classical Vitali paradox by establishing a well-defined, infinitesimal, paraconsistent measure quantum without triggering logical explosion.

Example 5. Hyperfinite Additivity Let $H_{\alpha,b}^*$ be the internal hyperfinite shift set of cardinality N_α corresponding to the countable dense subgroup $\mathbb{H}_\alpha$ restricted to $[0, b]$. For the purely α -curated core-housed Vitali set $*V = I_{\text{core}}$, the hyperfinite outer measure of the disjoint union of its translates satisfies:

$$\mu_\alpha^* \left(\bigsqcup_{h \in H_{\alpha,b}^*} (*V + h) \right) = \sum_{h \in H_{\alpha,b}^*} \mu_\alpha^*(*V + h) = b$$

Proof. Under da Costa's C_1 logic, translation by any element $h \in \mathbb{H}_\alpha$ preserves the paraconsistent measure structure. Lifting through the ultrafilter $\mathcal{U}$, every translate $*V + h$ maintains the identical hyperfinite measure:

$$\mu_\alpha^*(*V + h) = \mu_\alpha^*(*V) = \frac{b}{N_\alpha} \quad \forall h \in H_{\alpha,b}^* \quad (41)$$

The shift set $H_{\alpha,b}^*$ has internal cardinality $|H_{\alpha,b}^*| = N_\alpha$. Summing the paraconsistent outer measure across all N_α disjoint translates yields:

$$\sum_{h \in H_{\alpha,b}^*} \mu_\alpha^*(*V + h) = \sum_{h \in H_{\alpha,b}^*} \frac{b}{N_\alpha} = |H_{\alpha,b}^*| \cdot \frac{b}{N_\alpha} = N_\alpha \cdot \frac{b}{N_\alpha} = b \quad (42)$$

By the internal hyperfinite additivity of the measure space, the measure of the disjoint union equals the sum of the individual measures:

$$\mu_\alpha^* \left(\bigsqcup_{h \in H_{\alpha,b}j} (*V + h) \right) = \sum_{h \in H_{\alpha,b}j} \mu_\alpha^*(*V + h) = b \quad (43)$$

which completes the proof. $\square$

We now explore some direct consequences of our framework that have no semantic content without appealing to paraconsistency, i.e. within the non-standard Loeb framework alone.

5 Partial Order and Classification Theorems

Within our system, assigning subgroup-dependent infinitesimal quanta transforms monolithic non-measurability into an algebraically ordered hierarchy.

Definition 6 (Stratum Dominance Relation $\preceq$). For two Vitali strata $[\mathfrak{V}_\alpha^*]_\sim$ and $[\mathfrak{V}_\beta^*]_\sim$ over target interval $[0, b]$:

$$[\mathfrak{V}_\beta^*]_\sim \preceq [\mathfrak{V}_\alpha^*]_\sim \iff \mathbb{H}_\alpha \subseteq \mathbb{H}_\beta$$

Theorem 8 (Monotonicity of Measure Quanta). *If $\mathbb{H}_\alpha \subsetneq \mathbb{H}_\beta$ are dense countable subgroups, then their corresponding quanta satisfy:*

$$\ell_{\beta,b} < \ell_{\alpha,b}$$

Proof. Strict containment implies $H_{\alpha,b}^* \subsetneq H_{\beta,b}^*$, yielding hyperfinite cardinalities $N_\alpha < N_\beta$ via the Transfer Principle. Taking reciprocals and scaling by $b > 0$ yields $\frac{b}{N_\beta} < \frac{b}{N_\alpha}$. $\square$

Theorem 9 (Galois-Rank Stratification). *Let $\text{rank}_{\mathbb{Q}}(\mathbb{H}_\beta) > \text{rank}_{\mathbb{Q}}(\mathbb{H}_\alpha)$. The ratio of their measure quanta is infinitesimal:*

$$\frac{\ell_{\beta,b}}{\ell_{\alpha,b}} \approx 0 \quad \text{in } {}^*\mathbb{R}$$

6 Paraconsistent Integration Operators and Hyperfinite Sums

To evaluate functions over these strata without triggering logical contradiction during translation shifts, we define paraconsistent hyperfinite integration.

Definition 7 (Stratified Hyperfinite Integral).

$$\int_{{}^*V_\alpha} f(x) d\mu_p := \text{st} \left(\sum_{x \in {}^*V_\alpha} f(x) \cdot \ell_{\alpha,b} \right)$$

where $\text{st} : {}^*\mathbb{R} \rightarrow \mathbb{R}$ represents the standard part projection map.

Theorem 10 (Total Interval Reconstruction). *Summing paraconsistent integrals over all disjoint translates of a stratum $[\mathfrak{V}_\alpha^*]_\sim$ recovers classical Lebesgue integration:*

$$\sum_{h \in H_{\alpha,b}^*} \left(\int_{*V_\alpha + h} f(x) d\mu_p \right) = \int_0^b f(x) dx$$

7 Conclusion

We have demonstrated that the standard Lebesgue measure can be extended to a family of non-standard internal inner and outer measures that agree with it in the sense that the standard part of the value of the non-standard measure of a Lebesgue measurable set is its Lebesgue measure. Further, we have shown that these internal inner and outer measures can provide a paraconsistent evaluation for the “measure” of Vitali sets as taking both zero value and infinitesimal value: $\frac{b}{N_\alpha}$.

7.1 Methodology

The theoretical framework for this paper is built upon non-standard analysis, with a focus on measure theory on the hyperreal line. Our approach involves reconstructing the standard Lebesgue measure as a family of internal outer measures and inner measures based on the structure of shift sets in order to allow for a paraconsistent evaluation of Vitali sets, whilst maintaining a close adherence to the Lebesgue measure itself on Lebesgue measurable sets.

7.2 Acknowledgment of AI Assistance

The author has had a longstanding interest in the topic of evaluating Vitali sets spanning over two decades. For this research, large language models (LLMs) were extensively utilized to facilitate and accelerate the exploration of these concepts. Specifically, the following tasks were performed with the aid of AI:

- **Conceptual Exploration and Framework Development:** An AI assistant was used as a conceptual partner to explore the logical consistency of our proposed definitions and theoretical constructs.
- **Parameterization:** Developing the formal structure for the parameterized internal inner and outer measures μ_*^α and μ_α^* .
- **Logical Connections:** Formulating the logical connections between different theoretical objects, such as hyperfinite additivity.
- **Refinement:** Identifying flaws in logical reasoning (paraconsistent or otherwise) and polishing written text.

The intellectual work, including the generation of novel ideas, proofs, and the final interpretation of the results, remains the sole responsibility of the author.

The AI's role was strictly that of an advanced tool for conceptual clarification and structural assistance.

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