NO SET CARRIES EXACTLY TWO DENSE LINEAR ORDERS WITHOUT ENDPOINTS

A CUT-ROTATION PROOF IN A WEAK ZERMELO THEORY WITHOUT CHOICE OR REPLACEMENT

LIOR ISTHMUS

Abstract

Let Z_{sep} consist of Extensionality, Pairing, Infinity, Union, Power Set, and the full Separation schema, and write $s(X) = n$ when X carries exactly n isomorphism types of dense linear orders without endpoints. No form of Choice, Replacement, or Foundation is assumed. We prove

$$Z_{\mathrm{sep}} \vdash \neg\exists X\,(s(X) = 2).$$

Using countable-carrier uniqueness and the Dedekind-infinite four-type alternative from the exact-three companion, exact two forces X to be neither at most countable nor Dedekind-infinite and every DLO on X to be rigid. A type-count-free dyadic reflection argument eliminates the possibility that both types are self-dual, leaving one rigid non-self-dual dual pair $\{[L], [L^*]\}$.

For each $c \in L$, four one-point cut orders form an antipodal two-colored square. Vertical monochromaticity would self-dualize both rays and then the whole carrier; hence

$$L \cong \mathrm{Rot}_c(L).$$

Rigidity gives a unique isomorphism $\rho_c : L \to \mathrm{Rot}_c(L)$. Define $\delta(c) = \rho_c^{-1}(c)$. Its graph is obtained by Separation inside $X \times X$, without forming a Replacement-generated family $(\rho_c)_{c\in X}$. Opposite-ray exchange and hereditary Dedekind-finiteness force δ to be an injective strictly decreasing surjection. Thus δ is an order reversal of L, contradicting $L \not\cong L^*$.

Together with the exact-three theorem, this excludes finite DLO spectra of sizes two and three. Spectra of size at least four are not decided here.

Keywords. dense linear orders; axiom of choice; Dedekind-finite sets; cut rotation; categoricity; weak set theory.

2020 Mathematics Subject Classification. Primary 03E25; Secondary 03E30, 03C35, 03C64, 06A05.

Contents

1. Introduction

A **dense linear order without endpoints**, abbreviated **DLO**, is a nonempty strict linear order $(X, <)$ which is dense and has neither a least nor a greatest point. Throughout this paper,

$$X \text{ is at most countable} \quad \iff \quad X \hookrightarrow \omega,$$

and

$$X \text{ is Dedekind-infinite} \quad \iff \quad \omega \hookrightarrow X.$$

Without Choice these conditions need not be complementary.

In the language $\{<\}$, the DLO axioms form the complete theory $\mathrm{Th}(\mathbb{Q}, <)$; see Marker 2002. Thus the finite spectra considered here are precisely the numbers of models of this theory, up to internal isomorphism, on one fixed underlying set. For background on cardinal comparison and Dedekind-finite sets without Choice, see Jech 1973 and Howard–Rubin 1998.

The companion paper Isthmus 2026a proves in the same weak theory that no set carries exactly three DLO types, thereby answering Shelah's Question 7.11 negatively Shelah 2009. Two global spectrum results are used directly below: at-most-countable DLO uniqueness and the four-type alternative on every Dedekind-infinite carrier. Several elementary order-theoretic and foundational lemmas are also reused, including rigidity facts, finite ranking, binary-word coding, finite block sums, and bounded recursion. Appendix B of the present paper extracts and proves the type-count-free dyadic reflection engine needed for exact two, rather than appealing to an exact-three theorem with altered hypotheses. The earlier Lean 4 development formalizes the exact-three theorem package Isthmus 2026b.

A companion Lean 4 development formalizes all 45 numbered items in this paper and the claimed proof-theoretic endpoint Isthmus 2026c. It includes the object-language theory Z_{sep}, a natural-deduction calculus, and the completeness bridge used to produce the final derivation. Its final declaration is `ExactTwoDLO.FO.zsep_proves_not_spec2Sentence`; its type is the derivation assertion

$$\text{Provable } Z_{\mathrm{sep}}(\sim \text{spec2Sentence}).$$

The exact-three release is imported there as a pinned, read-only dependency; the exact-two reflection engine and cut-rotation argument are implemented in the separate `ExactTwoDLO` development.

Exact two has a different obstruction from exact three. On three types, reversal has a fixed point and therefore supplies a self-dual type. On two types, reversal may exchange the two types and have no fixed point. The main task is therefore to break one rigid non-self-dual dual pair without first assuming a self-dual representative.

The proof is organized around two statements.

Theorem 1.1 (Exact-two structural reduction). If $s(X) = 2$, then:

(1) $X \not\hookrightarrow \omega$;
(2) $\omega \not\hookrightarrow X$;
(3) every DLO on X is rigid;
(4) reversal exchanges the two types;
(5) no DLO on X is self-dual;
(6) after naming one representative L, the two types are

$$\{[L], [L^*]\}, \qquad L \ncong L^*.$$

Theorem 1.2 (Cut-preimage reversal). Let $L = (X, <)$ satisfy the residual conditions of Theorem 1.1. If every one-point cut rotation $\mathrm{Rot}_c(L)$ is isomorphic to L, then the unique cut-rotation isomorphisms determine a decreasing bijection

$$\delta : X \longrightarrow X.$$

Consequently $L \cong L^*$.

The finite cut square proves the hypothesis of Theorem 1.2. The contradiction with Theorem 1.1 then excludes exact two.

The new mechanism is local. Exact-three exclusion uses a self-dual representative and a full dyadic reflection tree. Exact-two exclusion uses the tree only to eliminate the all-self-dual reversal pattern. Once the remaining dual pair is isolated, a four-corner cut square and the point-indexed preimage map δ close the proof.

The paper is a genuine companion rather than a duplicate. The foundational coding, countable benchmark theorem, and Lindenbaum theorem are imported from Isthmus 2026a. The present paper proves the exact-two reduction, a parameterized type-count-free reflection-tree engine, the cut-square theorem, and the cut-preimage reversal.

2. Formal setting and imported results

Definition 2.1 (The base theory). Z_{sep} consists of Extensionality, Pairing, Infinity, Union, Power Set, and the full Separation schema. No use is made of Choice, Countable Choice, Dependent Choice, Replacement, Collection, or Foundation.

All orders are subsets of $X \times X$, and all witnessing maps are internal sets. The construction of ω, bounded unique recursion, finite functions, finite block sums, and the relevant arithmetic are available in this theory by Isthmus 2026a, Appendices A–B.

Definition 2.2 (Finite spectrum formulas). For every standard finite $n \geq 1$, let $\mathrm{Spec}_{\geq n}(X)$ assert that there are n pairwise nonisomorphic DLO relations on X. Put

$$\mathrm{Spec}_{=n}(X) := \mathrm{Spec}_{\geq n}(X) \wedge \neg\, \mathrm{Spec}_{\geq n+1}(X). \tag{1}$$

The notation $s(X) \geq n$ abbreviates $\mathrm{Spec}_{\geq n}(X)$, and $s(X) = n$ abbreviates $\mathrm{Spec}_{=n}(X)$. No quotient set of isomorphism classes is formed.

If $s(X) = n$ and $R_0, \ldots, R_{n-1}$ are pairwise nonisomorphic DLOs on X, then every DLO on X is isomorphic to exactly one R_i. Otherwise an additional type would witness $s(X) \geq n + 1$.

Definition 2.3 (Order notation). For sets A, B, the notation $A \hookrightarrow B$ means that an injection from A to B exists, and $A \not\hookrightarrow B$ means that no such injection exists. For $m \in \omega$, the notation $|A| \leq m$ abbreviates the existence of an injection $A \to m$, and $|A| = m$ abbreviates the existence of a bijection $A \to m$.

For a linear order $L = (X, <)$:

- L^* is the reverse order;
- $(-\infty, c)_L$ and $(c, \infty)_L$ are the strict rays at c;
- an **order reversal** of L is an anti-isomorphism $L \to L$;
- L is **self-dual** if $L \cong L^*$;
- L is **rigid** if its increasing automorphism group is trivial.

Finite order sums are taken over disjoint blocks. Empty blocks are omitted, and every new adjacent boundary is checked against the finite block-sum DLO criterion of Isthmus 2026a, Lemma 2.7.

Theorem 2.4 (At-most-countable DLO uniqueness). If X carries a DLO and $X \hookrightarrow \omega$, then

$$s(X) = 1. \tag{2}$$

Proof. This is Isthmus 2026a, Lemma 3.11. The proof first shows that an infinite subcountable set is bijective with ω, then carries out a deterministic least-index back-and-forth construction inside a fixed power set. □

Theorem 2.5 (Dedekind-infinite four-type alternative). If X carries a DLO and $\omega \hookrightarrow X$, then

$$X \hookrightarrow \omega \quad \text{or} \quad s(X) \geq 4. \tag{3}$$

Proof. This is Isthmus 2026a, Theorem 3.12. A choice-free monotone-subsequence construction gives a countable benchmark whose DLO complement is either subcountable or supports four pairwise nonisomorphic same-carrier DLOs. □

Lemma 2.6 (A nontrivial increasing automorphism gives a countable orbit). If a linear order on X has a nonidentity increasing automorphism, then

$$\omega \hookrightarrow X. \tag{4}$$

Proof. This is Isthmus 2026a, Lemma 6.1. Choose $x < g(x)$ after replacing g by g^{-1} if necessary, and recursively form the strictly increasing orbit $g^n(x)$. □

Corollary 2.7 (Common-host self-duality criterion). If one linear order contains initial segments isomorphic to both A and A^*, or final segments isomorphic to both, then

$$A \cong A^*. \tag{5}$$

Proof. This is Isthmus 2026a, Corollary 4.3, obtained from Lindenbaum's initial/final-segment theorem; see also Ervin 2017, §2.3. □

Lemma 2.8 (Rigidity and reversals on convex suborders). Let L be rigid.

(1) Every convex suborder of L is rigid.
(2) Every reversal of a rigid order is involutive.
(3) A rigid self-dual order has a unique reversal.

Proof. This is Isthmus 2026a, Lemma 4.8. A convex-suborder automorphism extends by the identity, two reversals differ by an automorphism, and the square of a reversal is an automorphism. $\square$

3. Exact-two structural reduction

Throughout this section assume

$$s(X) = 2,$$

and fix pairwise nonisomorphic representatives

$$L_0 = (X, <_0), \qquad L_1 = (X, <_1).$$

Lemma 3.1 (An exact-two carrier is not at most countable). One has

$$X \not\hookrightarrow \omega.$$

Proof. Otherwise Theorem 2.4 would give $s(X) = 1$. $\square$

Lemma 3.2 (An exact-two carrier is not Dedekind-infinite). One has

$$\omega \not\hookrightarrow X.$$

Proof. If $\omega \hookrightarrow X$, Theorem 2.5 gives either $X \hookrightarrow \omega$ or $s(X) \geq 4$. The first alternative contradicts Lemma 3.1. In the second, discarding one of four witnesses gives $s(X) \geq 3$, contradicting $s(X) = 2$. $\square$

Lemma 3.3 (Every exact-two DLO is rigid). Every DLO on X has trivial increasing automorphism group.

Proof. A nontrivial increasing automorphism would give $\omega \hookrightarrow X$ by Lemma 2.6, contradicting Lemma 3.2. $\square$

Lemma 3.4 (Hereditary Dedekind-finiteness). Let $Y \subseteq X$. Every injective map

$$j : Y \longrightarrow Y$$

is surjective.

Proof. Suppose j were injective and not surjective. Choose

$$y_0 \in Y \setminus j[Y]$$

and define

$$y_{n+1} = j(y_n)$$

by bounded unique recursion inside the fixed set $\omega \times Y$. If $n < m$ and $y_n = y_m$, injectivity cancels the first n iterates and gives

$$y_0 = j^{m-n}(y_0) \in j[Y],$$

contrary to the choice of y_0. Thus $n \mapsto y_n$ is an injection $\omega \hookrightarrow Y \hookrightarrow X$, contradicting Lemma 3.2. $\square$

Definition 3.5 (Reversal action on the two types). For each $i < 2$, let $\sigma(i) < 2$ be the unique index satisfying

$$L_i^* \cong L_{\sigma(i)}.$$

The graph of σ is the Separation-defined subset

$$\{(i,j) \in 2 \times 2 : L_i^* \cong L_j\};$$

exactness makes this relation a function. Reversing twice gives

$$\sigma^2 = \mathrm{id}_2 .$$

Hence either $\sigma = \mathrm{id}_2$ or $\sigma = (01)$.

Lemma 3.6 (Symmetric flank localization when all global types are self-dual). Assume every DLO on X is self-dual. Let P be a DLO on X with an involutive reversal k, and suppose

$$P = A + M + k[A], \qquad k[M] = M, \tag{6}$$

where A and $k[A]$ are nonempty DLOs without endpoints. Then

$$A \cong A^*.$$

Proof. On the same carrier define

$$R = A + M + k[A]^*, \qquad S = A^* + M + k[A].$$

The displayed decomposition of P makes M convex; if it has at least two points, it is therefore internally dense. After empty blocks are omitted, every reduced boundary is dense because the flank on the required side has no endpoint. Hence the finite block-sum criterion makes R and S DLOs.

The map k reverses the block order. On the reversed right flank, for example,

$$x <_{k[A]^*} y \quad\Longleftrightarrow\quad y <_{k[A]} x \quad\Longleftrightarrow\quad k(x) <_A k(y) \quad\Longleftrightarrow\quad k(y) <_{A^*} k(x).$$

The other blocks and mixed comparisons are immediate from $k[M] = M$ and the fact that k reverses P. Thus $k : R \to S$ is an anti-isomorphism, and

$$R^* \cong S.$$

Every DLO on X is self-dual, hence

$$R \cong R^* \cong S.$$

Under an isomorphism $R \to S$, the image of the initial block A is an initial segment of type A, while the literal first block of S is an initial segment of type A^*. Corollary 2.7 gives $A \cong A^*$. ☐

Proposition 3.7 (All-self-dual rigid carrier dichotomy). Assume every DLO on X is self-dual and let L be a rigid DLO on X. Then

$$X \hookrightarrow \omega \quad \text{or} \quad \omega \hookrightarrow X.$$

Proof. Choose a reversal h of L. By rigidity it is unique and involutive. Lemma B.1 gives the convex decomposition

$$X = H + F_h + h[H],$$

where H and $h[H]$ are nonempty DLOs without endpoints and F_h is empty or a singleton. Lemma 3.6, applied with $A = H$ and $M = F_h$, gives

$$H \cong H^*.$$

The half H is rigid because it is convex in the rigid order L. Thus the standing hypotheses of the type-count-free dyadic engine in Appendix B are satisfied. Corollary B.8 yields

$$X \hookrightarrow \omega \quad \text{or} \quad \omega \hookrightarrow X.$$

□

Theorem 3.8 (Exact-two residual normal form). If $s(X) = 2$, then:

(1) $X \not\hookrightarrow \omega$;
(2) $\omega \not\hookrightarrow X$;
(3) every DLO on X is rigid;
(4) reversal exchanges the two types;
(5) no DLO on X is self-dual;
(6) after naming one representative L, the spectrum is

$$\{[L], [L^*]\}, \qquad L \ncong L^*. \tag{7}$$

Proof. The first three assertions are Lemmas 3.1–3.3. If $\sigma = \mathrm{id}_2$, then every DLO on X is self-dual: if $U \cong L_i$, then

$$U^* \cong L_i^* \cong L_i \cong U.$$

Choosing either rigid representative and applying Proposition 3.7 gives $X \hookrightarrow \omega$ or $\omega \hookrightarrow X$, contradicting Lemmas 3.1 and 3.2. Hence $\sigma = (01)$. A self-dual DLO would represent a fixed point of σ, so none exists. The final description follows. □

4. The one-point cut square

Fix the representative

$$L = (X, <)$$

from Theorem 3.8.

Definition 4.1 (The antipodal cut square). For $c \in X$, put

$$A_c := (-\infty, c)_L, \qquad B_c := (c, \infty)_L.$$

Define four relations on the carrier X by

$$\mathcal{C}^c_{00} = A_c + \{c\} + B_c,$$

$$\mathcal{C}^c_{01} = A_c^* + \{c\} + B_c^*,$$

$$\mathcal{C}^c_{10} = B_c + \{c\} + A_c,$$

and

$$\mathcal{C}^c_{11} = B^*_c + \{c\} + A^*_c.$$

We display the square as

$$\begin{matrix} \mathcal{C}^c_{00} & \mathcal{C}^c_{10} \\ \mathcal{C}^c_{01} & \mathcal{C}^c_{11}. \end{matrix}$$

Thus the vertical edges are $\mathcal{C}^c_{00}$–$\mathcal{C}^c_{01}$ and $\mathcal{C}^c_{10}$–$\mathcal{C}^c_{11}$, while the horizontal edges are $\mathcal{C}^c_{00}$–$\mathcal{C}^c_{10}$ and $\mathcal{C}^c_{01}$–$\mathcal{C}^c_{11}$.

The order $\mathcal{C}^c_{10}$ is the **cut rotation** of L at c and is denoted

$$\mathrm{Rot}_c(L). \tag{8}$$

Lemma 4.2 (The cut square is actual and antipodal). For every $c \in X$, all four relations in Definition 4.1 are DLOs on X. Moreover,

$$\mathcal{C}^c_{11} = (\mathcal{C}^c_{00})^*, \qquad \mathcal{C}^c_{10} = (\mathcal{C}^c_{01})^*. \tag{9}$$

Proof. Each strict ray of a DLO is a convex DLO without endpoints. Reversing a ray preserves this property. Inserting one singleton between two endpointless DLO blocks satisfies the finite block-sum criterion. Each displayed relation is defined by Separation on $X \times X$. Reversing the block lists gives the two identities. ☐

Lemma 4.3 (Antipodal two-color square). Color the four vertices of a 2×2 square with two colors. If antipodal vertices have opposite colors, then either both vertical edges are monochromatic or both horizontal edges are monochromatic.

Proof. Fix the color of the upper-left vertex. The lower-right vertex has the other color. The upper-right vertex has one of the two colors, and the lower-left vertex, being antipodal to it, has the other. The two possible choices give the two conclusions. ☐

Lemma 4.4 (Two rigid self-dual DLOs and one point self-dualize their union). Let A and B be disjoint carriers of rigid self-dual DLOs, and let $c \notin A \cup B$. Then $A \sqcup \{c\} \sqcup B$ carries a self-dual DLO.

Proof. Let r_A and r_B be the unique reversals. By rigidity they are involutions. Lemma B.1 shows that the two reflection halves of each order are nonempty DLOs without endpoints and that each fixed-point set is empty or a singleton. Put

$$A^- := \{a : a < r_A(a)\}, \qquad A^+ := \{a : r_A(a) < a\},$$

$$F_A := \{a \in A : r_A(a) = a\},$$

and define B^-, B^+ and F_B analogously. The two halves in each pair are exchanged by the relevant reversal.

Put

$$F := F_A \sqcup F_B \sqcup \{c\}.$$

Map a point of F_A to 0, the point c to 1, and a point of F_B to 2. Because the three pieces are disjoint and the two fixed-point sets are empty or singletons, this is an injection $F \to 3$. Finite ranking Isthmus 2026a, Lemma 2.8 therefore gives an enumeration

$$F = \{z_0, \dots, z_{t-1}\}, \qquad 1 \le t \le 3.$$

Give the carrier one of the following block orders:

$$A^- + B^- + \{z_0\} + B^+ + A^+ \qquad (t = 1),$$

$$A^- + \{z_0\} + B^- + B^+ + \{z_1\} + A^+ \qquad (t = 2),$$

or

$$A^- + \{z_0\} + B^- + \{z_1\} + B^+ + \{z_2\} + A^+ \qquad (t = 3).$$

Every displayed order is a DLO: it starts and ends with endpointless DLO blocks, and no two singleton blocks are adjacent. Define

$$h = \big(r_A \restriction (A^- \cup A^+)\big) \cup \big(r_B \restriction (B^- \cup B^+)\big) \cup \{(z_i, z_{t-1-i}) : i < t\}. \tag{10}$$

The three domains in this union are disjoint, so h is a function. Each restricted reversal is involutive, and $i \mapsto t-1-i$ is an involution on t; hence

$$h^2 = \mathrm{id}_{A \sqcup \{c\} \sqcup B}\,.$$

In particular, h is bijective. It reverses the entire block list and every internal block order, so it is an order reversal of the constructed DLO. Hence that DLO is self-dual. $\square$

Theorem 4.5 (A vertical monochromatic cut edge is impossible). Under the residual normal form of Theorem 3.8, the two DLOs

$$\mathcal{C}^c_{00} \quad \text{and} \quad \mathcal{C}^c_{01}$$

are nonisomorphic for every $c \in X$.

Proof. Suppose

$$f : \mathcal{C}^c_{00} \longrightarrow \mathcal{C}^c_{01}$$

were an isomorphism.

If $f(c) <_{\mathcal{C}^c_{01}} c$, then $f(c) \in A_c$ and

$$f[A_c] = (-\infty, f(c))_{\mathcal{C}^c_{01}} = \{x \in A_c : f(c) <_L x < c\} \subsetneq A_c.$$

The inclusion is proper because $f(c) \in A_c$ but $f(c) \notin f[A_c]$. This is a set-theoretic statement; the target happens to carry the reversed order on that subset. Thus $f \restriction A_c$ is an injective nonsurjective selfmap of A_c, contradicting Lemma 3.4.

If $c <_{\mathcal{C}^c_{01}} f(c)$, then $f(c) \in B_c$ and

$$f[B_c] = (f(c), \infty)_{\mathcal{C}^c_{01}} = \{x \in B_c : c < x <_L f(c)\} \subsetneq B_c,$$

where the inclusion is proper because $f(c) \in B_c \setminus f[B_c]$. This gives the same contradiction. Therefore

$$f(c) = c.$$

It follows that $f[A_c] = A_c$ and $f[B_c] = B_c$. Since the target order reverses the inherited order on each ray, the restrictions of f are order reversals of the original DLOs on A_c and B_c. These ray orders are rigid by Lemma 2.8 because they are convex in the rigid order L. Lemma 4.4 therefore constructs a self-dual DLO on X, contradicting Theorem 3.8. $\square$

Theorem 4.6 (Every cut rotation has the original type). For every $c \in X$,

$$\mathrm{Rot}_c(L) \cong L. \tag{11}$$

The isomorphism is unique.

Proof. Color each corner $\mathcal{C}^c_{ij}$ by the unique exact-two type it represents. By Theorem 3.8, reversal exchanges the two types. The antipodal identities in Lemma 4.2 therefore make antipodal corners opposite in color. Lemma 4.3 gives a monochromatic direction.

The vertical direction is impossible by Theorem 4.5. Hence both horizontal edges are monochromatic. In particular,

$$\mathcal{C}^c_{00} = L \cong \mathcal{C}^c_{10} = \mathrm{Rot}_c(L).$$

If $f, g : L \to \mathrm{Rot}_c(L)$ are two isomorphisms, then $g^{-1} \circ f$ is an increasing automorphism of the rigid order L. Thus $f = g$. $\square$

5. The cut-preimage reversal

Throughout this section, let $L = (X, <)$ be a rigid DLO satisfying the following two hypotheses:

$$\text{(HDF)} \qquad \forall Y \subseteq X\ \forall j : Y \to Y, \big(j \text{ injective} \implies j[Y] = Y\big), \tag{12}$$

and

$$\text{(Rot)} \qquad \forall c \in X, \mathrm{Rot}_c(L) \cong L. \tag{13}$$

In the exact-two application, (HDF) is Lemma 3.4 and (Rot) is Theorem 4.6.

Definition 5.1 (The cut-preimage map). For a fixed $c \in X$, let

$$\rho_c : L \longrightarrow \mathrm{Rot}_c(L)$$

be the unique isomorphism supplied by (Rot) and rigidity. Define

$$\delta(c) := \rho_c^{-1}(c). \tag{14}$$

The graph of δ is the Separation-defined subset of $X \times X$

$$\left\{ \begin{aligned} (c, d) \in X \times X : \exists r \in \mathcal{P}(X \times X) &\text{ such that} \\ &r \text{ is an order isomorphism } L \to \mathrm{Rot}_c(L), \\ &(d, c) \in r \end{aligned} \right\}. \tag{15}$$

Rigidity gives uniqueness of r, and bijectivity of r gives uniqueness of d. No set of all maps ρ_c is formed.

Lemma 5.2 (Cut rotations exchange opposite rays). Let $d = \delta(c)$. Then ρ_c restricts to order isomorphisms

$$(-\infty, d)_L \cong (c, \infty)_L \tag{16}$$

and

$$(d, \infty)_L \cong (-\infty, c)_L. \tag{17}$$

Proof. In the rotated order

$$\mathrm{Rot}_c(L) = (c, \infty)_L + \{c\} + (-\infty, c)_L,$$

the strict initial segment below c is the original right ray and the strict final segment above c is the original left ray. Since $\rho_c(d) = c$, an order isomorphism sends the two rays at d to these two segments. □

Lemma 5.3 (The cut-preimage map is injective). The function $\delta : X \to X$ is injective.

Proof. Suppose

$$\delta(c) = \delta(e) = d$$

with $c < e$. Put

$$A := (-\infty, d)_L, \qquad T_c := (c, \infty)_L, \qquad T_e := (e, \infty)_L.$$

Lemma 5.2 gives isomorphisms

$$\rho_c \restriction A : A \to T_c, \qquad \rho_e \restriction A : A \to T_e.$$

Therefore

$$T_c \xrightarrow{(\rho_c \restriction A)^{-1}} A \xrightarrow{\rho_e \restriction A} T_e \hookrightarrow T_c$$

is an injective selfmap of T_c with range exactly T_e. The inclusion is proper because

$$e \in T_c \setminus T_e.$$

This contradicts (HDF). The case $e < c$ is symmetric. □

Lemma 5.4 (The cut-preimage map is strictly decreasing). If $c < e$, then

$$\delta(e) < \delta(c). \tag{18}$$

Proof. Suppose instead that

$$\delta(c) < \delta(e).$$

Put

$$I_c := (-\infty, \delta(c))_L, \qquad I_e := (-\infty, \delta(e))_L.$$

Then

$$I_c \subsetneq I_e,$$

with

$$\delta(c) \in I_e \setminus I_c.$$

Meanwhile,

$$(e, \infty)_L \subsetneq (c, \infty)_L.$$

Using Lemma 5.2, form the composite

$$I_e \xrightarrow{\rho_e} (e, \infty)_L \hookrightarrow (c, \infty)_L \xrightarrow{\rho_c^{-1}} I_c \hookrightarrow I_e. \tag{19}$$

This is an injective selfmap of I_e, and

$$\operatorname{ran}(\Psi_{c,e}) = \rho_c^{-1}[(e, \infty)_L] \subseteq I_c \subsetneq I_e, \tag{20}$$

where $\Psi_{c,e}$ denotes the displayed composite. Thus it is not surjective, contradicting (HDF). Hence $\delta(c) \not< \delta(e)$. Lemma 5.3 excludes equality, so $\delta(e) < \delta(c)$. □

Corollary 5.5 (The cut-preimage map is a decreasing bijection). The function $\delta : X \to X$ is bijective and strictly decreasing.

Proof. It is injective by Lemma 5.3. Hypothesis (HDF), applied to $Y = X$, makes every injective selfmap of X surjective. Strict decrease is Lemma 5.4. □

Theorem 5.6 (Cut-rotation self-duality theorem). Let $L = (X, <)$ be a rigid DLO satisfying (HDF) and (Rot). Then

$$L \cong L^*.$$

Proof. Rigidity makes every cut-rotation isomorphism unique, so Definition 5.1 gives the cut-preimage map. Lemmas 5.2–5.4 and Corollary 5.5 show that δ is a decreasing bijection of X. Hence, for all $x, y \in X$,

$$x <_L y \iff \delta(y) <_L \delta(x) \iff \delta(x) <_{L^*} \delta(y). \tag{21}$$

For the reverse implication in the first equivalence, suppose $\delta(y) <_L \delta(x)$. If $y <_L x$, strict decrease gives $\delta(x) <_L \delta(y)$, while $x = y$ contradicts the assumed strict inequality. Trichotomy therefore gives $x <_L y$. Hence δ is an order isomorphism from L to L^*. □

6. Exact-two exclusion

Theorem 6.1 (No exact-two DLO spectrum). Z_{sep} proves

$$\neg\exists X\,(s(X) = 2). \tag{22}$$

Proof. Assume $s(X) = 2$. Theorem 3.8 gives a rigid representative L satisfying

$$L \ncong L^*,$$

Lemma 3.4 gives (HDF), and Theorem 4.6 gives (Rot). Theorem 5.6 therefore gives $L \cong L^*$, a contradiction. □

Corollary 6.2 (No exact two or exact three). One has

$$Z_{\text{sep}} \vdash \neg\exists X\,(\mathrm{Spec}_{=2}(X) \vee \mathrm{Spec}_{=3}(X)). \tag{23}$$

Proof. The exact-two case is Theorem 6.1. The exact-three case is Isthmus 2026a, Theorem 6.3. □

Corollary 6.3 (ZF consequence). ZF proves that no set carries exactly two DLO isomorphism types.

Proof. ZF contains every axiom of Z_{sep}. □

7. Scope and further questions

For each standard finite $n \geq 2$, let $(\mathrm{FS})_n$ denote the metatheoretic assertion

$$Z_{\mathrm{sep}} \vdash \neg\exists X \ \mathrm{Spec}_{=n}(X).$$

The present paper and its exact-three companion prove $(\mathrm{FS})_2$ and $(\mathrm{FS})_3$. They do not prove $(\mathrm{FS})_n$ for $n \geq 4$.

The exact-two mechanism is specific to a two-color antipodal square. Once reversal is known to exchange the two types, opposite cut-square corners have opposite colors, and one of the two edge directions is forced to be monochromatic. Vertical monochromaticity is incompatible with the absence of self-dual types, so every cut rotation has the original type. For larger spectra, a four-corner cut square need not force any same-type edge, and the argument stops before the cut-preimage map can be defined globally.

Theorem 5.6 is independent of finite spectrum language. It says that a rigid DLO on a hereditarily Dedekind-finite carrier cannot be isomorphic to all of its one-point cut rotations unless it is self-dual. This cut-rotation rigidity principle may be useful in other choiceless classification problems.

The present methods do not decide whether any standard finite spectrum of size at least four can occur, or which such spectra are consistent. The countable-benchmark theorem gives only the lower bound $s(X) \geq 4$ on a non-subcountable Dedekind-infinite carrier; that bound does not exclude exact four or any larger finite value.

Appendix A. Weak-theory audit of the new construction

Proposition A.1 (Uniform formation of the cut rotations). For each $c \in X$, the graph of $\mathrm{Rot}_c(L)$ is a set obtained by Separation on $X \times X$.

Proof. For $x, y \in X$, the relation $x <_{\mathrm{Rot}_c(L)} y$ is the finite disjunction saying that either:

(1) $c < x < y$ in L;
(2) $c < x$ and $y = c$;
(3) $c < x$ and $y < c$;
(4) $x = c$ and $y < c$;
(5) $x < y < c$ in L.

This formula defines the block order

$$(c, \infty)_L + \{c\} + (-\infty, c)_L$$

inside the fixed set $X \times X$. □

Proposition A.2 (The finite type table requires no Choice). For a fixed c, assigning to each of the four cut-square relations its unique exact-two representative is a finite set construction in Z_{sep}.

Proof. The four relations form a set by Pairing and Union. For each relation, exactness gives a unique index in 2. The graph of the assignment is the Separation subset of the finite product 4×2 satisfying the internal isomorphism predicate. No choice from an infinite family is involved. □

Proposition A.3 (The cut-preimage graph uses no Replacement). The graph in Equation (15) is a set in Z_{sep}.

Proof. Every candidate is a pair in the fixed set $X \times X$, and every candidate isomorphism is an element of the fixed power set $\mathcal{P}(X \times X)$. The formula in Equation (15) therefore defines the graph by Separation. Uniqueness turns it into a function. At no point is the range

$$\{\rho_c : c \in X\}$$

formed. □

Proposition A.4 (The ray composites are internal maps). Every restriction, inverse, inclusion, and finite composite used in Lemmas 5.3 and 5.4 is an internal set.

Proof. Restrictions and inclusions are Separation-defined subsets of $X \times X$. The inverse of a bijection is obtained by reversing ordered pairs. The graph of a composite of two maps is defined by an existential formula on the appropriate fixed Cartesian product. Only finitely many maps are composed in each argument, so Pairing, Union, products, and Separation suffice. □

Appendix B. A type-count-free dyadic reflection engine

This appendix isolates the reflection-tree mechanism needed in Proposition 3.7. It is parameterized by self-duality inputs rather than by an exact finite spectrum. In addition to the standing infrastructure recorded in §2, it uses Lemma 2.8 and Lemma 3.6. Its remaining imported inputs are elementary foundational lemmas from the exact-three companion: finite ranking, finite unions of subcountable sets, explicit binary-word coding, and internally finite block sums Isthmus 2026a, Lemmas 2.8, 3.4(1), B.3, and Proposition B.5. No result whose statement assumes $s(X) = 3$ is used below.

To keep word sets distinct from finite ordinals, write

$$\mathrm{Bin}_{<\omega} := 2^{<\omega}, \qquad \mathrm{Bin}_n := \{s \in \mathrm{Bin}_{<\omega} : |s| = n\},$$

$$\mathrm{Bin}_{<n} := \{s \in \mathrm{Bin}_{<\omega} : |s| < n\}, \qquad \mathrm{Bin}_{\leq n} := \{s \in \mathrm{Bin}_{<\omega} : |s| \leq n\},$$

and let

$$N_n := 2^n.$$

Here the right-hand side is the finite ordinal obtained by exponentiation, so $N_n \in \omega$. We use the explicit bijections

$$v_n : \mathrm{Bin}_n \longrightarrow N_n$$

and

$$c_n : \mathrm{Bin}_{<n} \longrightarrow N_n - 1$$

from Isthmus 2026a, Lemma B.3.

Lemma B.1 (Geometry of one reflection split). Let I be a nonempty rigid self-dual DLO and let r be its unique reversal. Put

$$I_0 := \{x \in I : x < r(x)\},$$

$$I_1 := \{x \in I : r(x) < x\},$$

and

$$F_r := \{x \in I : r(x) = x\}.$$

Then:

(1) r is involutive;
(2) F_r is empty or a singleton;
(3) I_0 and I_1 are nonempty proper convex DLOs without endpoints;
(4) $I = I_0 \sqcup F_r \sqcup I_1$;
(5) r anti-isomorphically exchanges I_0 and I_1.

Proof. Lemma 2.8 gives involutivity. An order reversal has at most one fixed point: if $x < y$ were both fixed, reversal would give $y = r(y) < r(x) = x$. The two strict-inequality sets are respectively initial and final, are convex, and are exchanged by r.

They are nonempty because r cannot fix every point of a nontrivial strict order, by the same two-point argument. To show that I_0 has no greatest point, take $x \in I_0$ and choose

$$x < z_1 < z_2 < r(x).$$

At most one of z_1, z_2 is fixed by r, so choose a nonfixed z.

- If $z < r(z)$, then $z \in I_0$ and $x < z$.
- If $r(z) < z$, reversing $z < r(x)$ gives $x < r(z)$, and $r(z) \in I_0$.

The remaining endpoint statements are symmetric or follow from initiality and finality. Density is inherited from convexity, and trichotomy gives the displayed decomposition. $\square$

Lemma B.2 (Sharp singleton placement). Let $q \geq 1$. Let $C_0, \dots, C_{q-1}$ be nonempty DLO blocks without endpoints and let $z_0, \dots, z_{t-1}$ be singleton blocks. If $t \leq q$, the order

$$z_0 + C_0 + z_1 + C_1 + \cdots + z_{t-1} + C_{t-1} + C_t + \cdots + C_{q-1} \tag{24}$$

with the initial alternating part omitted when $t = 0$ and the final tail omitted when $t = q$, has no greatest point and contains no adjacent singleton blocks.

If $t = q + 1$, no ordering of the q DLO blocks and the t singleton blocks can simultaneously avoid adjacent singletons and avoid a singleton final block.

Proof. The displayed arrangement proves the first assertion. For the second, q nonsingleton blocks provide at most q separators between singleton blocks. Accommodating $q+1$ singleton blocks without adjacency forces the alternating pattern to begin and end with a singleton. $\square$

For the rest of the appendix, fix a rigid DLO $L = (X, <)$ with an involutive reversal h, and write

$$X = H + F_h + h[H] \tag{25}$$

as in Lemma B.1. Assume that H is self-dual and that every DLO on X is self-dual.

Lemma B.3 (All-self-dual finite localization). Let $n < \omega$. Suppose that:

(1) H is partitioned into cells $(I_s)_{s \in \mathrm{Bin}_n}$ and a center set $P_{<n}$;
(2) every I_s is a nonempty convex rigid self-dual DLO;
(3) there is an injective tag map

$$\tau_n : P_{<n} \longrightarrow \mathrm{Bin}_{<n}.$$

For each $s \in \mathrm{Bin}_n$, let r_s be the unique reversal of I_s and put

$$F_s := \{x \in I_s : r_s(x) = x\}.$$

Split I_s by r_s, and, when F_s is nonempty, tag its unique point by $s \in \mathrm{Bin}_n$. Then every resulting level-$(n+1)$ child cell D satisfies

$$D \cong D^*. \tag{26}$$

Proof. The child cells are indexed by Bin_{n+1}. Old center tags lie in $\mathrm{Bin}_{<n}$ and new center tags lie in Bin_n. Old centers lie outside every level-n cell, while a new center lies inside its parent cell; new centers from distinct parents lie in disjoint cells. Hence the combined tag assignment is injective into $\mathrm{Bin}_{<n+1}$.

Let $d \in \mathrm{Bin}_{n+1}$ index the target cell D, put $a = v_{n+1}(d)$, and define

$$\rho(s) = \begin{cases} v_{n+1}(s), & v_{n+1}(s) < a, \\ v_{n+1}(s) - 1, & a < v_{n+1}(s). \end{cases}$$

Then

$$\rho : \{s \in \mathrm{Bin}_{n+1} : s \neq d\} \longrightarrow q, \qquad q := N_{n+1} - 1, \tag{27}$$

is a bijection. Thus the other child cells have a canonical enumeration

$$C_0, \dots, C_{q-1}.$$

Composing the center tags with $c_{n+1} : \mathrm{Bin}_{<n+1} \to q$ injects the total center set into q. Finite ranking Isthmus 2026a, Lemma 2.8 therefore gives some $t \leq q$ and an enumeration

$$z_0, \dots, z_{t-1}$$

of all accumulated center points. Lemma B.2 gives a block order W_D on $H \setminus D$ in which every center is followed by a DLO cell and the final block is a DLO:

$$W_D = z_0 + C_0 + \cdots + z_{t-1} + C_{t-1} + C_t + \cdots + C_{q-1}. \tag{28}$$

Here the initial alternating part is omitted when $t = 0$, and the final tail is omitted when $t = q$. As sets,

$$H = D \sqcup W_D, \qquad X = D \sqcup W_D \sqcup F_h \sqcup h[W_D] \sqcup h[D]. \tag{29}$$

The combined tag graph is the union of the old tag map with the Separation-defined graph

$$\{(x,s) \in H \times \mathrm{Bin}_n : x \in F_s\},$$

and the enumeration $i \mapsto C_i$ is the Separation-defined subset of $q \times \mathcal{P}(H)$ determined by ρ. Here and below, W_D denotes both the displayed ordered block sum and its carrier $H \setminus D$.

Let

$$\mathfrak{B}_D \subseteq (q+t) \times \mathcal{P}(H)$$

be the graph of the displayed block family, and let

$$\mathfrak{R}_D \subseteq (q+t) \times \mathcal{P}(H \times H)$$

be the graph assigning to each singleton block the empty relation and to each cell block C_i its order inherited from L. Both graphs are formed by Separation from the two finite enumerations. Applying Isthmus 2026a, Proposition B.5 to $(\mathfrak{B}_D, \mathfrak{R}_D)$ yields the graph of the internally finite block-sum order W_D.

Give D and $h[D]$ their orders inherited from L. Give $h[W_D]$ the mirror order

$$u <_{h[W_D]} v \quad \iff \quad h(v) <_{W_D} h(u). \tag{30}$$

Put

$$M_D := W_D + F_h + h[W_D]$$

and

$$L_D := D + M_D + h[D]. \tag{31}$$

After empty blocks are omitted, the following list covers every boundary.

Boundary	Range	Reason for density
$D \mid z_0$	$t > 0$	D has no greatest point
$D \mid C_0$	$t = 0$	D has no greatest point
$z_i \mid C_i$	$0 \le i < t$	C_i has no least point
$C_i \mid z_{i+1}$	$0 \le i < t-1$	C_i has no greatest point
$C_j \mid C_{j+1}$	$0 \le j < q-1$ if $t = 0$; $t-1 \le j < q-1$ if $0 < t < q$; none if $t = q$	C_j has no greatest point

Boundary	Range	Reason for density
$W_D \mid F_h$	$F_h \neq \emptyset$	W_D has no greatest point
$F_h \mid h[W_D]$	$F_h \neq \emptyset$	$h[W_D]$ has no least point
$W_D \mid h[W_D]$	$F_h = \emptyset$	W_D has no greatest point and $h[W_D]$ has no least point
internal mirror boundaries	all corresponding indices	duals of the preceding internal cases
$h[W_D] \mid h[D]$	always	$h[D]$ has no least point

The first block D has no least point and the last block $h[D]$ has no greatest point. Hence the finite block-sum criterion makes L_D a DLO on the same carrier X.

The map h reverses L_D, exchanges D with $h[D]$, and satisfies

$$h[M_D] = M_D.$$

Every DLO on X is self-dual, so Lemma 3.6 applies to

$$L_D = D + M_D + h[D]$$

and gives $D \cong D^*$. $\square$

Definition B.4 (Complete dyadic level states). Put

$$U_H := \mathrm{Bin}_{<\omega} \times \mathcal{P}(H) \times \mathcal{P}(H \times H) \times \mathcal{P}(H). \tag{32}$$

A record $(s, I, r, F) \in U_H$ consists of a node, a cell, a reversal graph, and its fixed-point set.

For $n < \omega$, a subset $\mathcal{S} \subseteq U_H$ is a **valid level-n state** if:

(1) for every $s \in \mathrm{Bin}_{\leq n}$, exactly one record (s, I_s, r_s, F_s) belongs to $\mathcal{S}$, and no record with longer node belongs to $\mathcal{S}$;
(2) $I_\emptyset = H$;
(3) r_s is the unique reversal of I_s, and

$$F_s = \{x \in I_s : r_s(x) = x\};$$

(4) for $|s| < n$,

$$I_{s0} = \{x \in I_s : x < r_s(x)\},$$

$$I_{s1} = \{x \in I_s : r_s(x) < x\};$$

(5) every I_s is nonempty, convex in L, rigid, self-dual, and a DLO;
(6) for every $k \leq n$,

$$H = \Bigl(\bigsqcup_{|s|=k} I_s\Bigr) \sqcup P_{<k}, \tag{33}$$

where

$$P_{<k} := \{x \in H : \exists s\ (|s| < k \text{ and } x \in F_s)\},$$

and there is an injection

$$P_{<k} \longrightarrow N_k - 1.$$

Write $\mathrm{Valid}_H(n, \mathcal{S})$ for the conjunction of these clauses.

Theorem B.5 (Type-count-free dyadic actualization). For every $n < \omega$, there is a unique valid level-n state $\mathcal{S}_n$. The states are coherent under restriction, and the complete tree

$$\mathcal{T}_H := \bigl\{u \in U_H : \exists n \in \omega\, \exists \mathcal{S} \in \mathcal{P}(U_H)\, \bigl(\mathrm{Valid}_H(n, \mathcal{S}) \wedge u \in \mathcal{S}\bigr)\bigr\} \tag{34}$$

is a set in Z_{sep}.

Proof. For $n = 0$, the standing hypotheses make H rigid and self-dual. Its reversal and fixed-point set are unique, so $\mathcal{S}_0$ exists uniquely.

Assume $\mathcal{S}_n$ exists uniquely. Every old center has a unique node tag: a point in F_s belongs to neither child of s and hence to no descendant cell, while cells at incomparable nodes lie in disjoint branches. Since each F_s is empty or a singleton, the map

$$P_{<n} \longrightarrow \mathrm{Bin}_{<n}, \qquad x \longmapsto \text{the unique } s \text{ with } x \in F_s, \tag{35}$$

is injective.

Lemma B.1 determines the two geometric children of every level-n cell. Lemma B.3, applied with the preceding tag map, makes every child self-dual. Each child is convex in L and therefore rigid by Lemma 2.8. Its reversal and fixed-point set are consequently unique.

For $u \in U_H$, let $\mathrm{Old}_{\mathcal{S}_n}(u)$ mean $u \in \mathcal{S}_n$. Let $\mathrm{New}_{\mathcal{S}_n}(u)$ mean that

$$u = (s, I, r, F), \qquad s = t^\frown \varepsilon, \qquad |t| = n, \qquad \varepsilon \in 2,$$

there is a parent record $(t, I_t, r_t, F_t) \in \mathcal{S}_n$, and:

- if $\varepsilon = 0$, then $I = \{x \in I_t : x < r_t(x)\}$;
- if $\varepsilon = 1$, then $I = \{x \in I_t : r_t(x) < x\}$;
- r is the unique reversal of I;
- $F = \{x \in I : r(x) = x\}$.

Put

$$\mathcal{S}_{n+1} = \{u \in U_H : \mathrm{Old}_{\mathcal{S}_n}(u) \vee \mathrm{New}_{\mathcal{S}_n}(u)\}. \tag{36}$$

This is a Separation-defined subset of the fixed set U_H.

Tag old centers by their old nodes and a possible new center of I_s by $s \in \mathrm{Bin}_n$. This gives an injection

$$P_{<n+1} \longrightarrow \mathrm{Bin}_{<n+1} \xrightarrow{c_{n+1}} N_{n+1} - 1.$$

Clauses 1–5 of Definition B.4 hold by construction. For $k \leq n$, the partition and center bound are inherited from $\mathcal{S}_n$. For $k = n + 1$, Lemma B.1 partitions every parent into its two children and possible fixed point, while the displayed tag injection gives the required bound. Hence clause 6 holds as well. The geometric children are uniquely determined, and rigidity makes their reversals and fixed-point sets unique, so $\mathcal{S}_{n+1}$ is the unique valid level-$(n + 1)$ state.

If $m > n$, the restriction of any valid level-m state to nodes of length at most n is a valid level-n state and hence equals $\mathcal{S}_n$. Thus the states are coherent. Finally, $\mathrm{Valid}_H(n, \mathcal{S})$ is a formula on the fixed set $\omega \times \mathcal{P}(U_H)$, and the displayed definition of $\mathcal{T}_H$ is Separation on U_H; no Replacement-generated range is formed. □

Lemma B.6 (The center set is at most countable). Define

$$Z_H := \{x \in H : \exists (s, I, r, F) \in \mathcal{T}_H \ (x \in F)\}. \tag{37}$$

Then

$$Z_H \hookrightarrow \mathrm{Bin}_{<\omega} \hookrightarrow \omega. \tag{38}$$

Proof. Every center belongs to F_s for a unique node s: centers are excluded from all descendant cells, and incomparable nodes lie in disjoint cells. Two centers with the same tag lie in the empty-or-singleton set F_s and are equal. Thus $x \mapsto s$ is an injection obtained by Separation from $H \times \mathrm{Bin}_{<\omega}$. The explicit finite-word code of Isthmus 2026a, Lemma B.3 gives $\mathrm{Bin}_{<\omega} \hookrightarrow \omega$. □

Theorem B.7 (An off-center point generates an injective sequence). Let $x \in H \setminus Z_H$. For every n, let s_n be the unique word of length n whose cell contains x, and put

$$y_n := r_{s_n}(x). \tag{39}$$

Then $n \mapsto y_n$ is an injection $\omega \hookrightarrow X$.

Proof. The level partition and $x \notin Z_H$ give the unique cell I_{s_n}. Since x is not fixed by r_{s_n}, the points x and y_n lie in opposite children. The next chosen cell $I_{s_{n+1}}$ is the child containing x, so

$$y_n \notin I_{s_{n+1}}.$$

If $m > n$, then

$$y_m \in I_{s_m} \subseteq I_{s_{n+1}},$$

and hence $y_m \neq y_n$.

The graph is the Separation-defined subset of $\omega \times X$

$$G_x := \left\{ (n,y) \in \omega \times X : \exists s, I, r, F \begin{bmatrix} (s,I,r,F) \in \mathcal{T}_H, \\ |s| = n,\ x \in I,\ (x,y) \in r \end{bmatrix} \right\}. \tag{40}$$

Coherence and record uniqueness make G_x a function, and the preceding argument makes it injective. □

Corollary B.8 (Type-count-free rigid carrier dichotomy). Under the standing hypotheses of this appendix,

$$X \hookrightarrow \omega \quad \text{or} \quad \omega \hookrightarrow X. \tag{41}$$

Proof. If $H \hookrightarrow \omega$, then $h[H] \hookrightarrow \omega$ and F_h is empty or a singleton. The finite-union coding of Isthmus 2026a, Lemma 3.4(1) gives

$$X = H + F_h + h[H] \hookrightarrow \omega.$$

If $H \not\hookrightarrow \omega$, Lemma B.6 implies $H \neq Z_H$. Choose one point in $H \setminus Z_H$ and apply Theorem B.7. □

AI-use disclosure

AI systems were used extensively in an iterative, human-directed workflow. OpenAI ChatGPT was used for exploratory proof development, counterexample search, drafting, and virtual review; Anthropic Claude Code was used for Lean formalization, proof repair, and validation; and OpenAI Codex was used for mathematical and formalization audits, release review, and editorial checks. AI assistance was also used to translate and polish English prose. The author set the research objectives, decided which candidate arguments to retain or reject, reviewed and approved the final manuscript and formalization, and takes responsibility for all mathematical claims and any remaining errors. AI systems are tools, not authors.

References

- **Ervin 2017.** Garrett Ervin. "Every Linear Order Isomorphic to Its Cube Is Isomorphic to Its Square." *Advances in Mathematics* 313 (2017): 237–281. DOI: 10.1016/j.aim.2017.04.010.
- **Howard–Rubin 1998.** Paul Howard and Jean E. Rubin. *Consequences of the Axiom of Choice.* Mathematical Surveys and Monographs 59. American Mathematical Society, 1998.
- **Isthmus 2026a.** Lior Isthmus. "No Set Carries Exactly Three Dense Linear Orders without Endpoints: A Proof in a Weak Zermelo Theory without Choice or Replacement." Preprint, version `paper-rc6`, 2026. DOI: 10.5281/zenodo.21735262. GitHub.
- **Isthmus 2026b.** Lior Isthmus. *No Set Carries Exactly Three Dense Linear Orders without Endpoints: Lean 4 Formalization.* Software release. GitHub. DOI: 10.5281/zenodo.21729423.
- **Isthmus 2026c.** Lior Isthmus. *No Set Carries Exactly Two Dense Linear Orders without Endpoints: Lean 4 Formalization.* Software release candidate. GitHub.
- **Jech 1973.** Thomas J. Jech. *The Axiom of Choice.* Studies in Logic and the Foundations of Mathematics 75. North-Holland, 1973.
- **Marker 2002.** David Marker. *Model Theory: An Introduction.* Graduate Texts in Mathematics 217. Springer, 2002.
- **Shelah 2009.** Saharon Shelah. "Model Theory without Choice? Categoricity." *The Journal of Symbolic Logic* 74, no. 2 (2009): 361–401. arXiv:math/0504196. DOI: 10.2178/jsl/1243948319.