An Effective $O((\log n)/n)$ Bound for Complete-Graph Reliability Roots Near -1

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Abstract

Let $R_n(q)$ be the all-terminal reliability polynomial of the complete graph K_n, written in the edge-failure variable. We prove that, for every $n \geq 256$ with $n \equiv 0, 3 \pmod 4$, the polynomial R_n has a real zero q_n satisfying

$$-1 < q_n < -n^{-2/n}.$$

Consequently, $0 < q_n + 1 < 2 \log n/n$. More precisely, $R_n(-n^{-2/n}) > 1/3$ throughout this range and $R_n(-n^{-2/n}) \to 1$. The endpoint sign is supplied by an exact sign theorem, while positivity at the moving point follows from Möbius inversion on the partition lattice and explicit estimates for partitions with and without a block larger than $n/2$. This gives an effective quantitative refinement of the known accumulation of complete-graph reliability roots at -1.

1 Introduction

For a connected finite graph $G = (V, E)$, the all-terminal reliability polynomial records the probability that the surviving spanning subgraph is connected when every edge fails independently with probability q. Although its probabilistic meaning concerns $0 \leq q \leq 1$, the real and complex zeros of this polynomial encode substantial graph structure. This paper studies the real zeros just to the right of the boundary point -1.

Write $R_n(q) = \mathrm{Rel}(K_n; q)$ for the complete graph on n vertices, in the edge-failure convention. The main result gives an explicit root window and an effective threshold.

Theorem 1 (Main theorem)**.** *For every integer $n \geq 256$ with $n \equiv 0, 3 \pmod 4$, the polynomial R_n has a real zero q_n satisfying*

$$-1 < q_n < -n^{-2/n}.$$

Consequently,

$$0 < q_n + 1 < 1 - n^{-2/n} \leq \frac{2 \log n}{n}.$$

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The proof separates two signs. An exact theorem of Brown and DeGagné gives $R_n(-1) < 0$ in the two stated congruence classes. At the moving point $-n^{-2/n}$, a partition-lattice expansion expresses R_n as 1 plus an alternating sum. Absolute estimates for partitions with a giant block and for partitions with no block larger than $n/2$ show that this correction has magnitude less than $2/3$ for every $n \geq 256$ and tends to zero. The intermediate value theorem then proves Theorem 1.

1.1 Prior work and motivation

Brown and McMullin study real all-terminal reliability roots of simple graphs and ask for their closure [1, Sections 3–4]. Their final paragraph reports numerical evidence that complete-graph roots approach -1 and asks whether -1 belongs to the closure [1, Section 4, p. 11]. The qualitative endpoint statement has since been settled: Buys proves that, for every fixed $q \in (-1, 0)$ and all sufficiently large $n \equiv 0, 3 \pmod 4$, R_n has a zero in $(-1, q)$ [3, Corollary 2.3, p. 6], while Omar proves the global closure identity $\overline{\mathcal{Z}_{\text{simple}} \cap \mathbb{R}} = [-1, 0] \cup \{1\}$ [4, Theorem 1.1, p. 2]. Theorem 1 is a quantitative refinement for the complete-graph family: its contribution is the moving endpoint, the effective $O((\log n)/n)$ bound, and the explicit threshold.

2 Definitions

Throughout, a graph is finite, connected, and simple. This convention avoids a degeneracy: for a disconnected graph the all-terminal reliability polynomial defined below is identically zero, so every complex number would be a root. For a connected simple graph $G = (V, E)$, its all-terminal reliability polynomial in the edge-failure variable is

$$\operatorname{Rel}(G; q) = \sum_{\substack{A \subseteq E \\ (V,A) \text{ connected}}} (1-q)^{|A|} q^{|E|-|A|}.$$

The one-vertex graph is regarded as connected. Put

$$R_n(q) = \operatorname{Rel}(K_n; q), \qquad M_n = \binom{n}{2}.$$

It is convenient to use the connected labelled-graph polynomial

$$C_n(v) = \sum_{\substack{H \text{ connected} \\ V(H)=[n]}} v^{e(H)}.$$

For $q \neq 0$, direct expansion over the connected spanning subgraphs of K_n gives

$$R_n(q) = q^{M_n} C_n\left(\frac{1-q}{q}\right). \tag{1}$$

3 The sign at $q = -1$

The endpoint value is controlled by an exact result rather than an asymptotic calculation.

Theorem 2 (Brown–DeGagné [2, Theorem 3.1])**.** *If a connected simple graph G has n vertices and m edges, then* $\operatorname{Rel}(G; -1) \neq 0$ *and*

$$\operatorname{sgn} \operatorname{Rel}(G; -1) = (-1)^{m-n+1}.$$

Corollary 3. *For every $n \geq 1$,*

$$\operatorname{sgn} R_n(-1) = (-1)^{\binom{n}{2}-n+1}.$$

In particular, $R_n(-1) < 0$ whenever $n \equiv 0, 3 \pmod 4$.

Proof. Apply Theorem 2 to K_n, which has $\binom{n}{2}$ edges. A direct parity check shows that $\binom{n}{2} - n + 1$ is odd precisely when $n \equiv 0, 3 \pmod 4$. $\square$

4 Positivity Near -1

Let Π_n be the set of set partitions of $[n]$. For $\pi \in \Pi_n$, let $|\pi|$ be its number of blocks and let

$$\operatorname{cr}(\pi) = |\{\{i, j\} : i, j \in [n] \text{ lie in different blocks of } \pi\}|.$$

If the block sizes are $n_1, \ldots, n_b$, then

$$\operatorname{cr}(\pi) = \sum_{1 \leq i < j \leq b} n_i n_j = \binom{n}{2} - \sum_{i=1}^{b} \binom{n_i}{2}.$$

Lemma 4 (partition expansion)**.** *For every $n \geq 1$ and $r > 0$,*

$$R_n(-r) = \sum_{\pi \in \Pi_n} (-1)^{|\pi|-1+\operatorname{cr}(\pi)} (|\pi| - 1)! r^{\operatorname{cr}(\pi)}.$$

Consequently,

$$|R_n(-r) - 1| \leq \sum_{\substack{\pi \in \Pi_n \\ |\pi| \geq 2}} (|\pi| - 1)! r^{\operatorname{cr}(\pi)}.$$

Proof. For a partition π, let

$$F_\pi(v) = \prod_{B \in \pi} (1 + v)^{\binom{|B|}{2}}.$$

This is the total $v^{e(H)}$-weight of all graphs on $[n]$ with no edges between distinct blocks of π. If $G_\sigma(v)$ is the total weight of graphs whose connected-component partition is exactly σ, then

$$F_\pi(v) = \sum_{\sigma \leq \pi} G_\sigma(v),$$

where $\sigma \leq \pi$ means that every block of σ is contained in a block of π. Möbius inversion in the partition lattice gives

$$C_n(v) = \sum_{\pi \in \Pi_n} (-1)^{|\pi|-1} (|\pi| - 1)! \prod_{B \in \pi} (1 + v)^{\binom{|B|}{2}},$$

because the interval from π to the one-block partition is a partition lattice on $|\pi|$ elements.

Set $q = -r$. Then $(1 - q)/q = -1 - 1/r$, so

$$1 + \frac{1 - q}{q} = -\frac{1}{r}.$$

Writing $s(\pi) = \sum_{B\in\pi} \binom{|B|}{2}$, the contribution of π to $R_n(-r) = (-r)^{M_n} C_n((1+r)/(-r))$ is

$$(-r)^{M_n}(-1)^{|\pi|-1}(|\pi|-1)! \left(-\frac{1}{r}\right)^{s(\pi)}.$$

Since $M_n - s(\pi) = \mathrm{cr}(\pi)$ and

$$M_n + s(\pi) \equiv M_n - s(\pi) = \mathrm{cr}(\pi) \pmod 2,$$

this is exactly the displayed term. The one-block partition contributes 1, and taking absolute values of the remaining terms gives the inequality. □

Lemma 5 (endpoint positivity)**.** *Set $r_n = n^{-2/n}$. Then*

$$R_n(-r_n) > \frac{1}{3} \qquad (n \geq 256).$$

Moreover, $R_n(-r_n) \to 1$ as $n \to \infty$.

Proof. By Lemma 4,

$$|R_n(-r_n) - 1| \leq U_n, \qquad U_n = \sum_{\substack{\pi\in\Pi_n \\ |\pi|\geq 2}} (|\pi|-1)! r_n^{\mathrm{cr}(\pi)}.$$

We will prove both $U_n < 2/3$ for $n \geq 256$ and $U_n \to 0$. Split the nontrivial partitions according to whether or not they have a block of size greater than $n/2$.

Suppose first that π has a block L of size $n - s > n/2$. This block is unique, and $1 \leq s < n/2$. If the partition induced on $S = [n] \setminus L$ has j blocks, order those blocks and denote their sizes by $m_1, \ldots, m_j$. Ordering converts the factorial weight $j!$ exactly into the number of orders, while

$$\mathrm{cr}(\pi) = s(n-s) + \frac{s^2 - \sum_{i=1}^j m_i^2}{2}.$$

Consequently, the contribution $A_{n,s}$ of all partitions with this value of s is

$$\begin{aligned} A_{n,s} &= \binom{n}{s} \sum_{j\geq 1} \sum_{\substack{m_1+\cdots+m_j=s \\ m_i\geq 1}} \frac{s!}{m_1!\cdots m_j!} n^{-2s(n-s)/n} n^{-(s^2-\sum_i m_i^2)/n} \\ &= \binom{n}{s} s! n^{-2s+s^2/n} B_{n,s}, \end{aligned}$$

where

$$B_{n,s} = \sum_{j\geq 1} \sum_{\substack{m_1+\cdots+m_j=s \\ m_i\geq 1}} \prod_{i=1}^{j} \frac{n^{m_i^2/n}}{m_i!}.$$

Since $\binom{n}{s} s! \leq n^s$ and $s < n/2$,

$$A_{n,s} \leq n^{-s+s^2/n} B_{n,s} \leq n^{-s/2} B_{n,s}.$$

Define

$$\Phi_n(z) = \sum_{m=1}^{\lfloor n/2 \rfloor} \frac{n^{m^2/n}}{m!} z^m.$$

For $n \geq 256$, both $n^{1/n}$ and $n^{4/n}$ are at most 11/10: the function $x^{a/x}$ decreases for $x \geq e$, and the assertion follows by checking $n = 256$. The first two terms of $\Phi_n(1/4)$ are therefore at most 11/40 and 11/320, respectively. For $3 \leq m \leq n/2$, consider $h(x) = n^{x/n}/x$. Its logarithmic derivative is

$$(\log h)'(x) = \frac{\log n}{n} - \frac{1}{x}, \qquad (\log h)''(x) = \frac{1}{x^2} > 0,$$

so $\log h$ is convex and its maximum on $[3, n/2]$ is attained at an endpoint. The two endpoint values $n^{3/n}/3$ and $2/\sqrt{n}$ are at most 1/2 for $n \geq 256$. The estimate $m! \geq (m/e)^m$ now gives

$$\frac{n^{m^2/n}}{m!4^m} \leq \left(\frac{en^{m/n}}{4m}\right)^m \leq \left(\frac{3}{8}\right)^m.$$

It follows that

$$\Phi_n(1/4) \leq \frac{11}{40} + \frac{11}{320} + \sum_{m \geq 3} \left(\frac{3}{8}\right)^m = \frac{63}{160} < \frac{1}{2}.$$

The number $B_{n,s}$ is the coefficient of z^s in $\Phi_n(z)/(1 - \Phi_n(z))$. Since all coefficients are nonnegative,

$$B_{n,s} \leq 4^s \frac{\Phi_n(1/4)}{1 - \Phi_n(1/4)} \leq 4^s.$$

Thus the total contribution from partitions with a block larger than $n/2$ satisfies

$$U_n^A \leq \sum_{s=1}^{\lfloor (n-1)/2 \rfloor} \left(\frac{4}{\sqrt{n}}\right)^s < \frac{1}{3} \qquad (n \geq 256),$$

and this bound tends to zero with n.

It remains to treat partitions all of whose blocks have size at most $n/2$. For a partition with b blocks, ordering the blocks gives each partition exactly $b!$ times, whereas its weight is $(b-1)! = b!/b$. Thus the exact ordered-block expression has a factor $1/b$; discarding that factor gives

$$U_n^B \leq \sum_{b=2}^{n} \sum_{\substack{k_1 + \cdots + k_b = n \\ 1 \leq k_i \leq n/2}} \frac{n!}{k_1! \cdots k_b!} n^{-n + \sum_i k_i^2/n}.$$

The standard Stirling bounds give, for every ordered composition in this sum,

$$\frac{n!}{k_1! \cdots k_b!} n^{-n + \sum_i k_i^2/n} \leq 3\sqrt{n} \prod_{i=1}^{b} \left(c k_i^{-1/2} \left(\frac{n^{k_i/n}}{k_i}\right)^{k_i} \right), \qquad c = (2\pi)^{-1/2}.$$

Indeed, one may use $n! \leq 3\sqrt{n}(n/e)^n$ and $k! \geq \sqrt{2\pi}\, k^{k+1/2} e^{-k}$. Define

$$\Psi_n(z) = \sum_{k=1}^{\lfloor n/2 \rfloor} c k^{-1/2} \left(\frac{n^{k/n}}{k}\right)^k z^k.$$

Summing over ordered compositions yields

$$U_n^B \leq 3\sqrt{n}\, [z^n] \frac{\Psi_n(z)^2}{1 - \Psi_n(z)}.$$

We bound this coefficient at $y = 6/5$. Since $c < 2/5$, and since $n^{1/n}, n^{2/n} \le 11/10$ for $n \ge 256$, the $k = 1$ and $k = 2$ terms of $\Psi_n(y)$ are at most

$$\frac{66}{125} \quad \text{and} \quad \frac{2}{5} \cdot \frac{3}{4} \left(\frac{33}{50}\right)^2 = \frac{3267}{25000},$$

respectively. Here again $(\log h)''(x) = 1/x^2 > 0$, so the endpoint argument for $h(x) = n^{x/n}/x$ shows that $n^{k/n}/k \le 2/5$ when $3 \le k \le n/2$ and $n \ge 256$. As $k^{-1/2} \le 3/5$ in this range, the remaining terms contribute at most

$$\frac{2}{5} \cdot \frac{3}{5} \sum_{k \ge 3} \left(\frac{12}{25}\right)^k = \frac{10368}{203125}.$$

Therefore

$$\Psi_n(6/5) \le \frac{66}{125} + \frac{3267}{25000} + \frac{10368}{203125} = \frac{1153299}{1625000} < \frac{3}{4}.$$

Nonnegativity of the coefficients now implies

$$[z^n] \frac{\Psi_n(z)^2}{1 - \Psi_n(z)} \le \left(\frac{6}{5}\right)^{-n} \frac{(3/4)^2}{1 - 3/4} = \frac{9}{4} \left(\frac{6}{5}\right)^{-n}.$$

It follows that

$$U_n^B \le \frac{27}{4} \sqrt{n} \left(\frac{5}{6}\right)^n < \frac{1}{3} \qquad (n \ge 256),$$

where the last inequality follows by checking $n = 256$; the displayed bound is decreasing thereafter and tends to zero. Combining the two classes gives $U_n < 2/3$ for $n \ge 256$ and $U_n \to 0$. Hence $R_n(-n^{-2/n}) > 1/3$ in the stated range and tends to 1. □

5 Proof of the main theorem

Proof of Theorem 1. Set $r_n = n^{-2/n}$. By Lemma 5, $R_n(-r_n) > 1/3$. By Corollary 3, $R_n(-1) < 0$ for $n \equiv 0, 3 \pmod 4$. Since $0 < r_n < 1$, the interval $(-1, -r_n)$ is nonempty. The intermediate value theorem gives a real zero of R_n in this interval.

Finally,

$$1 - n^{-2/n} = 1 - e^{-2 \log n / n} \le \frac{2 \log n}{n},$$

using $1 - e^{-x} \le x$ for $x \ge 0$. Taking $n \to \infty$ through the congruence classes $0, 3 \pmod 4$ gives roots tending to -1. □

6 Further questions

The one-sided interval in Theorem 1 does not provide a matching lower bound, uniqueness of the root in the boundary window, or an asymptotic equivalent. Natural next questions are whether one can choose q_n so that

$$q_n + 1 \sim \frac{2 \log n}{n},$$

whether the relevant real root is unique, and how the remaining real roots of K_n are distributed near the unit circle.

References

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