Characteristic-Compatible Lifting and Natural Direct-Sum Presentations for q-Matroids

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Abstract

A q-matroid is a rank function on the subspaces of a vector space over $\mathbb{F}_q$. A coordinate q-matroid on $\mathbb{F}_q^n$ is determined by an ordinary matroid on $[n]$ through their weighted cyclic flats. For $q = p^a$, we prove that such a q-matroid is representable over a finite extension of $\mathbb{F}_q$ if and only if the corresponding ordinary matroid is representable over a field of characteristic p. The sufficiency direction uses a generic diagonal rescaling of an ordinary representation, matroid intersection, and simultaneous specialization over a finite field. We also establish closure of transversal q-matroids under the Ceria–Jurrius direct sum. More precisely, for arbitrary presentations of two transversal q-matroids, the list obtained by inflating every presenting subspace by the other ambient summand presents their direct sum. Its presentation-rank minimum agrees, on every subspace, with the direct-sum rank minimum. Iteration gives a natural presentation and a flattened rank formula for every finite direct sum.

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1 Introduction

The theory of q-matroids replaces subsets of a finite ground set by subspaces of a finite-dimensional vector space. This change preserves the rank axioms but makes two familiar operations unexpectedly delicate. Representations must control the ranks of all subspaces, rather than only coordinate ones, and a subspace of a direct sum need not split along the two summands. This paper gives exact answers to one representability problem and one direct-sum problem arising from these two difficulties.

1.1 Coordinate q-matroids and characteristic-compatible lifting

Throughout, vector spaces are finite-dimensional. Unless a different field is specified explicitly, all ambient spaces and all q-matroids appearing in a common construction are over the same field $\mathbb{F}_q$. A *q-matroid* $M = (E, \rho)$ is an ambient space E with a rank function ρ on subspaces that is integer valued with $0 \le \rho(S) \le \dim S$, monotone ($S \le T \Rightarrow \rho(S) \le \rho(T)$), and submodular ($\rho(S+T) + \rho(S \cap T) \le \rho(S) + \rho(T)$). A subspace I is *independent* if $\rho(I) = \dim I$. These are the standard rank-function conventions for q-matroids.

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Fix $q = p^a$ and the standard basis $e_1, \dots, e_n$ of $\mathbb{F}_q^n$. Write

$$\phi(S) = \langle e_i : i \in S \rangle \qquad (S \subseteq [n]).$$

A subspace in the image of ϕ is *coordinate*, and a q-matroid is coordinate if all its cyclic flats are coordinate [6, Defs. 3.1–3.2]. A q-matroid is determined by its weighted cyclic-flat data [1, Prop. 3.26]. Under ϕ, the cyclic-flat axioms give a two-way correspondence: replacing each weighted cyclic flat $Z \subseteq [n]$ of an ordinary matroid by $\phi(Z)$ produces a coordinate q-matroid, and applying ϕ^{-1} to the weighted cyclic flats of a coordinate q-matroid produces its *corresponding matroid* [6, Lem. 3.3 and Def. 3.4]. If this corresponding matroid is M, their coordinate ranks agree:

$$r_M(S) = \rho_N(\phi(S)) \qquad (S \subseteq [n]). \tag{1}$$

We shall also use the cyclic-flat reconstruction formula

$$\rho_N(V) = \min_{Z \in \mathcal{Z}(N)} \big(\rho_N(Z) + \dim V - \dim(V \cap Z)\big), \tag{2}$$

which is the cyclic-flat reconstruction formula [1, Def. 4.5 and Cor. 4.12(3)].

A q-matroid N on $\mathbb{F}_q^n$ is *representable* if there are a finite extension $K/\mathbb{F}_q$ and a matrix G over K such that

$$\rho_N(V) = \operatorname{rank}_K(GY_V) \tag{3}$$

for every $V \leq \mathbb{F}_q^n$, where the columns of Y_V are an $\mathbb{F}_q$-basis of V.

Our first main result shows that the characteristic visible on coordinate subspaces is the only obstruction to representing all subspaces.

Theorem 1 (Exact representability criterion)**.** *Let N be a coordinate q-matroid over $\mathbb{F}_q$, where $q = p^a$, and let M be its corresponding ordinary matroid. Then N is representable as a q-matroid if and only if M is representable over some field of characteristic p.*

For the nontrivial implication, start with a characteristic-p matrix for M and rescale its columns by algebraically independent parameters. A Cauchy–Binet expansion and the matroid-intersection theorem identify the rank on every $\mathbb{F}_q$-subspace with the cyclic-flat reconstruction formula. Since there are only finitely many such subspaces, one simultaneous specialization over a sufficiently large finite extension preserves all required ranks. This argument is carried out in Section 2.

1.2 Transversal presentations and direct sums

The relevant notions of transversality were introduced by Saaltink [11]. A *presentation* is a finite list $\mathcal{X} = (X_1, \dots, X_k)$ of subspaces of E (repetitions allowed). A linearly independent set B has an *avoidance transversal* of $\mathcal{X}$ if its members can be ordered $b_1, \dots, b_\ell$ and matched to distinct indices so that the i-th chosen subspace omits b_i; a subspace X is a *partial q-transversal* of $\mathcal{X}$ if every basis of X has an avoidance transversal [6, Def. 2.12]. The partial q-transversals of $\mathcal{X}$ are the independent spaces of a q-matroid $M_{\mathcal{X}}$, the *transversal q-matroid* presented by $\mathcal{X}$ [6, Defs. 2.12–2.13]. For a subspace $U \leq E$ it is convenient to write

$$f_{\mathcal{X}}(U) = \big|\{\, i : \; U \not\leq X_i \,\}\big|,$$

the number of presentation subspaces not containing U.

The *direct sum* of $M_1 = (E_1, \rho_1)$ and $M_2 = (E_2, \rho_2)$ is the q-matroid $M_1 \oplus M_2 = (E_1 \oplus E_2, \rho)$ with

$$\rho(X) = \min_{Y \leq X} \big(\rho_1(\pi_1 Y) + \rho_2(\pi_2 Y) + \dim X - \dim Y\big) \tag{4}$$

[6, Def. 2.7]; this is the direct-sum operation developed by Ceria and Jurrius [4]. Here π_1, π_2 are the coordinate projections. Given presentations $\mathcal{X}_1 = (A_1, \dots, A_k)$ of M_1 and $\mathcal{X}_2 = (B_1, \dots, B_\ell)$ of M_2, the *natural direct-sum presentation* is

$$\mathcal{X}_1 \oplus \mathcal{X}_2 = (A_1 \oplus E_2, \dots, A_k \oplus E_2,\ E_1 \oplus B_1, \dots, E_1 \oplus B_\ell), \tag{5}$$

modelled on the classical presentation of a direct sum.

Our second main result identifies this natural presentation with the direct-sum rank on every subspace, including subspaces that do not split along the two ambient summands.

Theorem 2 (Natural direct-sum presentation)**.** *Let $M_1 = (E_1, \rho_1)$ and $M_2 = (E_2, \rho_2)$ be transversal q-matroids over the same field, with presentations $\mathcal{X}_1$ and $\mathcal{X}_2$. Then $\mathcal{X}_1 \oplus \mathcal{X}_2$ presents $M_1 \oplus M_2$. More precisely, for every $X \le E_1 \oplus E_2$,*

$$r_{\mathcal{X}_1 \oplus \mathcal{X}_2}(X) = \min_{Y \le X} \big(\rho_1(\pi_1 Y) + \rho_2(\pi_2 Y) + \dim X - \dim Y\big).$$

The proof uses Saaltink's Hall-type characterization and presentation-rank formula to express the left-hand side as a second minimum. Intersecting a candidate subspace with $U \oplus V$ and applying Grassmann's identity gives the less immediate inequality between the minima. The same argument iterates: Corollary 17 gives an explicit flattened rank formula and an inflated presentation for every finite direct sum.

1.3 Relation to earlier results and motivating questions

The two theorems above sharpen questions posed by Fulcher in the study of cyclic-flat embeddings [6]. His cyclic-flat embedding theorem shows that a coordinate q-matroid is transversal and representable whenever its corresponding ordinary matroid is transversal [6, Thm. 3.15]. He asked whether ordinary representability alone would suffice.

Conjecture 3 ([6, Conj. 3.17])**.** *If the corresponding matroid of a coordinate q-matroid is representable, then the q-matroid is representable.*

As stated, this is false because the characteristic of the ordinary representation need not agree with the base characteristic. The necessary implication and Fano/non-Fano examples were already established in the more general induced-matroid framework of Gluesing-Luerssen and Jany [7, Thm. 4.4 and Exs. 4.8(b),(c)]. Thus the negative answer is prior work. Theorem 1 supplies the missing converse and gives the exact characteristic-compatible replacement.

For direct sums, Fulcher stated the coordinate case [6, Thm. 4.7]; its cyclic-flat proof applies directly to minimal presentations. He conjectured that coordinates are unnecessary and that the natural presentation (5) works for arbitrary given presentations.

Conjecture 4 ([6, Conj. 4.8])**.** *Let M_1 and M_2 be transversal q-matroids with presentations $\mathcal{X}_1$ and $\mathcal{X}_2$. Then $M_1 \oplus M_2$ is a transversal q-matroid with presentation $\mathcal{X}_1 \oplus \mathcal{X}_2$.*

Theorem 2 proves this statement for the actual given presentations, with no coordinate or minimality assumption. This distinction matters because Saaltink's cyclic-flat statement is for minimal presentations, whereas redundant slots of a general presentation need only be flats [11, Thms. 21 and 23]. Associativity of the direct sum was already proved by Gluesing-Luerssen and Jany [8, Thm. 6.3 and Cor. 6.4]; the finite-sum contribution here is the explicit flattened formula together with its natural presentation.

Transversality and representability should not be conflated in this setting. Representable q-matroids can have a nonrepresentable direct sum [9], although direct sums of uniform q-matroids are representable over sufficiently large extensions [3]. Theorem 2 is therefore a closure theorem for a presentation class, not a representability theorem.

2 Characteristic-compatible representability

2.1 Known obstruction and explicit witnesses

For completeness, this subsection gives a self-contained specialization of the induced-matroid obstruction and the two examples from Gluesing-Luerssen and Jany [7, Thm. 4.4 and Exs. 4.8(b),(c)]. They are included to make the characteristic obstruction and its standard witnesses explicit; the results of this subsection are not new.

Lemma 5 (Characteristic obstruction, [7, Thm. 4.4]). *Let N be a coordinate q-matroid over $\mathbb{F}_q$, where $q = p^a$, and let M be its corresponding ordinary matroid. If N is representable, then M is representable over a field of characteristic p.*

Proof. Let $G \in K^{k\times n}$ represent N over a finite extension $K/\mathbb{F}_q$. Then $\operatorname{char} K = p$. For $S \subseteq [n]$, a basis matrix for $\phi(S)$ is the coordinate inclusion with columns e_i, $i \in S$, and hence $GY_{\phi(S)} = G_S$, the corresponding column submatrix. Therefore

$$r_M(S) = \rho_N(\phi(S)) = \operatorname{rank}_K(G_S)$$

by (1). Thus the columns of G represent M over K, a field of characteristic p. □

The small rank-three matroids needed below are recalled with proofs, so the counterexamples do not depend on an unstated representability convention.

Lemma 6 (Fano matroid). *The Fano matroid F_7 is representable over a field K if and only if* $\operatorname{char} K = 2$.

Proof. Use the labelling in which the seven three-point lines are

$$124, \quad 135, \quad 236, \quad 167, \quad 257, \quad 347, \quad 456.$$

If F_7 is represented over K, take 123 as a basis and normalize

$$v_1 = (1,0,0), \qquad v_2 = (0,1,0), \qquad v_3 = (0,0,1).$$

The lines $124, 135, 236$ allow the further normalization

$$v_4 = (1,1,0), \qquad v_5 = (1,0,1), \qquad v_6 = (0,1,c)$$

with $c \neq 0$; this uses rescaling of the coordinate axes and of the columns. From the line 167, the seventh column has the form $v_7 = (x, y, cy)$ with $x, y \neq 0$. The line 257 says that the first and third coordinates agree, so $x = cy$, while the line 347 says that the first and second coordinates agree, so $x = y$. Hence $c = 1$ and v_7 is projectively equal to $(1,1,1)$. The remaining line 456 is dependent exactly when

$$\det \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} = -2$$

vanishes, proving $\operatorname{char} K = 2$. Conversely, in characteristic two the seven nonzero vectors of $\mathbb{F}_2^3$ represent the projective plane $\mathrm{PG}(2,2)$ and hence F_7. □

Lemma 7 (Non-Fano matroid). *Let F_7^- be obtained from F_7 by relaxing the circuit-hyperplane 456. Then F_7^- is representable over K if and only if* $\operatorname{char} K \neq 2$.

Proof. The six three-point lines of F_7^- are

$$124, \quad 135, \quad 236, \quad 167, \quad 257, \quad 347,$$

whereas 456 is a basis. Any representation can be normalized exactly as in Lemma 6; the six displayed lines force

$$\begin{aligned} v_1 &= (1,0,0), & v_2 &= (0,1,0), & v_3 &= (0,0,1), \\ v_4 &= (1,1,0), & v_5 &= (1,0,1), & v_6 &= (0,1,1), \\ & & & & v_7 &= (1,1,1). \end{aligned}$$

Since 456 is independent, its determinant -2 must be nonzero. Thus $\operatorname{char} K \neq 2$.

Conversely, suppose $\operatorname{char} K \neq 2$ and use the seven displayed columns. The six stated dependencies follow from

$$v_4 = v_1 + v_2, \quad v_5 = v_1 + v_3, \quad v_6 = v_2 + v_3, \quad v_7 = v_1 + v_6 = v_2 + v_5 = v_3 + v_4.$$

The first three lines are the coordinate planes $z = 0$, $y = 0$, $x = 0$, and the last three are the planes $y = z$, $x = z$, $x = y$. Inspection shows that each contains exactly its stated three columns. These six triples contain every pair except $45, 46, 56$. A vector on the line spanned by v_4, v_5 is $(a + b, a, b)$. It is proportional to none of the other five columns: the only non-coordinate possibilities are v_6, which would require $a = b \neq 0$ and $a + b = 0$, contrary to $2 \neq 0$, and v_7, which would require $a = b = a + b \neq 0$. The coordinate cases force $a = b = 0$. Permuting the three coordinates gives the same conclusion for the pairs 46 and 56. Thus there are no additional dependent triples, while $\det(v_4, v_5, v_6) = -2 \neq 0$. The represented matroid is F_7^-. □

Proposition 8 (Prior-work counterexamples)**.** *Conjecture 3 is false over every prime-power base field* $\mathbb{F}_q$*.*

Proof. Write $q = p^a$. Lemma 3.3 of [6] ensures that the weighted cyclic flats of every ordinary matroid lift to a coordinate q-matroid. If p is odd, let N be the coordinate q-matroid corresponding to F_7. Its corresponding ordinary matroid is representable over $\mathbb{F}_2$, but not in characteristic p by Lemma 6. Lemma 5 therefore rules out representability of N.

If $p = 2$, instead take the coordinate q-matroid corresponding to F_7^-. The ordinary matroid F_7^- is representable over $\mathbb{F}_3$, but has no representation in characteristic two by Lemma 7. Again Lemma 5 shows that N is not representable. In both cases the hypothesis of Conjecture 3 holds and its conclusion fails. □

2.2 The exact lifting criterion

The converse to Lemma 5 requires more than checking coordinate subspaces. A characteristic-p representation of the ordinary matroid must be converted into a rank-metric representation whose rank is correct on every $\mathbb{F}_q$-subspace. A generic diagonal rescaling provides that conversion.

Lemma 9. *If a finite matroid M is representable over some field of characteristic p, then for every $q = p^a$ it is representable over a finite extension of $\mathbb{F}_q$.*

Proof. Let the ground set be $[n]$ and let $r = r_M([n])$. The case $r = 0$ is immediate. Otherwise take an $r \times n$ universal matrix $X = (x_{ij})$ over $\mathbb{F}_p[x_{ij}]$. For every r-subset $B \subseteq [n]$, let Δ_B be its maximal minor. Quotient by the ideal generated by Δ_B for all non-bases B, and then invert the product of Δ_B over all bases. The assumed representation over a characteristic-p field shows that this localized, finitely generated $\mathbb{F}_p$-algebra is nonzero. Choose a maximal ideal and pass to its residue field. Every nonbasis minor vanishes because it belongs to the defining ideal, whereas every basis minor survives because their product was inverted and is therefore a unit. Thus the specialized matrix has exactly the bases of M. The residue field is finitely generated as an $\mathbb{F}_p$-algebra, hence finite by Zariski's lemma. It is some $\mathbb{F}_{p^m}$. Extending scalars to $\mathbb{F}_{p^{\operatorname{lcm}(a,m)}}$ gives a representation over a finite extension of $\mathbb{F}_q$. □

Lemma 10 (Generic diagonal rank)**.** *Let $A \in k^{r\times n}$, let $B \in k^{n\times d}$, and let $D(t) = \operatorname{diag}(t_1, \dots, t_n)$, where the t_i are algebraically independent over k. Then, over $k(t_1, \dots, t_n)$,*

$$\operatorname{rank}(AD(t)B) = \min_{S\subseteq[n]} \big(\operatorname{rank}(A_S) + \operatorname{rank}(B_{[n]\setminus S})\big).$$

Proof. For $S \subseteq [n]$, write $\overline{S} = [n] \setminus S$. Since

$$AD(t)B = A_S D_S B_S + A_{\overline{S}} D_{\overline{S}} B_{\overline{S}},$$

subadditivity gives the displayed rank as at most $\operatorname{rank}(A_S) + \operatorname{rank}(B_{\overline{S}})$.

For the reverse inequality, an $\ell \times \ell$ minor of $AD(t)B$ expands by Cauchy–Binet as

$$\sum_{|I|=\ell} \det(A_{R,I}) \det(B_{I,C}) \prod_{i\in I} t_i.$$

The monomials are distinct. Thus some ℓ-minor is nonzero exactly when there is an ℓ-element set I independent both in the column matroid of A and in the row matroid of B. The rank is consequently the maximum size of a common independent set of these matroids. Edmonds' matroid-intersection theorem [5] identifies this maximum with the displayed minimum. □

Lemma 11. *Let $V \le \mathbb{F}_q^n$ have basis matrix Y. For $S \subseteq [n]$,*

$$\operatorname{rank}(Y_{[n]\setminus S}) = \dim V - \dim(V \cap \phi(S)).$$

Proof. The indicated submatrix represents the coordinate projection $V \to \mathbb{F}_q^{[n]\setminus S}$. Its kernel is precisely $V \cap \phi(S)$, so the formula is rank–nullity. □

Lemma 12. *Let M be a matroid on $[n]$, and let $\mathcal{Z}(M)$ be its cyclic flats. For every $V \le \mathbb{F}_q^n$,*

$$\begin{aligned}&\min_{S\subseteq[n]} \big(r_M(S) + \dim V - \dim(V \cap \phi(S))\big)\\ &\quad= \min_{Z\in\mathcal{Z}(M)} \big(r_M(Z) + \dim V - \dim(V \cap \phi(Z))\big).\end{aligned}$$

Proof. Fix $S \subseteq [n]$, let $F = \operatorname{cl}_M(S)$, and let Z be the cyclic part of the flat F, namely the set of non-coloops of $M|F$. Then Z is a cyclic flat. Indeed, it is a union of circuits. It is closed because an element outside F is not in $\operatorname{cl}_M(Z) \subseteq F$, while an element of $F \setminus Z$ is a coloop of $M|F$ and so is not in $\operatorname{cl}_M(F \setminus \{e\})$, which contains Z. We have $r_M(F) = r_M(S)$ and $\phi(S) \subseteq \phi(F)$, so replacing S by F cannot increase the expression being minimized.

Put $C = F \setminus Z$. The elements of C are coloops of $M|F$, whence $r_M(F) = r_M(Z) + |C|$. Moreover,

$$\dim(V \cap \phi(F)) - \dim(V \cap \phi(Z)) \le |C|.$$

It follows that the expression at Z is no larger than the expression at F, and hence no larger than the expression at S. Taking the minimum over S proves the equality. □

Theorem 13 (Characteristic-compatible lifting)**.** *Let M be a matroid on $[n]$ representable over some field of characteristic p, and let $q = p^a$. The coordinate q-matroid corresponding to M is representable over a finite extension of $\mathbb{F}_q$.*

Proof. By Lemma 9, choose a finite extension $k/\mathbb{F}_q$ and a matrix $A \in k^{r\times n}$ representing M. Let $D(t) = \operatorname{diag}(t_1, \dots, t_n)$ with the t_i algebraically independent. For $V \le \mathbb{F}_q^n$, choose a basis matrix Y_V. Lemmas 10, 11, and 12 give

$$\operatorname{rank}(AD(t)Y_V) = \min_{Z\in\mathcal{Z}(M)} \big(r_M(Z) + \dim V - \dim(V \cap \phi(Z))\big),$$

which is exactly (2) for the coordinate q-matroid corresponding to M.

There are only finitely many $\mathbb{F}_q$-subspaces V. For each V, choose a nonzero minor of $AD(t)Y_V$ whose size equals this rank (with the 0-minor understood as 1), and multiply all these minors together with $t_1 \cdots t_n$. The resulting polynomial is nonzero. Over a sufficiently large finite extension K/k it has a nonvanishing value: this follows, for example, by choosing $|K|$ larger than its degree in each variable and inducting on the number of variables. Choose such a point $s = (s_1, \ldots, s_n)$ and put $G = AD(s)$. The selected minors show that $\mathrm{rank}_K(GY_V)$ is at least the required rank for every V. The factor $t_1 \cdots t_n$ ensures that every s_i is nonzero, so G is obtained from A by nonzero column scalings and still represents M.

For the reverse inequality, fix a cyclic flat Z, put $W = V \cap \phi(Z)$, and extend a basis of W to one of V. The images of the basis vectors from W lie in the span of the columns G_Z, whose dimension is $r_M(Z)$; the remaining vectors increase rank by at most $\dim V - \dim W$. Thus

$$\mathrm{rank}_K(GY_V) \leq r_M(Z) + \dim V - \dim(V \cap \phi(Z)).$$

Taking the minimum over cyclic flats gives the opposite inequality. Hence $\mathrm{rank}_K(GY_V)$ equals the coordinate q-matroid rank for every $V \leq \mathbb{F}_q^n$, so G is a representation. $\square$

Proof of Theorem 1. Necessity is Lemma 5. Conversely, suppose that M is representable over a field of characteristic p. Since N is coordinate, its weighted cyclic flats are exactly the coordinate lifts of the weighted cyclic flats of M. Hence N is the coordinate q-matroid corresponding to M, and Theorem 13 makes it representable. $\square$

3 Hall criterion and presentation rank formula

The preceding result depends essentially on the coordinate correspondence. For direct sums, the correct tool is instead the union and presentation-rank machinery developed by Saaltink [11, Sec. 3.1 and Thm. 15]. We recall the needed statements, with proofs, because the equality of the two rank minima in the next section uses them explicitly. The results in this section are known and are not claimed as new.

3.1 A two-dimensional example

Let $E_1 = E_2 = \mathbb{F}_q$ and present each one-dimensional free q-matroid by the one-slot list (0). The natural presentation on $E_1 \oplus E_2$ is

$$(0 \oplus E_2,\ E_1 \oplus 0).$$

For every line $L \leq E_1 \oplus E_2$, at least one of these two coordinate axes does not contain L, while neither contains the whole plane. Hence $f(L) \geq 1$ and $f(E_1 \oplus E_2) = 2$. Lemma 14 below shows that every subspace is independent, including every oblique line $\langle(1, \alpha)\rangle$. Thus the presentation gives the rank-two free q-matroid, exactly the direct sum of its two rank-one factors. The argument below proves the same agreement for non-coordinate subspaces and summands with arbitrary rank defects.

3.2 Hall criterion and rank formula

Lemma 14 (Hall criterion, [11, Thm. 15])**.** *Let $\mathcal{X} = (X_1, \ldots, X_k)$ be a presentation in E and $I \leq E$. Then I is a partial q-transversal of $\mathcal{X}$ iff $\dim W \leq f_{\mathcal{X}}(W)$ for every $W \leq I$.*

Proof. $(\Rightarrow)$ Let $W \leq I$, extend a basis B_W of W to a basis B of I, and take an injective avoidance matching $\mu : B \to [k]$. For each $b \in B_W$ we have $b \in W$ and $b \notin X_{\mu(b)}$, so $W \not\leq X_{\mu(b)}$; the indices $\mu(B_W)$ are distinct and each is counted by $f_{\mathcal{X}}(W)$, giving $\dim W = |B_W| \leq f_{\mathcal{X}}(W)$.

($\Leftarrow$) Let B be any basis of I and form the bipartite graph on B and $\{1,\dots,k\}$ with $b \sim i$ iff $b \notin X_i$; an avoidance transversal is a matching saturating B. By Hall's marriage theorem [10], it is enough to check every subset of B. For $C \subseteq B$ put $W = \operatorname{span}(C)$, so $\dim W = |C|$. An index i is adjacent to some $c \in C$ iff some $c \notin X_i$, i.e. (as X_i is a subspace) iff $W \nleq X_i$; thus C has exactly $f_{\mathcal{X}}(W)$ neighbours, and Hall's condition $|C| \le f_{\mathcal{X}}(W)$ holds by hypothesis. So B has an avoidance transversal; as B was arbitrary, I is a partial q-transversal. □

Lemma 15 (rank determined by independents)**.** *A q-matroid rank function ρ on E satisfies $\rho(S) = \max\{\dim I : \ I \le S,\ \rho(I) = \dim I\}$ for every $S \le E$.*

Proof. $\rho(0) = 0$. If L is a line then $A \cap L = 0$ for $L \nleq A$, and submodularity gives $\rho(A+L) \le \rho(A) + \rho(L) \le \rho(A) + 1$; with monotonicity, adding a line raises rank by 0 or 1. If $I \le S$ with $\rho(I) < \rho(S)$, some line $L \le S$ has $\rho(I+L) = \rho(I)+1$: otherwise every line gives $\rho(I+L) = \rho(I)$, and induction on $\dim U$ (using submodularity on $A = I + U'$, $B = I + L$ with $U = U' + L$) yields $\rho(I+U) = \rho(I)$ for all $U \le S$, contradicting $U = S$. Start with the independent space $I = 0$. Whenever $\rho(I) < \rho(S)$, choose a line $L \le S$ with $\rho(I+L) = \rho(I)+1$. Then $L \nleq I$, and therefore

$$\dim(I+L) = \dim I + 1 = \rho(I) + 1 = \rho(I+L),$$

so independence is preserved. Iteration produces an independent subspace of dimension $\rho(S)$. Conversely, every independent $I \le S$ satisfies $\dim I = \rho(I) \le \rho(S)$. □

Lemma 16 (Presentation rank formula, [11, Sec. 3.1])**.** *If $\mathcal{X}$ presents the transversal q-matroid $M_{\mathcal{X}} = (E, \rho_{\mathcal{X}})$, then for every $S \le E$,*

$$\rho_{\mathcal{X}}(S) = r_{\mathcal{X}}(S) := \min_{W \le S} \big(\dim S - \dim W + f_{\mathcal{X}}(W)\big).$$

Proof. Write $g_i(W) = 0$ if $W \le X_i$ and 1 otherwise, so $f_{\mathcal{X}} = \sum_i g_i$. Each g_i is monotone. It is also submodular. If both U, V lie in X_i, every term in

$$g_i(U+V) + g_i(U \cap V) \le g_i(U) + g_i(V)$$

is zero. If exactly one lies in X_i, then $U + V \nleq X_i$ and $U \cap V \le X_i$, so both sides equal one. If neither lies in X_i, the right-hand side is two and the left-hand side is at most two. Summing over i proves that $f_{\mathcal{X}}$ is monotone and submodular.

We verify the rank axioms for $r_{\mathcal{X}}$. Since $f_{\mathcal{X}}(0) = 0$, the term $W = 0$ gives $r_{\mathcal{X}}(S) \le \dim S$, while every term is nonnegative. Suppose $S \le T$ and fix $W \le T$. The natural map $S/(S \cap W) \to T/W$ is injective, and monotonicity of $f_{\mathcal{X}}$ gives

$$\dim S - \dim(S \cap W) + f_{\mathcal{X}}(S \cap W) \le \dim T - \dim W + f_{\mathcal{X}}(W).$$

The left side is an admissible term for $r_{\mathcal{X}}(S)$. Taking the minimum over $W \le T$ therefore gives $r_{\mathcal{X}}(S) \le r_{\mathcal{X}}(T)$.

For submodularity choose minimisers $A \le S$ and $B \le T$. The subspaces $A + B \le S + T$ and $A \cap B \le S \cap T$ are admissible in the corresponding minima, whence

$$\begin{aligned}
r_{\mathcal{X}}(S+T) + r_{\mathcal{X}}(S \cap T) &\le \dim(S+T) + \dim(S \cap T) \\
&\quad - \dim(A+B) - \dim(A \cap B) \\
&\quad + f_{\mathcal{X}}(A+B) + f_{\mathcal{X}}(A \cap B) \\
&\le \dim S - \dim A + f_{\mathcal{X}}(A) + \dim T - \dim B + f_{\mathcal{X}}(B) \\
&= r_{\mathcal{X}}(S) + r_{\mathcal{X}}(T).
\end{aligned}$$

Here the second inequality uses the two dimension identities and submodularity of $f_{\mathcal{X}}$. Thus $r_{\mathcal{X}}$ is a q-matroid rank function.

Finally, because the term $W = 0$ equals $\dim I$, a subspace I is $r_{\mathcal{X}}$-independent exactly when $\dim I - \dim W + f_{\mathcal{X}}(W) \ge \dim I$ for every $W \le I$, or equivalently $\dim W \le f_{\mathcal{X}}(W)$ for every $W \le I$. Lemma 14 identifies these spaces with the partial q-transversals of $\mathcal{X}$, which are precisely the $\rho_{\mathcal{X}}$-independent spaces. The two rank functions agree by Lemma 15. □

4 Direct sums of arbitrary transversal q-matroids

For $U \le E_1$, $V \le E_2$ write $f_1(U) = |\{i : U \not\le A_i\}|$ and $f_2(V) = |\{j : V \not\le B_j\}|$, and let ρ_1, ρ_2 be the rank functions of M_1, M_2. For $Z \le E_1 \oplus E_2$ one has $Z \le A_i \oplus E_2 \iff \pi_1 Z \le A_i$ and $Z \le E_1 \oplus B_j \iff \pi_2 Z \le B_j$; counting slots of (5),

$$f_{\mathcal{X}_1 \oplus \mathcal{X}_2}(Z) = f_1(\pi_1 Z) + f_2(\pi_2 Z). \tag{6}$$

Proof of Theorem 2. By (6), $r_{\mathcal{X}_1 \oplus \mathcal{X}_2}(X) = \min_{Z \le X}\big(\dim X - \dim Z + f_1(\pi_1 Z) + f_2(\pi_2 Z)\big)$. Call this $R(X)$ and call the displayed direct-sum minimum $D(X)$.

$D \le R$. Fix $Z \le X$. Choosing $W = \pi_1 Z$ in Lemma 16 for M_1 gives $\rho_1(\pi_1 Z) \le f_1(\pi_1 Z)$, and likewise $\rho_2(\pi_2 Z) \le f_2(\pi_2 Z)$. Taking $Y = Z$ in the definition of D,

$$D(X) \le \dim X - \dim Z + \rho_1(\pi_1 Z) + \rho_2(\pi_2 Z) \le \dim X - \dim Z + f_1(\pi_1 Z) + f_2(\pi_2 Z).$$

Minimising over Z yields $D(X) \le R(X)$.

$R \le D$. Fix $Y \le X$ and set $P = \pi_1 Y \le E_1$, $Q = \pi_2 Y \le E_2$. Let $U \le P$, $V \le Q$ be arbitrary and put $Z = Y \cap (U \oplus V)$, so $Z \le X$ with $\pi_1 Z \le U$, $\pi_2 Z \le V$. Viewing Y and $U \oplus V$ inside $P \oplus Q$, Grassmann's identity and $Y + (U \oplus V) \le P \oplus Q$ give

$$\dim Z = \dim Y + \dim(U \oplus V) - \dim\big(Y + (U \oplus V)\big) \ge \dim Y + \dim U + \dim V - \dim P - \dim Q.$$

Using Z in R and monotonicity of f_1, f_2 ($\pi_1 Z \le U$, $\pi_2 Z \le V$),

$$\begin{aligned} R(X) &\le \dim X - \dim Z + f_1(U) + f_2(V) \\ &\le \dim X - \dim Y + (\dim P - \dim U + f_1(U)) + (\dim Q - \dim V + f_2(V)). \end{aligned}$$

This holds for all $U \le P$, $V \le Q$; minimising over U and V and applying Lemma 16 to M_1, M_2 gives $R(X) \le \dim X - \dim Y + \rho_1(P) + \rho_2(Q)$. Minimising over Y yields $R(X) \le D(X)$.

Thus $R = D$: the presentation rank of $\mathcal{X}_1 \oplus \mathcal{X}_2$ equals the direct-sum rank. In particular both sides have the same independent spaces, so the transversal q-matroid presented by $\mathcal{X}_1 \oplus \mathcal{X}_2$ is exactly $M_1 \oplus M_2$. □

5 Finite direct sums

Associativity of the Ceria–Jurrius direct sum is known from the cyclic-flat theory of Gluesing-Luerssen and Jany [8, Thm. 6.3 and Cor. 6.4]. The following corollary records an explicit flattened rank formula and, for transversal summands, the natural finite presentation supplied by Theorem 2.

Corollary 17 (Flattened finite-sum formula and presentation)**.** *For q-matroids $M_1, \dots, M_m$ over the same field $\mathbb{F}_q$, on ambient spaces $E_1, \dots, E_m$, every parenthesisation of the iterated direct sum has, after the canonical identification of ambient spaces with $E := E_1 \oplus \cdots \oplus E_m$, the rank function*

$$\Phi_{[m]}(X) = \min_{Y \le X}\Big(\dim X - \dim Y + \sum_{a=1}^{m} \rho_a(\pi_a Y)\Big).$$

Consequently, if the M_a are transversal with presentations $\mathcal{X}_a$, then $M_1 \oplus \cdots \oplus M_m$ is transversal with presentation the list of inflated subspaces $E_1 \oplus \cdots \oplus X_{a,s} \oplus \cdots \oplus E_m$ (only the a-th coordinate restricted), counted with multiplicity.

Proof. For every nonempty $A \subseteq [m]$, put $E_A = \bigoplus_{a \in A} E_a$ and define

$$\Phi_A(P) = \min_{Z \le P}\Big(\dim P - \dim Z + \sum_{a \in A} \rho_a(\pi_a Z)\Big) \qquad (P \le E_A).$$

First, $\Phi_{\{a\}} = \rho_a$. Indeed, if $Z \le P \le E_a$, a chain from Z to P may be written $Z = Z_0 < Z_1 < \cdots < Z_d = P$, where $d = \dim P - \dim Z$ and $Z_{t+1} = Z_t + L_t$ for a line $L_t \not\le Z_t$. Submodularity gives

$$\rho_a(Z_{t+1}) + \rho_a(0) \le \rho_a(Z_t) + \rho_a(L_t) \le \rho_a(Z_t) + 1.$$

Summing yields $\rho_a(P) \le \rho_a(Z) + \dim P - \dim Z$. Every term in the minimum is therefore at least $\rho_a(P)$, and equality is obtained at $Z = P$.

We next prove the partition identity. Let $A = I \sqcup J$ with both blocks nonempty, and set

$$\Gamma(X) = \min_{Y \le X} \bigl(\dim X - \dim Y + \Phi_I(\pi_I Y) + \Phi_J(\pi_J Y)\bigr).$$

For $T \le X$, choosing $\pi_I T$ and $\pi_J T$ themselves in the two inner minima gives

$$\Gamma(X) \le \dim X - \dim T + \sum_{a \in A} \rho_a(\pi_a T).$$

Minimising over T proves $\Gamma(X) \le \Phi_A(X)$.

For the reverse inequality, fix $Y \le X$, put $P = \pi_I Y$ and $Q = \pi_J Y$, and choose arbitrary $U \le P$, $V \le Q$. Set $H = Y \cap (U \oplus V)$. Inside $P \oplus Q$, Grassmann's identity gives

$$\dim H \ge \dim Y + \dim U + \dim V - \dim P - \dim Q.$$

Moreover $\pi_i H \le \pi_i U$ for $i \in I$ and $\pi_j H \le \pi_j V$ for $j \in J$. Using H in the minimum defining $\Phi_A(X)$ and then monotonicity of the component ranks yields

$$\begin{aligned}\Phi_A(X) &\le \dim X - \dim H + \sum_{a \in A} \rho_a(\pi_a H)\\ &\le \dim X - \dim Y + \left(\dim P - \dim U + \sum_{i \in I} \rho_i(\pi_i U)\right)\\ &\quad + \left(\dim Q - \dim V + \sum_{j \in J} \rho_j(\pi_j V)\right).\end{aligned}$$

Minimising independently over $U \le P$ and $V \le Q$ gives

$$\Phi_A(X) \le \dim X - \dim Y + \Phi_I(P) + \Phi_J(Q).$$

Finally minimising over $Y \le X$ gives $\Phi_A(X) \le \Gamma(X)$, proving the partition identity.

The outermost operation in any parenthesisation is the binary direct sum of two smaller blocks. Induction on m, using the partition identity and the singleton calculation, shows that every parenthesisation has rank $\Phi_{[m]}$.

Now assume that $\mathcal{X}_a = (X_{a,1}, \ldots, X_{a,k_a})$ presents M_a. We prove the presentation statement by induction on m. It is tautological for $m = 1$. For $m > 1$, split a chosen parenthesisation at its outermost operation into nonempty blocks I and J. By induction, the two block sums are presented by lists in which an original slot $X_{a,s}$ is unrestricted in all other coordinates of its block. Theorem 2 says that the natural presentation of the outer binary sum further inflates each slot from block I by all of E_J, and each slot from block J by all of E_I. Thus every original $X_{a,s}$ is unrestricted in every coordinate except a, which is exactly the displayed inflated list. Its rank is $\Phi_{[m]}$ by the first part of the proof, independently of the chosen parenthesisation. □

6 Concluding remarks

Coordinate structure has sharply different roles in the two settings studied here. For representability it exposes a necessary characteristic condition, and Theorem 1 shows that this condition is also sufficient. For transversal direct sums, by contrast, coordinate structure is unnecessary: Theorem 2 works with arbitrary presentations, including nonminimal ones, and identifies their presentation rank with the Ceria–Jurrius rank on every subspace. The finite-sum formula in Corollary 17 makes this compatibility explicit under iteration.

Determining when the resulting non-coordinate transversal q-matroids are representable remains separate. The characteristic criterion proved here applies to coordinate q-matroids. A recent multilinear notion of q-matroid representability, introduced by Alfarano and Degen [2], is a different relaxation based on almost affine rank-metric codes; none of the representability statements here use that broader notion.

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