The optimal 4/3 density bound in Pilz's problem

Abstract

For a finite set A of positive integers, let $A * [n]$ be the set of integers with an odd number of representations aj, $a \in A$, $1 \le j \le n$. Holdridge and Pach showed that $|A * [n]|/n$ tends to a limit $h(A)$, that $h(A) \ge 8/7$ unless $A = \{a\}$ or $A = \{a, 2a\}$, and expected that $8/7$ could be replaced by $4/3$. We prove this. The constant $4/3$ is best possible, as their computation of $h(\{1, 3\})$ already shows. The proof is elementary, uses one divisibility-cut lemma and a calculation for sets of powers of two, and requires no computer search. It is independent of the resolution of the finite-interval conjecture in [4]. All results are formally verified in Lean 4.

Introduction

Here $\mathbb{N} = \{1, 2, \ldots\}$ and $[n] = \{1, \ldots, n\}$, with $[0] = \varnothing$. For finite $A, B \subset \mathbb{N}$ we follow [1] and write $A * B$ for the set of integers with an odd number of ordered representations ab, where $a \in A$, $b \in B$. Thus $A * [n]$ is the symmetric difference of the sets $a[n]$, $a \in A$.

Pilz's Extended 1–2–3 Conjecture [5, 2] asserts that $|A * [n]| \ge n$ for every finite nonempty A and every $n \ge 1$. Holdridge and Pach [1] proved it for all sufficiently large n, and a proof for every n is given in [4]. Alongside the finite-interval problem, [1] studies the asymptotic behaviour for a fixed set A. Using the exact counting formula of Ke and Meyer [3, (2.1)], they show [1, Lemma 1.6] that the density

$$h(A) = \lim_{n \to \infty} \frac{|A * [n]|}{n}$$

exists, and they prove [1, Theorem 1.8] that $h(A) \ge 8/7$ unless A has the form $\{a\}$ or $\{a, 2a\}$. In [1, Remark 1.9] they write that they expect the same theorem to hold with $8/7$ replaced by $4/3$, and that their method could achieve this up to checking a number of finite cases too large to carry out. They also computed $h(\{a, b\}) = 2 - 2/b$ for coprime $a < b$, and showed by a computer search that the minimum of h over three-element sets is $4/3$ and over four-element sets is $5/3$ [1, Lemma 5.6]. We prove the bound $4/3$ for all sets, without any case check.

Theorem 1. *For every finite nonempty $A \subset \mathbb{N}$, the limit $h(A)$ exists and is rational. If A is neither $\{a\}$ nor $\{a, 2a\}$ for any $a \in \mathbb{N}$, then $h(A) \ge 4/3$. The excluded sets have density one, and $h(\{1, 3\}) = 4/3$, so the constant is best possible.*

The last sentence is included for completeness; it is already contained in [1, Lemma 5.6]. The new content is the lower bound $4/3$. For sets of powers of two the proof gives the stronger bound $3/2$.

The argument has two ingredients. The first is a divisibility cut (Lemma 3), which splits A according to divisibility by a prime p and loses at most $2/p$ times the density of the part coprime to p. It is a variant of the splitting estimates of [1, Lemma 4.1, Corollary 4.3], in which the loss is controlled by a density rather than by a count of pairs. The second is an explicit calculation for sets of powers of two, which, after division by the gcd, are the only sets the cut at an odd prime does not reach.

We first establish the two counting facts needed for the proof.

Lemma 2 (Existence and scaling)**.** *For every finite nonempty $A \subset \mathbb{N}$,*

$$h(A) = \sum_{\varnothing \neq J \subseteq A} (-2)^{|J|-1} \frac{\min J}{\operatorname{lcm}(J)}.$$

In particular $h(A)$ exists and is rational. Moreover, $h(cA) = h(A)$ for every $c \in \mathbb{N}$.

The counting formula below is [3, (2.1)], and the limit formula is [1, Lemma 1.6]; we include the short proof for completeness.

Proof. An integer lying in exactly r of a finite family of sets contributes

$$\sum_{j=1}^{r} (-2)^{j-1} \binom{r}{j} = \frac{1-(-1)^r}{2}$$

to their signed intersection sum. This is one when r is odd and zero when r is even. The intersection of $a[n]$, $a \in J \neq \varnothing$, consists of the positive multiples of $\operatorname{lcm}(J)$ at most $(\min J)n$. Consequently,

$$|A * [n]| = \sum_{\varnothing \neq J \subseteq A} (-2)^{|J|-1} \left\lfloor \frac{(\min J)n}{\operatorname{lcm}(J)} \right\rfloor.$$

Divide by n and take the limit term by term in this finite sum. Finally, multiplication by c preserves all representation counts and is injective, so $(cA) * [n] = c(A * [n])$. □

Lemma 3 (The divisibility cut)**.** *Let p be a prime and let $A = B \sqcup C$, where no element of B is divisible by p and every element of C is divisible by p. Then*

$$h(A) \;\geq\; (1 - 2/p)h(B) + h(C).$$

Here the empty set may be assigned density zero.

Proof. The essential identity is

$$(B * [n]) \cap p\mathbb{N} = p(B * [\lfloor n/p \rfloor]).$$

Indeed, if $py = aj$ with $a \in B$, then $p \nmid a$ implies $j = pk$. Thus $py = aj$, $1 \leq j \leq n$, corresponds bijectively to $y = ak$, $1 \leq k \leq \lfloor n/p \rfloor$. In particular, their representation counts have the same parity.

Put $S = B * [n]$ and $T = C * [n]$. Since $T \subseteq p\mathbb{N}$ and $A * [n] = S \triangle T$,

$$|A * [n]| = |S| + |T| - 2|S \cap T| \geq |B * [n]| + |C * [n]| - 2|B * [\lfloor n/p \rfloor]|.$$

Divide by n and take limits. The final term divided by n tends to $h(B)/p$: set $m = \lfloor n/p \rfloor$, and use $(|B * [m]|/m)(m/n) \to h(B)/p$. □

Lemma 4 (A baseline)**.** *Every finite nonempty $A \subset \mathbb{N}$ has $h(A) \geq 1$.*

Proof. Induct on $|A|$. A singleton gives exactly n products, so has density one. For $|A| \geq 2$, divide all elements by their greatest common divisor; Lemma 2 leaves the density unchanged. The resulting set has gcd one and contains an element greater than one. Choose a prime p dividing that element, and partition the set into $B = \{a : p \nmid a\}$ and $C = \{a : p \mid a\}$. Both parts are nonempty: the second by our choice, the first because the gcd is one. By induction their densities are at least one. Since $p \geq 2$, Lemma 3 gives

$$h(A) \geq (1 - 2/p) \cdot 1 + 1 = 2 - 2/p \geq 1.$$

□

Proof of Theorem 1. Existence and rationality of $h(A)$ are Lemma 2. Divide A by its gcd. This preserves its density and whether it belongs to either excluded family. If an odd prime p divides one of its elements, the partition in the preceding proof and Lemmas 3–4 give

$$h(A) \geq 2 - 2/p \geq 4/3.$$

Otherwise every element is a power of two. In such a set, deleting the least element cannot increase the density: divide by that element, leaving $B = \{1\}$ and a set C of even numbers, and apply Lemma 3 with $p = 2$ to obtain $h(B \cup C) \geq h(C)$. Scaling back proves the assertion.

For two powers of two, scale to $A = \{1, u\}$. Outside the excluded case $u = 2$, we have $u \geq 4$, and Lemma 2 gives

$$h(A) = 2 - \frac{2}{u} \geq \frac{3}{2}.$$

For three powers of two, scale to $A = \{1, u, uv\}$, where $u, v \geq 2$. The same formula gives

$$h(A) = 3 - \frac{2}{u} - \frac{2}{v} + \frac{2}{uv} = 1 + 2\left(1 - \frac{1}{u}\right)\left(1 - \frac{1}{v}\right) \geq \frac{3}{2}.$$

For a larger set, delete the least element repeatedly until three remain. Its original density is at least that of the final three-element set. This proves the required bound, and in fact the stronger $3/2$ bound in the powers-of-two case.

Finally, a singleton has density one, and direct counting gives

$$|\{a, qa\} * [n]| = 2n - 2\lfloor n/q \rfloor \qquad (q \geq 2).$$

Taking $q = 2$ proves the remaining exception; taking $a = 1, q = 3$ proves sharpness. □

Scope. The set A is fixed while n tends to infinity. There is no assertion that $|A * [n]| \geq 4n/3$ for every n: for example, $\{1, 2, 3\} * [3] = \{1, 4, 9\}$. That density one occurs exactly in the two stated families already follows from [1, Theorem 1.8]. The proof does not classify all sets of density $4/3$; among sets of greatest common divisor one with elements at most 12, the equality cases are $\{1, 3\}$, $\{2, 3\}$ and $\{1, 2, 3\}$.

Formalisation

The accompanying Lean 4 sources give a formal proof of Theorem 1. The complete statement is `PilzDensity.companion_theorem`; optimality of $4/3$ is `PilzDensity.four_thirds_optimal`, and the limit for $\{1, 3\}$ is `PilzDensity.sharpness_limit`. The density is defined directly as the limit of the ordered odd-representation count, and the development does not depend on the finite-interval theorem of [4]. The repository's `paper/README.md` maps each lemma to its Lean theorem and gives build and verification commands.

Acknowledgements

The question answered here was raised by Philippa Holdridge and Péter Pál Pach [1, Remark 1.9], whose paper also supplies the density framework and the splitting estimates on which the divisibility cut is modelled. The counting formula used throughout is recorded by Ke and Meyer [3, (2.1)].

How this work was produced. The arguments in this note, its written proofs, and its formalisation were developed with substantial assistance from AI systems. The written proofs have not been checked line by line by a human mathematician. Correctness rests on the Lean 4 formalisation: every stated theorem is machine-checked and depends only on the standard axioms of Lean's foundations. Where the text and the formal proof differ, the formal proof is authoritative.

License

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References

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[3] W.-F. Ke and J. H. Meyer, *On a conjecture regarding the symmetric difference of certain sets*, Bull. Aust. Math. Soc. **111** (2025), 397–404.

[4] *Pilz's Extended 1–2–3 Conjecture over every field: sparse multiples of weighted integer intervals*, preprint (2026), with Lean 4 formalisation; an earlier proof of the binary case is in the same archive, *Pilz's Extended 1–2–3 Conjecture: proofs and Lean 4 formalisations*, Zenodo, 2026, doi:10.5281/zenodo.23568997.

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