Paraconsistent Micro-Geometry: A Dialethic Non-Standard Approach to Subgroup-Stratified Vitali Measures

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September 11, 2026

Abstract

Classical measure theory segregates measurable sets from pathological entities like Vitali sets by enforcing the equality of inner and outer measures under countable subadditivity. When this equality fails, classical frameworks relegate the set to "non-measurable," discarding quantitative evaluation. This paper introduces a paraconsistent, dialethic framework within Non-Standard Analysis (NSA) that accommodates the inner-outer measure gap for subgroup-stratified Vitali selectors ($^*V$). By defining internal hyperfinite outer and inner measures governed by dense shift sets $H^*_{\alpha,b}$, we formalize an architecture where a set simultaneously exhibits an inner measure of 0 and a non-zero, subgroup-stratified infinitesimal outer measure $\ell_{\alpha,b} = b/N_\alpha$. Leveraging paraconsistent logic to prevent trivial explosion, this approach reframes the measure-theoretic defect as a true micro-geometric signature.

1 1. Introduction

The historical tension between translation invariance, countable additivity, and the axiom of choice culminates in the classical non-measurability of Vitali sets. In standard real analysis, a set $V \subset \mathbb{R}$ is deemed non-measurable because its Lebesgue outer measure $\lambda^*(V)$ and inner measure $\lambda_*(V)$ cannot be reconciled without violating additivity.Non-Standard Analysis (NSA), following Robinson and Nelson, provides an internal universe $^*V$ wherein infinitesimal scales become accessible. However, attempting to force hyperreal-valued measures onto standard external power sets $\mathcal{P}(\mathbb{R})$ via classical countable coverings ($\sum_{i=1}^{\infty}$) collapses under non-Archimedean scale mismatches.To resolve this without erasing fine-grained structure via the standard part operator (st), we substitute classical countable limits with internal hyperfinite shift sets ($H^*_{\alpha,b}$). This substitution yields a divergence between the internal hyperfinite outer measure μ^*_α and inner measure μ^α_*. Rather than treating this gap as an error requiring elimination, we invoke dialethism—a paraconsistent logical framework—to formally accept both statements as true semantic properties of the set, establishing a consistent arithmetic of inconsistent measures.

2 2. Preliminaries: Internal Hyperfinite Shift Architectures

Let $\mathbb{R}$ be extended to the hyperreal field ${}^*\mathbb{R}$, and let α denote a dense subgroup index governing a partition of an interval $[0, b]$. We construct an internal hyperfinite shift set $H^*_{\alpha,b}$ with internal cardinality $N_\alpha = |H^*_{\alpha,b}| \in^* \mathbb{N} \setminus \mathbb{N}$.The internal Vitali selector ${}^*V \subset^* [0, b]$ is constructed via internal choice over equivalence classes defined by the translation subgroup. To capture its micro-geometric architecture, we replace external countable covering classes with internal hyperfinite indexing.

- Definition 1 (Internal Hyperfinite Outer Measure)For any internal set $E \subseteq^* [0, b]$, the subgroup-stratified hyperfinite outer measure is defined as:

$$\mu^*_\alpha(E) = \inf \left\{ \sum_{h \in H^*_{\alpha,b}} \text{len}(I_h) \;\middle|\; E \subseteq \bigcup_{h \in H^*_{\alpha,b}} I_h \text{ where each } I_h \text{ is an internal interval} \right\}$$

- Definition 2 (Internal Hyperfinite Inner Measure)Dramatically mirroring the outer framework, the inner measure is defined via internal hyperfinite packings:

$$\mu^\alpha_*(E) = \sup \left\{ \sum_{h \in K^*} \text{len}(I_h) \;\middle|\; \bigcup_{h \in K^*} I_h \subseteq E \text{ where } K^* \subseteq H^*_{\alpha,b} \text{ and each } I_h \text{ is an internal interval} \right\}$$

- 3. Explicit Evaluation and the Measure GapEvaluating these formulations across measurable sets and pathological selectors reveals the exact mechanics of the stratification.

- Theorem 1 (Measurable Consistency)For any standard Lebesgue measurable set $A \subseteq [0, b]$ and its internal counterpart ${}^*A$:

$$\mu^\alpha_*({}^*A) = \mu^*_\alpha({}^*A) =^* \lambda({}^*A)$$

  Furthermore, for standard sets, the stratification parameter α is dormant: the sum over the hyperfinite grid yields $N_\alpha \cdot (b/N_\alpha) = b$, preserving macroscopic compatibility.

- Theorem 2 (The Vitali Infinitesimal Gap)For an internal Vitali selector ${}^*V$:Inner Measure Collapse: $\mu^\alpha_*({}^*V) = 0$. Because ${}^*V$ contains no nontrivial open internal intervals, no positive internal measure can be packed from within.

- Outer Measure Infinitesimal: $\mu_\alpha^*({}^*V) = \frac{b}{N_\alpha} = \ell_{\alpha,b} > 0$. The internal covering infimum resolves to the exact reciprocal of the shift-set cardinality.Thus, the internal framework produces a structural gap:

$$\mu_*^\alpha({}^*V) \neq \mu_\alpha^*({}^*V)$$

In classical mathematics, this inequality terminates analysis. Here, it defines the micro-geometric weight of the subgroup rank α.

3 4. Paraconsistent Integration: A Dialethic Framework

To prevent the inequality $\mu_*^\alpha({}^*V) \neq \mu_\alpha^*({}^*V)$ from triggering the Principle of Explosion ($ex falso quodlibet$) under a unified measure assignment $\mu({}^*V)$, we formalize a dialethic semantic evaluation.Let $\mathcal{L}$ be a paraconsistent logic (such as a da Costa system or relevant logic) where the presence of a truth glut—a proposition P and its negation $\neg P$ both taking designated truth values—does not trivialize deduction.
We assign dual predicates to the set ${}^*V$:$\mathfrak{T}_1({}^*V) := [\mu({}^*V) = 0]$ (Derived from inner measure packing)$\mathfrak{T}_2({}^*V) := [\mu({}^*V) = \ell_{\alpha,b}]$ (Derived from outer measure covering)Under classical logic, $\mathfrak{T}_1 \wedge \mathfrak{T}_2$ is a contradiction $\perp$. Under the dialethic non-standard extension, we define the paraconsistent measure space $\mathcal{M}_{\text{para}}$ such that:

$$\mathbf{Val}(\mu({}^*V)) = \{0, \ell_{\alpha,b}\}$$

This dual valuation is governed by hyperfinite subadditivity:

$$\mu^*\left(\bigcup_{k\in K^*} A_k\right) \leq \sum_{k\in K^*} \mu^*(A_k)$$

evaluated over internal hypernatural index sets $K^* \in^* \mathbb{N}$. Because the summation operator shares the non-Archimedean scale of the shift set, the partition of an interval into translated Vitali components satisfies:

$$\sum_{h\in H_{\alpha,b}j} \mu^*({}^*V + h) = N_\alpha \cdot \frac{b}{N_\alpha} = b$$

reconciling the macroscopic volume b with the microscopic infinitesimal constituents without trivializing the system.

4 5. Conclusion

By uniting internal hyperfinite outer/inner measures with paraconsistent logic, we bypass the traditional dead ends of non-measurable set theory. The measure of a subgroup-stratified Vitali set is no longer an undefined void or an arbitrary exclusion; it is an explicitly calculated, dialethic hyperreal value that captures both its internal null-packing and its external infinitesimal footprint.